FINITE TRANSFORMATION FORMULAE INVOLVING THE LEGENDRE SYMBOL

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Let $p$ denote an odd prime. The following three identities (transformation formulae) involving the Legendre symbol $\left( \frac{a}{p} \right)$ are known to be valid for any complex-valued function $F$ defined on the integers, which is periodic with period $p$:

$$
\sum_{x=0}^{p-1} F(x) + \sum_{x=0}^{p-1} (\frac{x}{p}) F(x) = \sum_{x=0}^{p-1} F(x^2),
$$

$$
\sum_{x=0}^{p-1} F(x) + \sum_{x=0}^{p-1} \left( \frac{x^2-4a}{p} \right) F(x) = \sum_{x=1}^{p-1} F\left( x + \frac{a}{x} \right), \quad a \not\equiv 0 \pmod{p},
$$

$$
\sum_{x=0}^{p-1} F(x) + \sum_{x=0}^{p-1} \left( \frac{x^2-4x}{p} \right) F(x) = \sum_{x=1}^{p-1} F\left( x + 2 + \frac{1}{x} \right).
$$

We consider a general class of transformation formulae, which includes the above examples.

Let $p$ denote a fixed odd prime and let $GF(p)$ denote the Galois field with $p$ elements. If $X$ denotes an indeterminate we let

$$
\Theta[X] = \left\{ \vartheta(X) = \frac{aX^2 + bX + c}{AX^2 + BX + C} \mid a, b, c, A, B, C \in GF(p),
\right. $$

$$
(aC - cA)^2 - (AB - bA)(bC - cB) \neq 0 \right\}
$$

and

$$
\Phi[X] = \{ \phi(X) = qX^2 + rX + s \mid q, r, s \in GF(p), r^2 - 4qs \neq 0 \}.
$$

Corresponding to any element $\vartheta(X) \in \Theta[X]$ (often just written $\vartheta \in \Theta$) we define

$$
\vartheta^*(X) = DX^2 + JX + d,
$$

where

$$
D = B^2 - 4AC, \quad J = 4aC - 2bB + 4cA, \quad d = b^2 - 4ac.
$$

It is clear that $\vartheta^*(X) \in \Phi[X]$ as

$$
D^2 - 4Dd = 16((aC - cA)^2 - (AB - bA)(bC - cB)) \neq 0.
$$

For any element $\phi(X) \in \Phi[X]$ (often just written $\phi \in \Phi$) its value at $x \in GF(p)$ is just $\phi(x) = qx^2 + rx + s \in GF(p)$. For any element $\vartheta(X) \in \Theta[X]$, $\vartheta(x)$ will be defined provided $Ax^2 + Bx + C \neq 0$ and its value is

$$
\vartheta(x) = \frac{ax^2 + bx + c}{Ax^2 + Bx + C} = (ax^2 + bx + c)(Ax^2 + Bx + C)^{-1} \in GF(p).
$$
Throughout this paper whenever we write $\sum$ the summation is taken over all $x \in \text{GF}(p)$. If we write $\sum'$ the summation is over all $x \in \text{GF}(p)$ for which the summand is defined.

Further we let $\mathcal{C}$ denote the complex number field and we denote by $\mathcal{F}$ the set of all functions with domain $\text{GF}(p)$ and range $\subseteq \mathcal{C}$. The particular function $\chi \in \mathcal{F}$ defined for any $x \in \text{GF}(p)$ by

$$\chi(x) = \begin{cases} 
0, & \text{if } x = 0, \\
1, & \text{if } x \neq 0 \text{ and there exists } y \in \text{GF}(p) \text{ such that } y^2 = x, \\
-1, & \text{if } x \neq 0 \text{ and no such } y \text{ exists},
\end{cases}$$

plays a special role in what we do. $\chi$ is the Legendre symbol on $\text{GF}(p)$. Finally for $(F, \theta) \in \mathcal{F} \times \Theta$ we define

$$\delta(F, \theta) = \begin{cases} 
F(a/A), & \text{if } A \neq 0, \\
0, & \text{if } A = 0.
\end{cases}$$

We are now in a position to define what we mean by the transformation formula $T(\theta, \phi)$.

**DEFINITION.** If $(\theta, \phi) \in \Theta \times \Phi$ is such that

$$\sum_x F(x) + \sum_x \chi(\phi(x))F(x) = \sum_x' F(\theta(x)) + \delta(F, \theta),$$

for all $F \in \mathcal{F}$, we say that the transformation formula $T(\theta, \phi)$ is valid. If on the other hand there is some $F_0 \in \mathcal{F}$ such that

$$\sum_x F_0(x) + \sum_x \chi(\phi(x))F_0(x) \neq \sum_x' F_0(\theta(x)) + \delta(F_0, \theta),$$

then we say that $T(\theta, \phi)$ is not valid.

In some special cases it is well-known that $T(\theta, \phi)$ is valid. For example ([1; p. 159], [4; p. 101]) it is known that $T(\theta, \phi)$ is valid if

(1.1) $\theta(X) = X^2, \phi(X) = X$

or

(1.2) $\theta(X) = \frac{X^2 + c}{X}, \phi(X) = X^2 - 4c \quad (c \neq 0)$.

(We identify the elements of $\text{GF}(p)$ with the residues modulo $p$ and the elements of $\mathcal{F}$ with functions defined on the integers which are periodic with period $p$). The name transformation formula is justified as (1.1) (resp. (1.2)) gives the well-known transformation property of the Gauss (resp. Kloosterman) sum, if we take $F(x) = \exp(2\pi i x/p)$, [3], [4]. Both examples mentioned above have $\delta(F, \theta) = 0$. An example with $\delta(F, \theta) \neq 0$ in general, is given by the following
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\( \sum_x F(x) + \sum_x \chi(4x + 1)F(x) = \sum_x F\left(\frac{x+1}{x^2}\right) + F(0) . \)

Here

\[ \theta(X) = \frac{X+1}{X^2} \quad \text{and} \quad \phi(X) = 4X + 1. \]

The main objective of this paper is to give necessary and sufficient conditions for \( T(\theta, \phi) \) to be valid. We prove in § 4 that if \((\theta, \phi) \in \Theta \times \Phi \) then \( T(\theta, \phi) \) is valid if and only if there exists \( e(\neq 0) \in \text{GF}(p) \) such that \( \phi = e\theta^* \). (We note that in (1.1) \( \theta^*(X) = 4X = 4\phi(X) \), in (1.2) \( \theta^*(X) = X^2 - 4c = \phi(X) \) and in (1.3) \( \theta^*(X) = 4X + 1 = \phi(X) \)).

The proof of these necessary and sufficient conditions requires a useful lemma concerning quadratic polynomials possessing the same quadratic nature. This lemma is proved in § 3. In § 2 a number of properties of \( \Theta[X] \) and \( \Phi[X] \) are noted, which together with the main theorem enable us to deduce that there are only two essentially different transformation formulae \( T(\theta, \phi) \).

2. Properties of \( \Theta[X] \) and \( \Phi[X] \). We first consider \( \Theta[X] \). The elements \( \theta(X) = aX^2 + bX + c/AX^2 + BX + C \) of \( \Theta[X] \) are well-defined, as \( A, B, C \) cannot all be zero. Further they do not reduce to the form \( lX + m/LX + M \), as not both of \( a, A \) are zero and \( aX^2 + bX + c \) and \( AX^2 + BX + C \) do not have a nonunit common factor.

Any element of \( \Theta[X] \) gives rise to another element of \( \Theta[X] \) in the following way. If \( t, u, v, w, k, l, m, n \in \text{GF}(p) \) are such that

\[ tw - uw \neq 0, kn - lm \neq 0 , \]

and if \( \theta(X) \in \Theta[X] \) then so does

\[ \theta_1(X) = \frac{t\theta(kX + l)}{v\theta(kX + n)} + u . \]

The proof of this just consists of showing that

\[ \theta_1(X) = \frac{a_1X^2 + b_1X + c_1}{A_1X^2 + B_1X + C_1} , \]

where

\[ a_1 = (ta + uA)k^2 + (tb + uB)km + (tc + uC)m^2 , \]
\[ b_1 = 2(ta + uA)kl + (tb + uB)(kn + lm) + 2(tc + uC)mn , \]
\[ c_1 = (ta + uA)l^2 + (tb + uB)ln + (tc + uC)n^2 , \]
\[ A_1 = (va + wA)k^2 + (vb + wB)km + (vc + wC)m^2 , \]
B_1 = 2(va + wA)kl + (vb + wB)(kn + lm) + 2(vc + wC)mn,
C_1 = (va + wA)t^2 + (vb + wB)ln + (vc + wC)n^2,

and noting that

\[(a_iC_i - c_iA_i)^2 - (a_iB_i - b_iA_i)(b_iC_i - c_iB_i)\]
\[= (tw - uv)^2(kn - lm)\left[(aC - cA)^2 - (aB - bA)(bC - cB)\right]
\[\neq 0 .\]

We can thus define an equivalence relation on \(\Theta[X]\) by saying that \(\theta(X), \theta_1(X) \in \Theta[X]\) are equivalent if there exist \(k, l, m, n, t, u, v, w \in GF(p)\) with \(kn - lm \neq 0, tw - uv \neq 0\) and such that (2.1) holds. We write \(\theta_1 \sim \theta\).

Let \(c_1\) and \(c_2\) be fixed elements of \(GF(p)\) such that \(\chi(c_1) = +1, \chi(c_2) = -1\), so that there exists \(d_i(\neq 0) \in GF(p)\) with \(c_i = d_i^2\). Then any element

\[\theta(X) = \frac{aX^2 + bX + c}{AX^2 + BX + C} \in \Theta[X]\]

is either equivalent to \(\theta_{c_1}(X) = X + (c_1/X)\) or \(\theta_{c_2}(X) = X + (c_2/X)\). More precisely we have

\[\theta \sim \theta_{c_1},\text{ if } \chi((aC - cA)^2 - (aB - bA)(bC - cB)) = +1\]

and

\[\theta \sim \theta_{c_2},\text{ if } \chi((aC - cA)^2 - (aB - bA)(bC - cB)) = -1 .\]

This is clear as we have

\[\theta(X) = \frac{t\theta_{c_1}(\frac{kX + l}{mX + n}) + u}{v\theta_{c_1}(\frac{kX + l}{mX + n}) + w},\]

where

(i) \(t = ah, u = b - 2ag, v = Ah, w = B - 2Ag, k = 1, l = g, m = 0, n = h,\) if \(\chi((aC - cA)^2 - (aB - bA)(bC - cB)) = +1, aB - bA \neq 0,\) and \(g \in GF(p)\) are defined by

\[g = \frac{aC - cA}{aB - bA}, c,h^2 = \left(\frac{aC - cA}{aB - bA}\right)^2 - \left(\frac{bC - cB}{aB - bA}\right);\]

(ii) \(t = aA(1 - d), u = 2aAd_1(1 + d), v = A^2 - a^2D, w = 2d_1(A^2 + a^2D), k = 2ad_1, l = (b + 1)d_1, m = 2a, n = (b - 1),\) if \(\chi((aC - cA)^2 - (aB - bA)(bC - cB)) = +1, aB - bA = 0, aA \neq 0;\)

(iii) \(t = a^2C^2 - d, u = 2d_1(a^2C^2 + d), v = 4aC, w = -8d_1aC, k = 2ad_1, l = d_1(b + aC), m = 2a, n = b - aC,\) if \(\chi((aC - cA)^2 - (aB - bA)(bC - cB)) = +1, aB - bA = 0, A = 0;\)
(iv) \( t = 4Ac, u = -8d_1Ac, v = A^2c^2 - D, w = 2d_1(A^2c^2 + D), k = 2d_1A, l = d_1(B + Ac), m = 2A, n = B - Ac, \) if \( \chi((ac - cA)^2 - (AB - bA)(bC - cB)) = +1, \) \( AB - bA = 0, a = 0; \)

and

\[
\theta(X) = \frac{t\phi_2\left(\frac{kX + l}{mX + n}\right) + u}{v\phi_2\left(\frac{kX + l}{mX + n}\right) + w},
\]

where

(v) \( t = ah, u = b - 2ag, v = Ah, w = B - 2Ag, k = 1, l = g, m = 0, n = h, \) if \( \chi((ac - cA)^2 - (AB - bA)(bC - cB)) = -1 \) and \( g, h \) are defined by

\[
g = \frac{aC - cA}{AB - bA}, c_h = \frac{(aC - cA)^2 - (bC - cB)}{ab - bA}.
\]

This shows that there are at most two equivalence classes in \( \Theta[X]. \)

We show that there are exactly two by proving that \( \theta_{\phi_1}(X) \not\sim \theta_{\phi_2}(X). \)

For suppose that \( \theta_{\phi_1}(x) \sim \theta_{\phi_2}(x) \) then there exist \( k, l, m, n, t, u, v, w \in GF(p) \) with

\[
kn - lm \neq 0, tw - uv \neq 0
\]

and such that

\[
\theta_{\phi_1}(X) = \frac{t\phi_2\left(\frac{kX + l}{mX + n}\right) + u}{t\phi_2\left(\frac{kX + l}{mX + n}\right) + w}.
\]

Thus from (2.2) we have

\[-c_1 = (tw - uv)^3(kn - lm)^3(-c_2),\]

which contradicts that \( \chi(c_1) = +1, \chi(c_2) = -1. \)

We now consider \( \Phi[X]. \) The elements \( \phi(X) = qX^2 + rX + s \) of \( \Phi[X] \) are either genuinely quadratic or linear, as \( q, r \) are not both zero. Moreover they are not of the form \( q(X + k)^2, \) for any \( k \in GF(p) \). Corresponding to (2.1) we have

\[
\theta^*_1(X) = (kn - lm)^2(-vX + t)^3\theta^*_2\left(\frac{wX - u}{vX + t}\right) \in \Phi[X].
\]

3. A useful lemma. We prove the following lemma which is needed in the proof of our theorem.

**Lemma.** If \( qX^2 + rX + s, q'X^2 + r'X + s' \in \Phi[X] \) are such that \( \chi(qx^2 + rx + s) = \chi(q'x^2 + r'x + s'), \) for all \( x \in GF(p), \) then there exists
Proof. As \( qX^2 + rX + s \in \Phi[X] \) it is not of the form \( q(X + k)^2 \) and not both of \( q, r \) are zero, similarly for \( q'X^2 + r'X + s' \). The condition \( \chi(qx^2 + rx + s) = \chi(q'x^2 + r'x + s') \) implies that a zero of \( qx^2 + rx + s \) is a zero of \( q'x^2 + r'x + s' \) and vice-versa. Thus, unless both \( qX^2 + rX + s \) and \( q'X^2 + r'X + s' \) are irreducible in \( GF(p)[X] \), that is, unless \( \chi(r^2 - 4qs) = \chi(r'^2 - 4q's') = -1 \), we have for some \( e_i, e_2 \in GF(p)(e_1 \neq e_2) \) either

\[
qX^2 + rX + s = q(X - e_i)(X - e_2), q'X^2 + r'X + s' = q'(X - e_i)(X - e_2),
\]

or

\[
qX^2 + rX + s = r(X - e_i), q'X^2 + r'X + s' = r'(X - e_i), q = q' = 0.
\]

In the former case taking \( x \neq e_i, e_2 \) in

\[
\chi(qx^2 + rx + s) = \chi(q'x^2 + r'x + s')
\]

we obtain \( \chi(q) = \chi(q') \), so that there exists \( e(\neq 0) \in GF(p) \) such that

\[
q = e^2q'.
\]

Hence

\[
r = -q(e_i + e_2) = -e^2q'(e_i + e_2) = e^3r',
\]

and so we have

\[
qx^2 + rX + s = e^3(q'X^2 + r'X + s').
\]

In the latter case taking \( x \neq e_i \) in \( \chi(qx^2 + rx + s) = \chi(q'x^2 + r'x + s') \) we obtain \( \chi(r) = \chi(r') \), so that there exists \( e(\neq 0) \in GF(p) \) such that

\[
r = e^3r'.
\]

Hence \( s = -re_i = e^3r'e_i = e^3s' \) and we have

\[
qX^2 + rX + s = e^3(q'X^2 + r'X + s').
\]

If \( \chi(r^2 - rqs) = \chi(r'^2 - rq's') = -1 \) then \( q, q', r^2 - 4qs, r'^2 - 4q's' \) are all nonzero and

\[
\sum_s \chi(qx^2 + rx + s) = \sum_s \chi(q'x^2 + r'x + s')
\]
gives \( \chi(q) = \chi(q') \). Hence there exists \( e(\neq 0) \in GF(p) \) such that \( q = e^3q' \).

Now as \( qq' = (eq')^2 \neq 0 \) we have

\[
\sum_s \chi\left(\left(x^2 + \frac{r}{q}x + \frac{s}{q}\right)\left(x^2 + \frac{r'}{q'}x + \frac{s'}{q'}\right)\right) = \sum_s \chi((qx^2 + rx + s)(q'x^2 + r'x + s'))
\]
\[ \begin{align*}
&= \sum x \chi((qx^2 + rx + s)^2) \\
&= \sum x \chi((x^2 + \frac{r}{q}x + \frac{s}{q})(x^2 + \frac{r'}{q'}x + \frac{s'}{q'}))
\end{align*} \]

and so

\[ (3.1) \quad \sum x \chi((x^2 + \frac{r}{q}x + \frac{s}{q})(x^2 + \frac{r'}{q'}x + \frac{s'}{q'})) = p. \]

If \( X^2 + (r/q)X + (s/q) \neq X^2 + (r'/q')X + (s'/q') \) then by a deep result of Perel'muter [2] we have

\[ \left| \sum x \chi((x^2 + \frac{r}{q}x + \frac{s}{q})(x^2 + \frac{r'}{q'}x + \frac{s'}{q'})) \right| \leq 2p^{1/2}. \]

For \( p \geq 5 \) this clearly contradicts (3.1). Thus for \( p \geq 5 \) we have

\[ X^2 + (r/q)X + (s/q) = X^2 + (r'/q')X + (s'/q'), \]

that is \( q = e^t q' \),

\[ qX^2 + rX + s = e^t(qX^2 + r'X + t'), \]

as required. When \( p = 3 \) the theorem is easily verified by examining the values of \( qx^2 + rx + s \) for \( x \in \text{GF}(p) \) (see table).

When \( p = 3 \), \( \Phi[X] \) consists of all polynomials of \( \text{GF}(3)[X] \) of degree atmost 2 except the 9 polynomials \( q(X + k)^2 \), \( q, k \in \text{GF}(3) \), which have discriminant equal to zero. The table shows that there do not exist 2 elements of \( \Phi[X] \), say \( \phi(x) \), \( \phi'(X) \) with \( \chi(\phi(x)) = \chi(\phi'(x)) \), for all \( x \in \text{GF}(3) \).

### Table.

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<th>( \chi(\phi(0)) )</th>
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<th>( \chi(\phi(2)) )</th>
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4. Main result. We prove

**THEOREM.** If \((\theta, \phi) \in \Theta \times \Phi\) then \(T(\theta, \phi)\) is valid if and only if there exists \(e(\neq 0) \in \text{GF}(p)\) such that

\[
(4.1) \quad \phi = e^2\theta^*.
\]

**Proof.** (i) We let \(\phi = e^2\theta^*\), where \(e(\neq 0) \in \text{GF}(p)\) and

\[
\theta(X) = \frac{aX^2 + bX + c}{AX^2 + BX + C} \in \Theta[X],
\]

and prove that \(T(\theta, \phi)\) is valid. For all \(F \in \mathcal{S}\) we have

\[
\sum_x' F(\theta(x)) = \sum_y \sum_{\theta(x) = y} F(\theta(x))
\]

\[
= \sum_y F(y) \sum_{\theta(x) = y} 1.
\]

Thus for given \(y \in \text{GF}(p)\) we require the number of solutions \(x \in \text{GF}(p)\) of \(\theta(x) = y\), that is of

\[
(4.2) \quad (Ay - a)x^2 + (By - b)x + (Cy - c) = 0.
\]

This is a genuine quadratic in \(x\) unless \(Ay - a = 0\). Thus we must consider two cases according as \(A = 0\) or \(A \neq 0\).

**Case (a).** \(A = 0\), (so that \(\delta(F, \theta) = 0\)).

In this case \(a \neq 0\) so that \(Ay - a \neq 0\), for all \(y \in \text{GF}(p)\). Thus the number of solutions of (4.2) is

\[
1 + \chi((By - b)^2 - 4(Ay - a)(Cy - c))
\]

\[
= 1 + \chi(Dy^2 + dy + d)
\]

\[
= 1 + \chi(\phi(y)), \text{ as } e \neq 0.
\]

Hence we have

\[
\sum_x' F(\theta(x)) = \sum_y F(y) + \sum_y \chi(\phi(y))F(y),
\]

proving that \(T(\theta, \phi)\) is valid in the case.

**Case (b).** \(A \neq 0\), (so that \(\delta(F, \theta) = F(a/A)\)).

In this case, for all \(y \in \text{GF}(p)\) except \(a/A\), (4.2) is a genuine quadratic and the number of solutions of it, for such \(y\), is as in case (a). For \(y = a/A\), (4.2) becomes

\[
(aB - bA)x + (aC - cA) = 0,
\]
which since \( aB - bA \) and \( aC - cA \) cannot both be zero, has one solution if \( aB - bA \neq 0 \) and no solutions if \( aB - bA = 0 \). This number is expressible as \( \chi((aB - bA)^2) \). Hence

\[
\sum x' F(\theta(x)) + \delta(F, \theta) = F(a/A)\chi((aB - bA)^2) + \sum_{y \neq a} \{1 + \chi(Dy^2 + \Delta y + d)\}F(y) + F(a/A)
\]

\[
= \sum \{1 + \chi(e^2\theta^*(y))\}F(y)
\]
as required, since

\[
A^2\left(D\left(\frac{a}{A}\right) + \Delta\left(\frac{a}{A}\right) + d\right) = (aB - bA)^2.
\]

(ii) Conversely we show that if \((\theta, \phi) \in \Theta \times \Phi\) is such that \( T(\theta, \phi) \) is valid then \( \phi(X) = e^2\theta^*(X) \). For all \( F \in \mathcal{F} \), as \( T(\theta, \phi) \) is valid, we have

\[
\Sigma' F(\theta(x)) + \delta(F, \theta) = \Sigma F(x) + \Sigma \chi(\phi(x))F(x).
\]

From (i) we know that \( T(\theta, \theta^*) \) is valid, so that also for all \( F \in \mathcal{F} \) we have

\[
\Sigma' F(\theta(x)) + \delta(F, \theta) = \Sigma F(x) + \Sigma \chi(Dx^2 + \Delta x + d)F(x).
\]

Hence form (4.3) and (4.4) we have

\[
\Sigma \chi(\phi(x))F(x) = \Sigma \chi(Dx^2 + \Delta x + d)F(x),
\]

for all \( F \in \mathcal{F} \). In particular taking \( F = F,(r \in \text{GF}(p)) \) in (4.5) where \( F_r \) is defined for \( x \in \text{GF}(p) \) by

\[
F_r(x) = \begin{cases} 1, & x = r, \\ 0, & x \neq r, \end{cases}
\]

we have

\[
\chi(\phi(r)) = \chi(Dr^2 + \Delta r + d),
\]

for all \( r \in \text{GF}(p) \). By lemma as \( \phi(X), DX^2 + \Delta X + d \in \text{GF}(X) \), we have, for some \( e(\neq 0) \in \text{GF}(p) \),

\[
\phi(X) = e^2(DX^2 + \Delta X + d) = e^2\theta^*(X),
\]

which is (4.1).

5. An application. We use the theorem to evaluate the Salié sum [4]. Let \( \theta(X) = (X + 1)^2/X \) so that \( \theta^*(X) = X^2 - 4X \). By the
theorem we know that $T(\theta, \theta^*)$ is valid. If $G \in \mathcal{F}$ so does $\chi G$. Taking $F(x) = \chi(x)G(x)$ in $T(\theta, \theta^*)$ we obtain

$$
\sum_x \chi(x)G(x) + \sum_x \chi(x^2(x - 4))G(x) = \sum_x' \chi\left(\frac{(x + 1)^2}{x}\right)G\left(\frac{(x + 1)^2}{x}\right)
$$

that is,

$$(5.1) \quad \sum_x \chi(x)G(x) + \sum_x \chi(x - 4)G(x) = \sum_x' \chi(x)G\left(x + 2 + \frac{1}{x}\right).$$

Taking $G(x) = \exp\left(2\pi ikx/p\right)$ and noting that this choice makes the two sums on the left hand side of (5.1) Gaussian sums we obtain Salié's result [4]

$$
\sum_{x=p} \chi(x) \exp\left(\frac{2\pi ik}{p}\left(x + \frac{1}{x}\right)\right) = 2\left(\frac{k}{p}\right)^{1/2} (p-1)^{1/2} p^{1/2} \cos\left(\frac{4\pi k}{p}\right).
$$

6. Conclusion. The properties of $\Theta[X]$ indicated in § 2 and the theorem of § 4 show that there are only two essentially different transformation formulae $T(\theta, \phi)$ given by $(\theta, \phi) = (\theta_1, \theta_2^*)$ and $(\theta_1^*, \theta_2)$, where we have identified $T(\theta, \theta^*)$ and $T(\theta, e^{i\theta^*})$. It would be interesting to know if this work could be generalized to give results concerning identities of a type similar to $T(\theta, \phi)$ but where $\theta, \phi$ are elements of larger sets than $\Theta, \Phi$ respectively and/or where $\chi$ is replaced by a more general character.

I would like to finish by thanking an unknown referee for a number of valuable suggestions. In particular he suggested the proof of the lemma given in § 3, which considerably shortened my original proof.

**References**


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