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### EXTREMAL ELEMENTS OF THE CONVEX CONE $A_n$ OF FUNCTIONS

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Let  $A_1$  be the set of nonnegative real functions f on [0, 1]such that  $\nabla_h^1 f(x) = f(x) - f(x+h) \leq 0$ , h > 0, for  $[x, x+h] \subset [0, 1]$ , and let  $A_n$ , n > 1, be the set of functions belonging to  $A_{n-1}$  such that  $\nabla_h^n f(x) = \nabla_h^{n-1} f(x) - \nabla_h^{n-1} f(x+h) \leq 0$  for

$$[x, x + nh] \subset [0, 1]$$
.

Since the sum of two functions in  $A_n$  belongs to  $A_n$  and since a nonnegative real multiple of an  $A_n$  function is an  $A_n$  function, the set of  $A_n$  functions forms a convex cone. It is the purpose of this paper to give the extremal elements (i.e., the generators of extreme rays) of this cone, to prove that they form a closed set in a compact convex set that does not contain the origin but meets every ray of the cone, and to show that for the functions of the cone an integral representation in terms of extremal elements is possible. The intersection of the  $A_n$  cones is the class of functions alternating of order<sup> $\infty$ </sup>. Thus, the set of these functions, which will be denoted by  $A_{\infty}$ , forms a convex cone also. The extremal elements for the convex cone  $A_{\infty}$  are given too.

Let f be a function in  $A_1$  which assumes exactly one positive value in [0, 1]; that is, f(x) = 0,  $x \in [0, \xi)$ , f(x) = c > 0,  $x \in [\xi, 1]$  where  $0 \leq \xi \leq 1$ , if  $f = f_1 + f_2$ , where  $f_1$  and  $f_2 \in A_1$ , then  $0 = \mathcal{V}_h^1 f(x) =$  $\mathcal{V}_h^1 f_1(x) + \mathcal{V}_h^1 f_2(x)$  implies  $\mathcal{V}_h^1 f_i(x) = 0$  for i = 1, 2 and  $[x, x + h] \subset [\xi, 1]$ . Therefore,  $f_i(x) = 0$ ,  $x \in [0, \xi)$ ,  $f_i(x) = c_i \geq 0$ ,  $x \in [\xi, 1]$ , i = 1, 2, where  $c_1 + c_2 = c$ . Hence, f is an extremal element of  $A_1$ . On the other hand, if f assumes at least two positive values in [0, 1], then a nonproportional decomposition can be given by taking

$$f_1(x) = \min \{f(x), (1/2)[f(0) + f(1)]\}$$

and  $f_2 = f - f_1$ . Therefore, the extremal elements of  $A_1$  are precisely the functions in  $A_1$  which assume exactly one positive value in [0, 1].

Since a function in  $A_n$ , n > 1, is nonnegative, nondecreasing, and concave on [0, 1], it must be continuous on (0, 1] (cf. [6], p. 148). It follows that the only extremal elements of  $A_1$  which are in  $A_n$  are those functions f such that f = c > 0 on (0, 1] while f(0) = 0 or f(0) = c and these functions are again extremal in  $A_n$ . If  $f \in A_n$ , n > 1, f is not constant and f(0) > 0, then a nonproportional decomposition can be given by taking  $f_1 = f(0)$  and  $f_2 = f - f_1$ . If  $f \in A_n$ , f(0) = 0, f is not constant on (0, 1] and f is not continuous at 0 (that is, f(0+) > 0), then take  $f_1 = f(0+)$  on  $(0, 1], f_1(0) = 0$  and  $f_2 = f - f_1$ . In so doing,  $f_1$  and  $f_2 \in A_n$  and  $f_1$  and  $f_2$  are not proportional to f. Therefore, for n > 1, the only extremal elements of  $A_n$  such that f(0) > 0 are the positive constant functions, and the only extremal elements of  $A_n$  which are discontinuous at 0 are those functions fsuch that f(0) = 0 and f = c > 0 on (0, 1]. It will be shown that the remaining extremal elements of  $A_n$ , n > 1, are indefinite integrals of the extremal elements of a cone which is similar to  $A_1$ . This cone is given in the following definitions.

DEFINITION 1. If g is a real function continuous almost everywhere on (0, 1] and n is a positive integer, then g is said to satisfy property P(n) if

$$\lim_{\delta\to 0} \inf \int_{1}^{\delta} \int_{1}^{t_{n-1}} \cdots \int_{1}^{t_2} \int_{1}^{t_1} g(t) dt dt_1 \cdots dt_{n-1}$$

exists and is finite.

DEFINITION 2. Let K(n) denote the convex cone of nonnegative, nonincreasing real functions on (0, 1] which satisfy property P(n).

In the same manner that the extremal elements of  $A_1$  were found, it can be shown that the extremal elements of K(n) are precisely these functions which assume exactly one positive value in (0,1]. Preliminary to the determination of the extremal elements of  $A_n$ , it is shown in the following two lemmas how the  $A_n$  functions are related to the functions in K(n-1), where n > 1.

LEMMA 1. If 
$$f \in A_n$$
, then  $(-1)^n f_+^{(n-1)} \in K(n-1)$ , where  $n > 1$ .

*Proof.* The proof will be by induction on n. If  $f \in A_2$ , then f is nonnegative, nondecreasing and concave. It follows that  $f'_+$  is nonnegative and nonincreasing on (0, 1], where  $f'_+(1) = f'_+(1-)$  [6]. Also,

$$f(x) = \int_0^x f'_+(t) dt + f(0+)$$

which implies that  $f'_+$  satisfies property P(1) [4].

Assume that  $f \in A_n$  implies  $(-1)^n f_+^{(n-1)} \in K(n-1)$  for  $n \ge 2$ . If  $f \in A_{n+1}$ , then

$$\nabla_h^2 \nabla_h^{n-1} f(x) = \nabla_h^{n+1} f(x) \leq 0$$

for  $[x, x + (n + 1)h] \subset [0, 1]$ , which implies that

$$\nabla_h^2 \nabla_{\delta_1}^1 \nabla_{\delta_2}^1 \cdots \nabla_{\delta_{n-1}}^1 f(x) \leq 0$$

for 
$$[x, x+2h+\delta_1+\delta_2+\cdots+\delta_{n-1}]\subset [0,1]$$
 [2]. It then follows that $(-1)^{n-1}\mathcal{V}_h^2f_+^{(n-1)}(x)\leq 0$ 

for  $[x, x + 2h] \subset (0, 1]$ , and hence,  $(-1)^{n-1}f_{+}^{(n-1)}$  is concave on (0, 1]. Therefore,  $f^{(n-1)} = f_{+}^{(n-1)}$ , since  $f_{+}^{(n-1)}$  is continuous. It follows that  $f_{+}^{(n)}$  exists on (0, 1], where  $f_{+}^{(n)}(1) = f_{+}^{(n)}(1-)$ , and  $(-1)^{n+1}f_{+}^{(n)}$  is non-negative and nonincreasing. It remains only to show that  $f_{+}^{(n)}$  satisfies property P(n). If  $f \in A_{n+1}$ , then  $f \in A_n$ , and

$$\begin{split} \lim_{\delta \to 0} & \lim_{\delta \to 0} \int_{1}^{\delta} \int_{1}^{t_{n-1}} \cdots \int_{1}^{t_{2}} \int_{1}^{t_{1}} f_{+}^{(n)}(t) dt dt_{1} \cdots dt_{n-1} \\ & = \lim_{\delta \to 0} \int_{1}^{\delta} \int_{1}^{t_{n-1}} \cdots \int_{1}^{t_{3}} \int_{1}^{t_{2}} f^{(n-1)}(t_{1}) dt_{1} dt_{2} \cdots dt_{n-1} \\ & - f^{(n-1)}(1) \int_{1}^{0} \int_{1}^{t_{n-1}} \cdots \int_{1}^{t_{3}} \int_{1}^{t_{2}} dt_{1} dt_{2} \cdots dt_{n-1} \end{split}$$

exists and is finite, since  $f^{(n-1)}$  satisfies property P(n-1) by the induction hypothesis.

DEFINITION 3. If g is a real function on (0, 1] which satisfies property P(n), then define the function I(g, n;) by the equation

$$I(g, 1; x) = \int_{0}^{x} g(t) dt ,$$
  
$$I(g, n; x) = \int_{0}^{x} \int_{1}^{i_{n-1}} \cdots \int_{1}^{t_{2}} \int_{1}^{t_{1}} g(t) dt dt_{1} \cdots dt_{n-1} ,$$

 $n = 2, 3, 4, \dots, \text{ for } x \in [0, 1].$ 

LEMMA 2. If  $g \in K(n-1)$ , then  $(-1)^n I(g, n-1;) \in A_n$ , where n > 1.

*Proof.* The proof will be by induction on n. If  $g \in K(1)$ , then

$$I(g,\,1;\,x)=\int_{_0}^{^x}\!g(t)dt\geqq 0$$
 ,

for  $x \in [0, 1]$ . If  $[x, x + h] \subset [0, 1]$ , then

$$\mathcal{V}_h^{\scriptscriptstyle 1}I(g,1;x) = \int_{x+h}^x g(t)dt \leq 0$$

since  $g(t) \ge 0$ , where  $x \le t \le x + h$ . Since g is nonincreasing on (0, 1], then I(g, 1;) is concave on [0, 1] and it follows that  $\mathcal{P}_{h}^{2}I(g, 1; x) \le 0$ , for h > 0 and  $[x, x + 2h] \subset [0, 1]$  [4]. Hence,  $I(g, 1;) \in A_{2}$  whenever  $g \in K(1)$ .

Assume that  $(-1)^n I(g, n-1;) \in A_n$  for  $g \in K(n-1)$  and n > 1. If  $g \in K(n)$ , then let

$$f(x) = \int_{1}^{x} g(t) dt ,$$

for  $x \in (0, 1]$ . Since  $g \in K(n)$ , it is easily seen that  $-f \in K(n-1)$  and it follows from the induction hypothesis that

$$(-1)^{n+1}I(g,\,n;)=(-1)^nI(-f,\,n-1;)\in A_n$$
 .

By a repeated application of the mean value theorem for a Riemann integral, it can be shown that

$$\nabla_{h}^{n-1}I(g, n; x) = (-h)^{n-1}f(\xi)$$

for  $[x, x + (n-1)h] \subset [0, 1]$ , where  $x < \xi < x + (n-1)h$ . It follows that

$$egin{array}{l} & arPi_h^{n+1}(-1)^{n+1}I(g,\,n;\,x) \ &= (-1)^{n+1}arPi_h^2arPi_h^{n-1}I(g,\,n;\,x) \ &= (-1)^{2n}h^{n-1}arV_h^2f(\hat{\xi}) \leq 0 \end{array}$$

for  $[x, x + (n + 1)h] \subset [0, 1]$ , since f is concave on (0, 1] [4]. This inequality, together with the fact that  $(-1)^{n+1}I(g, n;) \in A_n$ , implies that  $(-1)^{n+1}I(g, n;) \in A_{n+1}$ .

It is a consequence of Lemmas 1 and 2 that  $f = I(f_{+}^{(n-1)}, n-1;)$ whenever  $f \in A_n$ , n > 1, f(0+) = 0 and  $f^{(k)}(1) = 0$  for  $1 \le k \le n-2$ . If  $f \in A_2$ , then f is concave on [0, 1] and

$$f(x) = \int_0^x f'_+(t) dt = I(f'_+, 1; x) .$$

If  $f \in A_n$ , n > 2, then  $(-1)^{n-2} f^{(n-2)}$  is concave on (0, 1] (cf. proof of Lemma 1). It follows that

$$f^{(n-2)}(x) = \int_{1}^{x} f^{(n-1)}_{+}(t) dt$$
 ,

which implies that  $f = I(f_{+}^{(n-1)}, n-1;)$  [4].

**PROPOSITION 1.** The function f defined by

$$f(x) = m[\hat{z}^{n-1} - (\hat{z} - x)^{n-1}]$$

for  $x \in [0, \xi]$  and  $m\xi^{n-1}$  for  $x \in [\xi, 1]$ , where  $0 < \xi \leq 1$  and m > 0, is an extremal element of  $A_n$ , n > 1.

*Proof.* If f is such a function, then

$$f_{+}^{(n-1)}(x) = (-1)^n m(n-1)!$$
,

 $x \in (0, \xi)$  and 0 for  $x \in [\xi, 1]$ , which implies that  $(-1)^n f_+^{(n-1)}$  is an ex-

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tremal element of K(n-1). Since f(0) = 0 and  $f^{(k)}(1) = 0$  for  $1 \le k \le n-2$  (whenever n > 2), then  $f = I(f_+^{(n-1)}, n-1;)$  and it follows from Lemma 2 that  $f \in A_n$ .

If g and  $h \in A_n$  such that f = g + h, then  $(-1)^n g_+^{(n-1)}$  and

$$(-1)^n h_+^{(n-1)} \in K(n-1)$$

and  $f_{+}^{(n-1)} = g_{+}^{(n-1)} + h_{+}^{(n-1)}$ . Since  $(-1)^n f_{+}^{(n-1)}$  is extremal in K(n-1), there are constants  $\lambda_i \ge 0$ , i = 1, 2, such that  $g_{+}^{(n-1)} = \lambda_1 f_{+}^{(n-1)}$  and  $h_{+}^{(n-1)} = \lambda_2 f_{+}^{(n-1)}$ . Since f(0) = 0 and  $f^{(k)}(1) = 0$  for  $1 \le k \le n-2$ , it follows that  $g(0) = g^{(k)}(1) = 0$  and  $h(0) = h^{(k)}(1) = 0$  for  $1 \le k \le n-2$ . Hence

$$egin{aligned} g &= I(g_+^{(n-1)},\,n-1;) = I(\lambda_1 f_+^{(n-1)},\,n-1;) \ &= \lambda_1 I(f_+^{(n-1)},\,n-1;) = \lambda_1 f \;, \end{aligned}$$

and similarly,  $h = \lambda_2 f$ . Thus, if  $f(x) = m[\xi^{n-1} - (\xi - x)^{n-1}], x \in [0, \xi]$ and  $m\xi^{n-1}$  for  $x \in [\xi, 1]$ , where  $0 < \xi \leq 1$  and m > 0, then f is extremal in  $A_n, n > 1$ . Denote this latter function by  $e(m, \xi, n - 1;)$ .

If  $f \in A_2$  such that f(0+) = f(0) = 0,  $f \neq 0$  and  $f \neq e(m, \xi, 1;)$ , for m > 0 and  $0 < \xi \leq 1$ , then  $f'_+$  is not extremal in K(1), since  $f'_+$  assumes at least two positive values in (0, 1]. It follows that there are functions  $g_1$  and  $g_2 \in K(1)$  such that  $f'_+ = g_1 + g_2$  and  $g_1$  and  $g_2$  are not proportional to  $f'_+$ . Since f(0) = 0, then  $f = I(f'_+, 1;)$  and it follows that

$$f = I(f'_+, 1;) = I(g_1 + g_2, 1;) = I(g_1, 1;) + I(g_2, 1;)$$
.

Thus, if  $f_i = I(g_i, 1;)$ , then  $f_i \in A_2$ , i = 1, 2, and  $f = f_1 + f_2$ . This gives a nonproportional decomposition of f. Therefore, the extremal elements of  $A_2$  are the positive constant functions, the functions which are a positive constant on (0, 1] and zero at 0 and the functions  $e(m, \xi, 1;)$ , where m > 0 and  $0 < \xi \leq 1$ . The remaining extremal elements of  $A_n$ , n > 2, are given in the next proposition.

PROPOSITION 2. If m > 0, the function e(m, 1, k;) is an extremal element of  $A_n$  for n > 2 and  $1 \le k \le n - 2$ .

*Proof.* Since  $A_n$  is a subcone of  $A_{k+1}$  and e(m, 1, k;) is an extremal element of  $A_{k+1}$ , it is sufficient to show that  $e(m, 1, k;) \in A_n$ . If f = e(m, 1, k;), then  $f = I(f^{(k)}, k;)$  where

$$f^{(k)}(x) = (-1)^{k+1} m(k!)$$

for  $0 < x \leq 1$ . Since  $f^{(k)}$  is constant on (0, 1], it follows from a repeated application of the mean value theorem for a Riemann integral that

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$$\nabla_{h}^{k+1} f(x) = \nabla_{h}^{1} \nabla_{h}^{k} f(x) = (-h)^{k} \nabla_{h}^{1} f^{(k)}(\xi) = 0$$

for h > 0,  $[x, x + (k + 1)h] \subset [0, 1]$ , and thus,  $\mathcal{P}_h^p f(x) = 0$  for h > 0,  $[x, x + ph] \subset [0, 1]$  and  $p \ge k + 1$ . Hence,  $f \in A_n$  for every n, which implies that f is extremal in  $A_p$ , for  $p \ge k + 1$ .

It will follow, as a consequence of the next three lemmas, that no other functions in  $A_n$  are extremal elements of  $A_n$ , n > 2.

LEMMA 3. Let  $f \in A_n$ , n > 2, such that f(0+) = f(0) = 0 and  $f \neq e(m, 1, k;)$ , where m > 0 and  $1 \leq k \leq n - 2$ . If there is an integer k such that  $1 \leq k \leq n - 2$  and  $f^{(k)}(1) \neq 0$ , then f is not an extremal element of  $A_n$ .

*Proof.* Let k denote the smallest integer such that  $f^{(k)}(1) \neq 0$ Then  $f \in A_n \subset A_{k+2}$  implies that  $(-1)^k f_+^{(k+1)} \in K(k+1)$ , and it follows from Lemma 2 that  $I(f_+^{(k+1)}, k+1;) \in A_{k+2}$ . Since f(0) = 0 and  $f^{(p)}(1) = 0$  for  $1 \leq p < k$ , then  $I(f_+^{(k+1)}, k+1;) = I(f^{(k)}, k;) - f^{(k)}(1)I(1, k;) = f - e(m, 1, k;)$ , where  $m = (-1)^{k-1}[1/(k!)]f^{(k)}(1) > 0$ . Since

$$\nabla_h^p e(m, 1, k; x) = 0$$

for h > 0,  $[x, x + ph] \subset [0, 1]$  and  $p \ge k + 1$  and  $f \in A_n$ , it follows that

$$abla_h^p I(f_+^{(k+1)}, k+1; x) = 
abla_h^p f(x) \leq 0,$$

for  $[x, x + ph] \subset [0, 1], k + 1 \leq p \leq n$ . Hence,  $f - e(m, 1, k;) \in A_n$ , where  $m = (-1)^{k-1}[1/(k!)]f^{(k)}(1)$ , and a nonproportional decomposition of f can be given by taking  $f_1 = e(m, 1, k;)$  and  $f_2 = f - f_1$ . Thus, fis not extremal.

LEMMA 4. Let  $f \in A_n$ , n > 2, such that  $f \neq 0$ , f(0+) = f(0) = 0and  $f \neq e(m, 1, k;)$ , where m > 0 and  $1 \leq k \leq n - 2$ . If  $f_{+}^{(n-1)} = 0$ on (0, 1], then f is not an extremal element of  $A_n$ .

*Proof.* If  $f_{+}^{(n-1)} = 0$ , then there is a positive integer  $k \leq n-2$  such that  $f^{(k)} \neq 0$  and  $f^{(k)}$  is constant on (0, 1]. Thus,  $f^{(k)}(1) \neq 0$  and it follows from Lemma 3 that f is not extremal.

It follows from Lemmas 3 and 4 that if f is an extremal element of  $A_n$ , n > 2, such that f(0+) = f(0) = 0 and either  $f_+^{(n-1)} = 0$  or  $f^{(k)}(1) \neq 0$  for some  $k, 1 \leq k \leq n-2$ , then f = e(m, 1, k;), where m > 0and  $1 \leq k \leq n-2$ .

**LEMMA 5.** Let  $f \in A_n$ , n > 2, such that f(0+) = f(0) = 0,  $f_+^{(n-1)} \neq 0$ and  $f^{(k)}(1) = 0$  for  $1 \leq k \leq n-2$ . If f is an extremal element of  $A_n$ , then  $f = e(m, \xi, n-1;)$ , where m > 0 and  $0 < \xi \leq 1$ .

# Proof. Since $f(0) = f^{(k)}(1) = 0$ for $1 \leq k \leq n-2$ , then $f = I(f^{(n-1)}_+, n-1;)$

and it follows from Lemma 1 that  $(-1)^n f_+^{(n-1)} \in K(n-1)$ . If  $g_1$  and  $g_2 \in K(n-1)$  such that  $(-1)^n f_+^{(n-1)} = g_1 + g_2$ , then

$$egin{aligned} f &= I(f_+^{(n-1)},\,n-1;) = (-1)^n I(g_1+g_2,\,n-1;) \ &= (-1)^n I(g_1,\,n-1;) + (-1)^n I(g_2,\,n-1;) \ . \end{aligned}$$

Then  $f_i = (-1)^n I(g_i, n-1;)$ , i = 1, 2, implies that  $f_1$  and  $f_2 \in A_n$  and  $f = f_1 + f_2$ . Since f is extremal in  $A_n$ , there are numbers  $\lambda_i \ge 0$  such that  $f_i = \lambda_i f$ , i = 1, 2, which implies that  $g_i = \lambda_i (-1)^n f_+^{(n-1)}$ , i = 1, 2, and  $(-1)^n f_+^{(n-1)}$  is therefore extremal in K(n-1). Thus,

$$(-1)^n f_+^{(n-1)}(x) = c > 0, \, x \in (0, \, \hat{\xi})$$

and 0 for  $x \in [\xi, 1]$ , which implies that

$$f=I(f_{+}^{_{(n-1)}},\,n-1;)=e(m,\,\xi,\,n-1;)$$
 ,

where m = c/(n - 1)!.

Therefore, the extremal elements of  $A_n$ , n > 2, are the positive constant functions, the functions which are a positive constant on (0, 1] and zero at 0, the functions e(m, 1, k;), where  $m > 0, 1 \le k \le n-2$ , and the functions  $e(m, \xi, n-1;)$ , where m > 0 and  $0 < \xi \le 1$ .

Since  $A_{\infty}$  is a subcone of  $A_n$ , it follows that the function e(m, 1, n;), m > 0, is an extremal element of  $A_{\infty}$  for every positive integer n. It is shown in the following proposition that  $A_{\infty}$  has no other extremal elements which are continuous and zero at 0.

PROPOSITION 3. If  $f \in A_{\infty}$  such that f(0+) = f(0) = 0 and  $f \neq e(m, 1, k)$ , where m > 0 and k is a positive integer, then f is not an extremal element of  $A_{\infty}$ .

*Proof.* Since  $f \in A_{\infty}$  is a function of class  $C^{\infty}$  on (0, 1], it follows from a theorem of Bernstein, Theorem 13-31 in [1], that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (x - 1)^n$$

for 0 < x < 1 by noting that the function g defined by

$$g(x) = f(1) - f(1 - x)$$

satisfies the hypothesis of the theorem. If there is a positive integer k such that  $f^{(k)}(1) \neq 0$ , then assume, without loss of generality, that

k is the least such integer. Then  $f \in A_{\infty} \subset A_{k+2}$  implies that

 $(-1)^{k} f^{(k+1)} \in K(k+1)$ 

from which it follows that  $I(f^{(k+1)}, k+1;) \in A_{k+2}$ . Hence,

$$I(f^{(k+1)}, k+1;) = I(f^{(k)}, k;) - f^{(k)}(1)I(1, k;) = f - e(m, 1, k;)$$

where  $m = (-1)^{k-1}[1/(k!)]f^{(k)}(1) > 0$ . If  $f_1 = e(m, 1, k;)$  and  $f_2 = f - f_1$ , then  $f_1 \in A_{\infty}$  since  $f_1 \in A_n$  for every *n* and  $f_2 \in A_{\infty}$  since  $f_2 \in A_{k+2}$  and

$$\nabla_{h}^{n} f_{2}(x) = \nabla_{h}^{n} [f(x) - e(m, 1, k; x)] = \nabla_{h}^{n} f(x) \leq 0,$$

for h > 0,  $[x, x + nh] \subset [0, 1]$  and  $n \ge k + 3$ . Since  $f_1$  is not proportional to f, this gives a nonproportional decomposition of f, and f is therefore not extremal. On the other hand, if  $f^{(k)}(0) = 0$  for each positive integer k, then f(x) = f(1) for  $0 < x \le 1$ , and f(0+) = f(0) = 0 implies that f = 0.

The results to this point are summarized in the following theorem.

THEOREM. The extremal elements of  $A_1$  are the functions which assume exactly one positive value in [0, 1]. The positive constant functions and the functions which are a positive constant on (0, 1]and zero at 0 are extremal elements of  $A_n$ , n > 1, and are therefore extremal in  $A_{\infty}$ . The functions  $e(m, \xi, n - 1; x) = m[\xi^{n-1} - (\xi - x)^{n-1}]$ ,  $x \in [0, \xi]$  and  $m\xi^{n-1}$  for  $x \in [\xi, 1]$ , where m > 0 and  $0 < \xi \leq 1$ , are extremal elements of  $A_n$ ,  $n \geq 2$ . The only other extremal elements of  $A_n$ ,  $n \geq 3$ , are those functions e(m, 1, k;),  $1 \leq k \leq n - 2$ . The extremal elements of  $A_{\infty}$  which are continuous and zero at 0 are the functions e(m, 1, k;),  $k \geq 1$ .

The set of functions  $A_n - A_n$ ,  $n \ge 1$ , forms the smallest linear space containing the convex cone  $A_n$ . With the topology of simple convergence,  $A_n - A_n$  is a Hausdorff locally convex space. Let  $C_n$  be the set of functions  $f \in A_n$  such that f(1) = 1. Then  $C_n$  is a convex set which meets every ray of  $A_n$  once and only once but does not contain the origin, that is the zero function. It then follows that fis an extreme point of  $C_n$  if, and only if, f is an extremal element of  $A_n$  which lies in  $C_n$ . A proof similar to that found on page 992 of [5] can be used here to show that  $C_n$  is compact. It follows from the next proposition that the set of extreme points of  $C_n$  is compact.

PROPOSITION 4. The set of extreme points of  $C_n$  is closed in  $C_n$ ,  $n \ge 1$ .

*Proof.* Since the topology of simple convergence is equivalent to

the topology of pointwise convergence, it will suffice to show that if  $\{f_i\}$  is a net of functions in ext  $C_n$  which converges pointwise to a function f, then  $f \in \text{ext } C_n$ ,  $n \ge 1$ , where  $\text{ext } C_n$  denotes the set of extreme points of  $C_n$ . The proof for n = 1 is obvious. Since all except a finite number of the functions in ext  $C_n$ , n > 1, are of the form  $e((1/\hat{\xi})^{n-1}, \hat{\xi}, n-1;)$ , where  $0 < \hat{\xi} \le 1$ , it can be assumed without loss of generality that  $f_i = e((1/\hat{\xi}_i)^{n-1}, \hat{\xi}_i, n-1;)$ , for each i.

If the net  $\{\xi_i\}$  of real numbers converges to 0, then it is easily seen that

$$\liminf_i f_i(x) = 1$$

for  $x \in (0, 1]$ . Since the topology is Hausdorff, it follows that f(0) = 0 and  $f(x) = 1, x \in (0, 1]$ , which implies that  $f \in \text{ext } C_n$ .

On the other hand, if  $\{\xi_i\}$  does not converge to 0, then there is a positive real number  $\xi_0$  and a subnet  $\{\xi_j\}$  of  $\{\xi_i\}$  such that  $\{\xi_j\}$  converges to  $\xi_0$ . If  $0 \leq x < \xi_0$ , then

$$\liminf_{j} f_{j}(x) = \frac{1}{\xi_{0}^{n-1}} \left[ \xi_{0}^{n-1} - (\xi_{0} - x)^{n-1} \right];$$

whereas

$$\lim_{j} f_j(x) = 1$$

if  $\xi_0 \leq x \leq 1$ . Therefore, since the topology is Hausdorff,

$$f = e((1/\hat{\xi}_0)^{n-1}, \hat{\xi}_0, n-1;)$$

and it follows that  $f \in \text{ext } C_n$ .

Since ext  $C_n$  and  $C_n$  are both compact subsets of the locally convex space  $A_n - A_n$ ,  $n \ge 1$ , it follows from Theorem 39.4 of Choquet [3] that for any function  $f_0 \in C_n$  there exists a probability measure  $\mu_0$  on ext  $C_n$  such that

$$f_{\scriptscriptstyle 0}(x) = \int f(x) d\mu_{\scriptscriptstyle 0}$$
 ,

for  $x \in [0, 1]$ . Since  $C_n$  meets every ray of  $A_n$  and does not contain the origin, it follows that each function of  $A_n$  is a scalar multiple of such a representation.

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