**THE PETURBATION OF THE SINGULAR SPECTRUM**

Richard Hindman Bouldin

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RICHARD BOULDIN

This paper designates a subset of the spectrum of a bounded self adjoint operator on a complex separable Hilbert space. The set is called the singular spectrum and is distinguished by the fact that it is a support for the singular part of the spectral measure of the operator. The behavior of the singular spectrum, when the operator is perturbed by a bounded self adjoint operator, is studied. The thrust of these results is to give conditions sufficient for the perturbed operator to have no singular spectrum.

Any nonnegative Borel measure can be written as the sum of two measures such that the first is absolutely continuous with respect to Lebesgue measure and the second is singular with respect to Lebesgue measure. The spectral measure $E(\cdot)$ of a self adjoint operator $T$ on a complex separable Hilbert space $H$ has a similar decomposition. In the Lebesgue decomposition for a nonnegative Borel measure there is no unique set which is the support of the singular measure in that decomposition. Although this is also true for the singular part of the spectral measure, in this research we show how to define a support for the singular part of the spectral measure $E(\cdot)$ so that a number of desirable properties result.

We call this support the singular spectrum of the operator $T$ and we denote it $\sigma_s(T)$. Having developed the basic properties of the singular spectrum, we study the behavior of this set under self adjoint perturbations. That is we derive information about $\sigma_s(T + V)$ in terms of $\sigma_s(T)$ and $V$. Since the singular spectrum of an operator $T$ always contains the eigenvalues of $T$, a perturbation theory for the singular spectrum would be a generalization of the theory for the perturbation of eigenvalues. A perturbation theory for the singular spectrum would be complementary to the so called "absolutely continuous" perturbation theory.

In § 1 we summarize our notation and some of the known results which we shall use. In § 2 we prove the basic properties which seem to make our definition of singular spectrum attractive. In § 3 we get a theory for finite dimensional perturbations which is analogous to the one dimensional theory given by Donoghue in [4]. The fourth section is devoted to theorems which conclude the absence of the singular spectrum under various hypotheses.

## 1. Preliminaries.

NOTATION. Throughout this paper $T$ will be a bounded self adjoint linear operator on a complex separable Hilbert space $H$. The perturbation will be a bounded self adjoint linear operator on $H$ and it will be denoted by $V$; in § 3 we further assume that $V$ is finite dimensional, i.e., $V = \sum_{j=1}^{r} < \cdot, h_j > c_j h_j$. We shall consistently denote the perturbed operator by $P$, i.e., $P = T + V$, and the spectral measures of the unperturbed and perturbed operators are denoted $E(\cdot)$ and $F(\cdot)$, respectively. The resolvent operators for $T$ and $P$ are written $R(z) = (T - xI)^{-1}$ and $S(z) = (P - zI)^{-1}$.

For a nonnegative Borel measure $m(\cdot)$ we define the symmetric derivative at $x$ to be $\lim m([x - \varepsilon, x + \varepsilon])/2\varepsilon$ as $\varepsilon \to 0$ and we denote this derivative by $D_{\mathrm{sym}} m(x)$. The real and imaginary parts of the complex number $z$ are written $\operatorname{Re} z$ and $\operatorname{Im} z$.

BOUNDARY VALUE THEORY. This paper uses in an essential way the following facts which are corollaries of well known results. We shall list these facts in itemized form and then comment on the proofs.

( 1 )  $\langle E(\cdot)f, g \rangle$ is a complex measure and $\operatorname{Re} \langle E(\cdot)f, g \rangle$, $\operatorname{Im} \langle E(\cdot)f, g \rangle$ are signed real measures. Each signed real measure can be written as the difference of positive measures.

( 2 )  For almost all real $x$ we have a finite limit, $\lim \langle R(x + ia)f, g \rangle$ as $a \to 0$ with $a > 0$.

( 3 )  If $D_{\mathrm{sym}} \langle E(x)f, f \rangle = \infty$ then $\operatorname{Im} \langle R(x + ia)f, f \rangle \to \infty$ as $a \to 0$.

( 4 )  If $m(\cdot)$ is a nonnegative Borel measure and $S$ is a measurable set then $m(S) = m_a(S) + m_s(S)$ with $m_s(S) = m(S \cap B)$ where

$$B = \{x \text{ real}: D_{\mathrm{sym}} m(x) = \infty\}, \quad m_a(\cdot) = m(\cdot) - m_s(\cdot)$$

and $m_a(\cdot)$, $m_s(\cdot)$ are absolutely continuous and singular measures with respect to Lebesgue measure, respectively.

*Proofs.*  ( 1 )  is an obvious consequence of the measure theoretic properties of the spectral measure $E(\cdot)$ and the Jordan decomposition for a signed real measure.

( 2 )  follows from the representaion

$$\langle R(z)f, g \rangle = \int (t - z)^{-1} \operatorname{Re} \langle E(dt)f, g \rangle + i \int (t - z)^{-1} \operatorname{Im} \langle E(dt)f, g \rangle$$

and items VIII and X of [2] and Theorem 8.6, p. 154, of [11].

( 3 )  is due to Donoghue in [3].

( 4 )  is the De la Vallee Poussin Theorem given in [12], p. 127.

**2. Definition and elementary properties of the singular spectrum.** Following Kato in [7], pp. 516-517, we define the absolutely continuous subspace, $H_a$, and the singular subspace, $H_s$, by

$$H_a = \{f \in H: \langle E(\cdot)f, f \rangle \text{ is absolutely continuous with}$$
$$\text{respect to Lebesgue measure}\}$$

$$H_s = \{f \in H: \langle E(\cdot)f, f \rangle \text{ is singular with respect to}$$
$$\text{Lebesgue measure}\}.$$

Kato shows that the orthogonal complement of $H_a$ is $H_s$ and if $Q$ is the orthogonal projection onto $H_a$ while $E$ is the orthogonal projection onto any subspace which reduces $T$ then $QE = EQ$.

In [4] Donoghue assumes that $T$ in $H$ has a cyclic vector $f$, that is $H$ has no nontrivial $T$-invariant subspace containing $f$. In that special case he defines a notion of singular spectrum. Donoghue relates his notion of singular spectrum to the boundary values of $R(z)$ but he does not relate it to the singular subspace.

In [10] Rosenblum constructs a Lebesgue decomposition for the spectral measure $E(\cdot)$; this gives rise in a natural way to a notion of singular spectrum and a notion of singular subspace. However, that notion of singular spectrum does not agree with the definition of Donoghue in the case that $T$ has a cyclic vector in $H$. Also the simple relationship to the boundary values of $R(z)$ is lost.

The definition of singular spectrum given below attempts to incorporate the best features of both of the approaches of Donoghue and of Rosenblum. Our definition makes sense pointwise and has a very simple relation to the boundary values of $R(z)$. Further, it is shown that the singular spectrum is always a Lebesgue null set, it is a support for the singular part of the spectral measure, and the spectral measure of the singular spectrum is the orthogonal projection onto the singular subspace. By definition the set of vectors, $\{f_b: b \in B\}$, is a generating basis for the operator $T$ in $H$ provided a dense subspace of $H$ is formed by the span of all vectors of the form $p(T)f_b$ where $p(x)$ is a complex polynomial. The role of generating bases in spectral theory is studied in [1], for example, p. 63.

DEFINITION. Let

$$S_{f,g} = \{x \text{ real}: |\langle R(x + ia_n)f, g \rangle| \to \infty \text{ as } n \to \infty \text{ for}$$
$$\text{some sequence, } \{a_n\}, \text{ of positive numbers}$$
$$\text{converging to } 0\}.$$

Define the singular spectrum of $T$ by $\sigma_s(T) = \cup \{S_{f,g}: f, g \in G\}$ where $G$ is some generating basis for $T$ in $H$.

It is clear that in the above definition the singular spectrum depends on the generating basis that is chosen. There is a natural choice for perturbation problems involving a compact self adjoint perturbation-namely the normalized eigenvectors of the perturbation.

THEOREM 1. *If $E_s(\cdot)$ and $E_a(\cdot)$ are defined by*

$$E_s(D) = E(D \cap \sigma_s(T))$$

*and $E_a(D) = E(D) - E_s(D)$ then $E(\cdot) = E_a(\cdot) + E_s(\cdot)$ and $E_a(\cdot), E_s(\cdot)$ are absolutely continuous and singular operator valued measures, respectively.*

*Proof.* Let $p(x)$ and $q(x)$ be complex polynomials and let $f$ and $g$ be elements of $G$, the generating basis used in the definition of $\sigma_s(T)$. If $h = p(T)f$ and $k = q(T)g$ then the measure $\langle E(\cdot)h, k\rangle$ is absolutely continuous with respect to the measure $\langle E(\cdot)f, f\rangle$. With this end in mind we let $D$ be a bounded Borel set such that

$$0 = \langle E(D)f, f\rangle = \|E(D)f\|^2.$$

Then

$$|\langle E(D)h, k\rangle| \leqq \|E(D)h\| \|k\|$$
$$= \|k\|[\langle p(T)^*E(D)p(T)f, f\rangle]^{1/2}$$
$$= \|k\|\left[\int_D |p(t)|^2\langle E(dt)f, f\rangle\right]^{1/2}$$
$$\leqq \|k\| \sup\{|p(t)| : t \in D\}\langle E(D)f, f\rangle = 0.$$

For $D$ an arbitrary Borel set such that $0 = \langle E(D)f, f\rangle$, let $D = \cup D_J$ where $\{D_j\}$ is a sequence of pairwise disjoint bounded Borel sets. Certainly $0 = \langle E(D_j)f, f\rangle$ for each $j$ and by what has been proved $0 = \langle E(D_j)h, k\rangle$ for each $j$. By the countable additivity of the complex measure $\langle E(\cdot)h, k\rangle$ we get that $0 = \langle E(D)h, k\rangle$, as desired.

It follows from what has been proved that if $h = \sum_{j=1}^r p_j(T)f_j$ with each $f_j \in G$ and each $p_j(t)$ a complex polynomial then $\langle E(\cdot)h, h\rangle$ is absolutely continuous with respect to $\sum_{j=1}^r \langle E(\cdot)f_j, f_j\rangle$. Because $G$ is a generating basis for $T$ in $H$ the subspace, $S$, formed by the span of all vectors with the form of $h$ is dense in $H$.

Let $f \in G$ and let $C$ denote the complement in the reals of the set $\sigma_s(T)$. If $D$ is a representative Borel set then the following defines a Lebesgue decomposition

(+)        $\langle E(D)f, f\rangle = \langle E(D \cap \sigma_s(T))f, f\rangle + \langle E(D \cap C)f, f\rangle$.

In order to prove the assertion about (+), observe that by item (2)

of the summary of § 1 for $f, g \in G$ we know that $S_{f,g}$ is a Lebesgue null set. Since $\sigma_s(T)$ is the countable union of a collection of Lebesgue null sets (because $H$ is separable), it is a Lebesgue null set. Thus the first measure on the right of $(+)$ is certainly singular with respect to Lebesgue measure. By item (3) of the aforementioned summary we have $\{x \text{ real}: D_{\mathrm{sym}} \langle E(x)f, f \rangle = \infty\} \subset S_{f,f} \subset \sigma_s(T)$. This and the fact that $\sigma_s(T)$ is a null set show that the first measure on the right of $(+)$ is measure theoretically equivalent to the singular part of $\langle E(\cdot)f, f \rangle$ as given in the Lebesgue decomposition of item (4) of the summary of § 1. This proves that $(+)$ is a Lebesgue decomposition. It follows almost immediately that the following is a Lebesgue decomposition when $\{f_j: j = 1, \cdots, r\} \subset G$

$$(++) \qquad \sum_{j=1}^{r} \langle E(D)f_j, f_j \rangle = \sum_{j=1}^{r} \langle E(D \cap \sigma_s(T))f_j, f_j \rangle \\ + \sum_{j=1}^{r} \langle E(D \cap C)f_j, f_j \rangle .$$

Take $h$ a vector from $S$, the dense subspace generated by $G$. In the second paragraph of this proof we showed that $\langle E(\cdot)h, h \rangle$ is absolutely continuous with respect to $\sum_{j=1}^{r} \langle E(\cdot)f_j, f_j, \rangle$ for some choice of $\{f_j: j = 1, \cdots, r\}$ from $G$. By the transitivity of absolute continuity and by $(++)$ above we see that $\langle E(\cap C)h, h \rangle$ is absolutely continuous with respect to Lebesgue measure. It is obvious that $\langle E(\cdot \cap \sigma_s(T))h, h \rangle$ is singular with respect to Lebesgue measure and therefore the following is a Lebesgue decomposition

$$\langle E(D)h, h \rangle = \langle E(D \cap \sigma_s(T))h, h \rangle + \langle E(D \cap C)h, h \rangle .$$

Let $D$ be any Lebesgue null set. Then for any $h \in S$,

$$\langle E(D \cap C)h, h \rangle = 0$$

and, since $S$ is dense in $H$, it follows that $E(D \cap C) = 0$. So $E(\cdot \cap C)$ is absolutely continuous with respect to Lebesgue measure as an operator valued measure. Hence the theorem is proved.

COROLLARY 1. $E(\sigma_s(T))H = H_s$ and thus $E(C)H = H_a$ where $C$ is the complement in the reals of $\sigma_s(T)$.

Proof. If $f \in E(\sigma_s(T))H$ then $f = E(\sigma_s(T))f$ and

$$\langle E(\cdot)f, f \rangle = \langle E(\cdot)E(\sigma_s(T))f, f) = \langle E(\cdot \cap \sigma_s(T))f, f \rangle .$$

Thus $f \in H_s$.

Assume that $f$ is a vector of $H_s$ which is orthogonal to $E(\sigma_s(T))H$.

Since $I = E(\sigma_s(T)) + E(C)$ it must be that $E(C)f = f$ and consequently we can choose a subset of $C$, say $D$, to be a support for $\langle E(\cdot)f, f\rangle$. However $f \in H_s$ implies that $D$ can be taken to be a Lebesgue null set and Theorem 1 shows that

$$E(D) = E_s(D) = E(D \cap \sigma_s(T)) = E(\varnothing) = 0 \ .$$

So $f = 0$ and the corollary is proved.

The next theorem shows that the point spectrum of $T$, $\sigma_{pt}(T)$, which is easily seen to be a subset of $\sigma_s(T)$ can be characterized in terms of the rate of growth of the resolvent operator $R(z)$.

THEOREM 2.   $x \in \sigma_{pt}(T)$ if and only if there exists some $f \in H$ such that $|a \operatorname{Im} \langle R(x + ia)f, f\rangle|$ approaches a finite nonzero limit as $a \to 0$ with $a > 0$.

*Proof.*  For $x \in \sigma_{pt}(T)$ take $f$ to be a corresponding normalized eigenvector. Then $\langle R(x + ia)f, f\rangle = i/a$ and so $|a \operatorname{Im}\langle R(x + ia)f, f\rangle| = 1$ for all $a > 0$.

Now assume that $x$ is not an eigenvalue of $T$ and take $f$ to be any unit vector of $H$. Note that for $\langle E(\cdot)f, f\rangle = m(\cdot)$ the measure of the set of real numbers is 1 and $m((x - \varepsilon, x + \varepsilon)) \to 0$ as $\varepsilon \to 0$ with $\varepsilon > 0$. Taking $a > 0$,

$$I_1 = (-\infty, x - a^{3/4}], \ I_2 = (x - a^{3/4}, x + a^{3/4}), \ I_3 = [x + a^{3/4}, \infty) \ ,$$

we use the spectral representation for $T$,

$$a \operatorname{Im} \langle R(x + a)f, f\rangle = \int [a^2/(t - x)^2 + a^2]m(dt)$$

$$= \int_{I_1} [a^2/(t - x)^2 + a^2]m(dt)$$

$$+ \int_{I_2} [a^2/(t - x)^2 + a^2]m(dt)$$

$$+ \int_{I_3} [a^2/(t - x)^2 + a^2]m(dt)$$

$$\leqq 2[a^2/a^{3/2} + a^2] + m(I_2) \ .$$

As $a \to 0$, $[a^2/a^{3/2} + a^2] = [a^{1/2}/1 + a^{1/2}] \to 0$ and $m(I_2) \to 0$. This proves that $|a \operatorname{Im} \langle R(x + ia)f, f|\to 0$ as $a \to 0$ and the theorem follows.

3.  A theory for finite-dimensional perturbations.   In this section we are concerned with the behavior of the singular spectrum when $T$ is perturbed by a self adjoint operator $V$ which has finite dimensional range.   The inquiry is motivated by two theorems.   The

first is the Weinstein-Aronszajn Theorem which gives a formula for the change in the multiplicity of an isolated eigenvalue according to the local behavior of the Weinstein-Aronszajn determinant, which is a meromorphic function in a neighborhood of the eigenvalue. The second theorem is due to Aronszajn and Donoghue and it states that the singular spectrum of $P = T + \langle \cdot, h \rangle ch$ is contained in the set

$$\{x \text{ real}: \langle R(x + ia)h, h \rangle \to 1/-c \text{ as } a \to 0\}$$

provided $h$ is a cyclic vector for $T$ in $H$. A corollary of this result is that the singular spectrum of $T$ is disjoint from the singular spectrum of $P$. In order to relate the two theorems we reformulate the second theorem in terms of the $W-A$ determinant. In the one is given by $w(z) = \det[I + VR(z)]/VH = 1 + c\langle R(z)h, h \rangle$. Thus the dimensional case the $W-A$ determinant singular spectrum of $P$ is contained in the set of real numbers where $w(z)$ takes the boundary value 0. It follows from the $W-A$ theorem that new isolated eigenvalues are contained in the set where the $W-A$ determinant takes the boundary value 0. This suggests the following theorem with $V = \sum_{j=1}^{r} \langle \cdot, h_j \rangle c_j h_j$.

THEOREM 3. *If* $x \in \sigma_s(P)$ *and* $x \notin \sigma_s(T)$ *then there exists a sequence of positive numbers converging to* 0, *say* $\{a_n\}$, *such that* $w(x + ia_n) \to 0$ *as* $a_n \to 0$.

*Proof.* We use an elementary inversion formula derived in [9], pp. 161–162,

$$\left[ I + \sum_{j=1}^{r} \langle \cdot, a_j \rangle b_j \right]^{-1} = I - \sum_{i,j=1}^{r} d(i, j)\langle \cdot, a_j \rangle b_i$$

where $d(i, j)$ is the quotient with numerator equal to the signed minor associated with the $(i, j)$th element of the matrix $(\delta_{ij} + \langle b_i, a_j \rangle)$ and the denominator of the quotient is the determinant of the above matrix. Thus

$$S(z) = (P - zI)^{-1} = ([I + VR(z)][T - zI])^{-1}$$
$$= R(z)[I + VR(z)]^{-1} = R(z)\left[ I + \sum_{j=1}^{r} \langle R(z)\cdot, h_j \rangle c_j h_j \right]^{-1}$$
$$= R(z)\left[ I + \sum_{j=1}^{r} \langle \cdot, c_j R(\bar{z})h_j \rangle h_j \right]^{-1}$$
$$= R(z)\left[ I - \sum_{i,j=1}^{r} d(i, j, z)\langle \cdot, c_j R(\bar{z})h_j \rangle h_i \right]$$
$$= R(z) - \sum_{i,j=1}^{r} d(i, j, z)\langle \cdot, R(\bar{z})h_j \rangle c_j R(z)h_i$$

where $d(i, j, z)$ is formed as in the above inversion formula from the

matrix with the following $(i, j)$th element

$$\delta_{ij} + \langle h_i, c_j R(\bar{z}) h_j \rangle = \delta_{ij} + \langle R(z) h_i, c_j h_j \rangle$$
$$= \delta_{ij} + \langle R(z) h_i, V h_j \rangle$$
$$= [I + VR(z)]/VH .$$

Thus the determinant involved in the definition of $d(i, j, z)$ is the $W-A$ determinant. It follows from the above formula that $\langle S(z) h_k, h_l \rangle$ is equal to

$$(+) \qquad \langle R(z) h_k, h_l \rangle - \sum_{i,j=1}^{r} d(i, j, z) \langle h_k, c_j R(\bar{z}) h_j \rangle \langle R(z) h_i, h_l \rangle .$$

Let $H_1, H'$ be the smallest subspaces of $H$ containing $VH$ and reducing the operators $T, P$, respectively. Since $VH \subset H_1$ we know $H_1$ is invariant under $V$ and since $V$ is self adjoint $H_1$ reduces $V$. Since $H_1$ reduces both $T$ and $V$ it must reduce $P$ and thus $H' \subset H_1$. By symmetry, i.e., $T = P + (-V)$, we get $H' = H_1$. Let $H''$ be the orthogonal complement of $H'$ in $H$. Then $\sigma_s(P) = \sigma_s(P/H') \cup \sigma_s(P/H'')$. Because the orthogonal complement of the kernel of $V$ is the closure of $VH$ which is contained in $H'$ we have $T/H'' = P/H''$ and

$$\sigma_s(T/H'') = \sigma_s(P/H'') .$$

So if $x \in \sigma_s(P/H'')$ then $x \in \sigma_s(T/H'')$ which is contained in $\sigma_s(T)$. This contradicts the hypothesis and so we may assume $x \in \sigma_s(P/H')$.

From the fact proved above, $H_1 = H'$, it follows that $\{h_j : j = 1, \cdots, r\}$ is a generating basis for both $T$ and $P$ in $H'$. Since $x \in \sigma_s(P/H')$ it must be that for some pair $h_k, h_l$ there is a sequence of positive numbers converging to 0, say $\{a_n\}$, such that

$$|\langle S(x + ia_n) h_k, h_l \rangle| \to \infty$$

as $a_n \to 0$. If for some $i, j$ there is a sequence of positive numbers converging to 0, say $\{b_n\}$, such that $|\langle R(x + ib_n) h_i, h_j \rangle| \to \infty$ as $b_n \to 0$ then $x \in \sigma_s(T)$ which is contrary to the hypothesis. Thus we may assume that $|\langle R(x + ia_n) h_i, h_j \rangle|$ is bounded uniformly in $n$. In view of the formula $(+)$ it must be that $w(x + id_n) \to 0$ as $d_n \to 0$ for $\{d_n\}$ some subsequence of $\{a_n\}$; otherwise every term of the equation would be bounded uniformly in $a_n$. This proves the theorem.

LEMMA 1. *It is clear that $VH$ is invariant under $[I + VR(z)]$ and under $[I - VS(z)]$. On $VH$ the following relations hold:*

$$[I + VR(z)][I - VS(z)] = I = [I - VS(z)][I + VR(z)] .$$

*Proof.* Note that $I + VR(z) = (P - zI) R(z)$ and

$$I - VS(z) = [P - zI - V]S(z) = (T - zI)S(z) .$$

COROLLARY 1 (*to Theorem* 3). *If* $x \in \sigma_s(T)$ *and* $x \notin \sigma_s(P)$ *then for some sequence of positive numbers converging to* 0, *say* $\{a_n\}$, *we have* $|w(x + ia_n)| \to \infty$ *as* $a_n \to 0$.

*Proof.* Apply Theorem 3 viewing $P$ as the unperturbed operator and $-V$ as the perturbation so that the $W-A$ determinant in this case is $V(z) = \det [I - VS(z)]/ - VH$. Since

$$\begin{aligned}
1 &= \det I/VH = \det \{[I - VS(z)]/VH[I + VR(z)]/VH\} \\
&= \det [I - VS(z)]/VH \det [I + VR(z)]/VH \\
&= v(z)w(z)
\end{aligned}$$

and since Theorem 3 guarantees a sequence of positive numbers converging to 0, say $\{a_n\}$, such that $v(x + ia_n) \to 0$ as $a_n \to 0$, it must be that $|w(x + ia_n)| \to \infty$ as $a_n \to 0$. This proves the corollary.

We want to emphasize the contrast between the multi-dimensional case and the one dimensional case of the Aronszajn-Donoghue theorem. In the multi-dimensional case the singular spectrum of the perturbed operator need not be disjoint from the singular spectrum of the unperturbed operator and this unfortunate fact is true for any reasonable definition of the singular spectrum. Indeed we provide an example where the dimension of the perturbation is two and the perturbed operator is unitarily equivalent to the unperturbed operator.

COUNTER-EXAMPLE. Let $u(\cdot)$ be an arbitrary probability measure on the reals. Let $H'$ and $H''$ both be equal to the space of "functions" square integrable with respect to the above measure. Define two multiplication operators on the two spaces according to $(M'f)(t) = tf(t)$ and $(M''f)(t) = tf(t)$ where $f$ is in the domain of such an operator if and only if $tf(t)$ is in the space. Let $H$ be the external direct sum of the two spaces, i.e., $H = H' \oplus H''$ and set

$$h_1 = (0, 1), \quad h_2 = (1, 0), \quad V = -\langle \cdot, h_1 \rangle h_1 + \langle \cdot, h_2 \rangle h_2 .$$

If $T = [M' + \langle \cdot, h_1 \rangle h_1] \oplus M''$ then $P = M' \oplus [M'' + \langle \cdot, h_2 \rangle h_2]$ which is clearly unitarily equivalent to $T$ by the operator $(f, g) \to (g, f)$. Thus the structure of the spectrum of $P$ is exactly the same as that of $T$.

4. A basic method for the perturbation of the singular spectrum. The basic method used in this section is to couple a certain simple factorization of the operator $I + VR(z)$ with inversion arguments

related to the second Neumann series (see [7], pp. 66–67). In order
to simplify the statements of the results we henceforth make the
following basic assumption:

$H$ has no proper subspace which
contains $VH$ and reduces $T$.

In a sense this is no assumption at all since it is always possible to
reduce to this situation by an argument due to Kato in [6] given
in detail in the second paragraph of the proof of Theorem 3. It is
also convenient to have an appropriate notion of neighborhood. If $B$
is a subset of the reals then we say that $N$ is an upper half plane
neighborhood (abbreviated u.h.p. nghd) of $B$ provided that for each
$x \in B$ there is a nontangential path ending at $x$ which is contained in
the intersection of $N$ with the upper half of the complex plane. The
purpose of these neighborhoods is that they permit free application
of the boundary value theory which requires nontangential limits.
Unless it is specifically stated to the contrary in the following $V$ is
a self adjoint bounded perturbation which is not necessarily finite
dimensional. Recall that Lemma 1 demonstrated that $VH$ is invariant
under the $W - A$ matrix $W(z) = I + VR(z)$ and its inverse $W(z)^{-1} = I - VS(z)$.

LEMMA 2. *Let $Q'$ be any orthogonal projection which commutes
with $T$ and set $Q = I - Q'$. Then $VH$ is invariant under $U(z) =
I + VR(z)Q$ and under $I - VQ'S(z)$. For any nonreal $z$ we have
$W^{-1}(z) = U^{-1}(z)[I - VQ'S(z)]$.*

*Proof.* An argument of Howland in [5] works in this generality
to show that $U(z)$ is one-to-one. Assume $0 = U(z)f = f + VR(z)Qf$.
Then

$$0 = Qf + QVR(z)Q^2f = Qf + QVQR(z)Qf$$
$$= (T + QVQ - zI)R(z)Qf .$$

Because $T$ and $T + QVQ$ are self adjoint for any nonreal $z$,

$$(T + QVQ - zI)R(z)$$

is one-to-one on $QH$. Thus $Qf = 0$ and by the first equation we see
that $f = 0$. Hence $U(z)$ is one-to-one.

Thus it suffices to show that $U(z)W^{-1}(z) = I - VQ'S(z)$. The
following simplication is based on the so-called second resolvent equa-
tion, i.e., $R(z)VS(z) = R(z) - S(z)$,

$$U(z)W^{-1}(z) = [I + VR(z)Q][I - VS(z)]$$
$$= I + VR(z)Q - VS(z) - VQ[R(z) - S(z)]$$

$$= I - VS(z) + V(I - Q')S(z)$$
$$= I - VQ'S(z) .$$

LEMMA 3. *Let* $S$ *be a subset of the reals and let* $N$ *be an upper half plane neighborhood of* $S$. *If* $\| VQ'S(z)/VH \|$ *and* $\| U^{-1}(z)/VH \|$ *are bounded for nonreal* $z \in N$ *then the intersection of* $S$ *with the singular spectrum of* $P$ *is empty.*

*Proof.* We note that for nonreal $z$

$$\| I - VS(z)/VH \| = \| W^{-1}(z)/VH \| \leqq \| U^{-1}(z)/VH \| \, \| I - VQ'S(z)/VH \| .$$

Thus for $g$, $h \in VH$ and nonreal $z \in N$ we get that $|\langle S(z)g, h \rangle|$ is bounded. Hence for each $x \in S$ we know that $x \notin \sigma_s(P)$ and the lemma is proved.

COROLLARY. *The above lemma remains true if the boundedness of* $U^{-1}(z)/VH$ *is replaced by*

$$(2') \qquad \| VR(z)Q/VH \| \leqq m < 1 \text{ for nonreal } z \in N .$$

*Proof.* By $(2')$ the following Neumann series converges in the uniform topology

$$U^{-1}(z)/VH = \{[I + VR(z)Q]/VH\}^{-1}$$
$$= \sum_{n=0}^{\infty} [VR(z)Q/VH]^n(-1)^n .$$

By the triangle inequality and the sum formula for a geometric progression we see that the norm of $U^{-1}(z)$ is bounded for nonreal $z \in N$. Thus the hypothesis of the lemma is satisfied.

THEOREM 4. *Let* $S$ *be a subset of the reals, let* $S'$ *be its complement, and let* $N$ *be an u.h.p. nghd of* $S$. *If for nonreal* $\bar{z} \in N$ *we have* (1) $\| S(z)/E(S)H \| \leqq M$ *and for nonreal* $z \in N$ *we have* (2) $\| R(z)/E(S')H \| \leqq m/\| V \|$ *with* $m < 1$ *then the intersection of* $S$ *with the singular spectrum of* $P$ *is empty.*

*Proof.* (1) above implies that $\| E(S)S(\bar{z}) \| = \| S(z)E(S) \|$ is bounded for nonreal $\bar{z} \in N$. Thus $\| VE(S)S(\bar{z}) \|$ is bounded for nonreal $\bar{z} \in N$ and the first condition of Lemma 3 is satisfied.

Using hypothesis (2) above we observe

$$\| VR(z)E(S')/VH \| \leqq \| VR(z)E(S') \|$$
$$\leqq \| V \| \, \| R(z)E(S') \|$$
$$= \| V \| \, \| R(z)/E(S')H \|$$
$$\leqq m < 1 .$$

Thus (2′) of the corollary to Lemma 3 is satisfied.

COROLLARY 1. *Let $x$ be a real number, let $\{x\}'$ be the complement in the reals of $\{x\}$; let $B$ be a nontangential path through $x$. If the norm of $S(z)/E\{x\}H$ is bounded for nonreal $\bar{z} \in B$ and if the norm of $R(z)/E(\{x\}')H$ is bounded by $m/\|V\|$ for $m < 1$ and nonreal $z \in B$ then $x$ is not an eigenvalue of $P$.*

COROLLARY 2. *If $x$ is an eigenvalue of $P$ then the distance from $x$ to the spectrum of $T$, denoted $\mathrm{dist}\,(x, \sigma(T))$, is not greater than $\|V\|$.*

*Proof.* Since our conclusion is trivial if $x \in \sigma(T)$, we may assume $x \notin \sigma(T)$ and so $\mathrm{dist}\,(x, \sigma(T)) \neq 0$. Since $E(\{x\})H = \{0\}$ the first hypothesis of Corollary 1 is certainly satisfied. Because the conclusion of that corollary is false, it must be that the second hypothesis is not satisfied, i.e., for every $m < 1$ and an infinite set of $z \in B$ converging to $x$ we have

$$\| R(z) \| = \| R(z)/E(\{x\}')H \| > m/\|V\| \,.$$

Thus $1/\mathrm{dist}\,(x, \sigma(T)) \geq \| R(x) \| \geq m/\|V\|$ and the desired conclusion follows.

Just as Theorem 4 was derived from Lemma 3 we shall also derive Theorem 5 from Lemma 3.

THEOREM 5. *Let $B$ denote the open interval $(a - \|V\|, b + \|V\|)$; $B'$ denotes the complement of $B$ in the reals. Let $N$ be an u.h.p. nghd of $(a, b)$. If the norm of $S(z)/E(B)H$ is bounded for all nonreal $\bar{z} \in N$ then $(a, b)$ has empty intersection with the singular spectrum of $P$.*

*Proof.* The norm of $S(z)/E(B)H$ is the same as the norms of $S(z)E(B)$ and $E(B)S(\bar{z})$ for nonreal $\bar{z} \in B$. Then for $Q' = E(B)$ the first hypothesis of Lemma 3 follows.

The space $E(B')H$ reduces $T$ and for $x \in (a, b)$ the operator $(T - xI)/E(B')H$ has an inverse operator with norm bounded by the reciprocal of $d = \mathrm{dist}\,(x, \sigma(T/E(B')H))$ which is strictly less than the reciprocal of $\|V\|$. It follows that for nonreal $z$ sufficiently close to $x$ the norm of $R(z)/E(B')H$ is less than or equal to $(1/2d) + (1/2\|V\|)$. Consequently

$$\begin{aligned}
\| VR(z)E(B') \| &\leq \| V \| \, \| R(z)E(B') \| \\
&= \| V \| \, \| R(z)/E(B')H \| \\
&\leq \| V \|(1/2d) + (1/2\|V\|) < 1 \,.
\end{aligned}$$

Thus the second hypothesis of Lemma 3 is satisfied and the desired conclusion follows.

A trivial nodification of the above proof gives the following corollary.

COROLLARY 1. *Let $m > 1$ and let $B$ be a nontangential path through $x$. If the norm of $S(z)/E((x - m\|V\|, x + m\|V\|))H$ is bounded for all nonreal $\bar{z} \in B$ then $x$ is not an eigenvalue for $P$.*

5. **Examples and remarks.** Our first objective in this section is to show how our main theorem on the behavior of the singular spectrum under finite dimensional perturbations can be applied to solve many concrete problems. The following theorem is an easy consequence of Theorem 3 and the classical $W - A$ formula; recall that $w(z)$ denotes the $W - A$ determinant.

THEOREM 6. *Let $T$ and $V$ be self adjoint operators on $H$ with $V$ finite dimensional. If there is a generating basis for $T$ on $H$, say $G$, and relative to this generating basis for each real number $x$ there exist positive numbers $d(x)$ and $e(x)$ such that $|w(x + ia)| \geqq e(x)$ for $d(x) > a > 0$ then $\sigma_s(P)$ is contained in $\sigma_s(T)$.*

Let $H$ be the space of square integrable "functions" with respect to Lebesque measure on the closed interval $[1, 2]$. Let $T$ be the operator that multiplies by the independent variable, i.e., $(Tf)(t) = tf(t)$. Let $V$ be $\langle \cdot, g_1 \rangle g_1 - \langle \cdot, g_2 \rangle g_2$ where $g_1(t) = 1$ and $g_2(t) = t - 3/2$ and note that $g_1$ is orthogonal to $g_2$. It is trivial that the set $G = \{g_1, g_2\}$ is a generating basis for $T$ on $H$ and we find that $\{1, 2\} = \sigma_s(T)$ when it is defined using $G$. In order to apply the above result we need to derive some information about the boundary values of the $W - A$ determinant.

These boundary values can be calculated by using some classical results summarized in [2]. We calculate the $W - A$ matrix, $W(z)$, with respect to the basis $\{g_1, g_2\}$ and we enumerate the four entries by moving left to right across the first row and then the second. These entires are $1 + \langle R(z)g_1, g_1 \rangle$, $-\langle R(z)g_1, g_2 \rangle$, $\langle R(z)g_2, g_1 \rangle$, and $1 - \langle R(z)g_2, g_2 \rangle$. These inner products have the following representations as integrals from 1 to 2,

$$\langle R(z)g_1, g_1 \rangle = \int 1/(t - z)dt$$

$$\langle R(z)g_1, g_2 \rangle = \langle R(z)g_2, g_1 \rangle = \int (t - 3/2)/(t - z)dt$$

$$\langle R(z)g_2, g_2 \rangle = \int (t - 3/2)^2/(t - z)dt \ .$$

The second and third integrals can be calculated in terms of the first and for any $x \in (1, 2)$ the limit of $\langle R(x + ia)g_1, g_1 \rangle$ as $a \rightarrow 0$ is

$$\text{p. v.} \int 1/(t - x)dt + i\pi$$

where the first integral is a principal value integral. This integral is easily calculated and it is found to be $\ln[(2 - x)/(x - 1)]$. Thus the boundary values of the $W - A$ determinant exist and they can be easily calculated. The imaginary part of the limit of $w(x + ia)$ as $a \rightarrow 0$, $x \in (1, 2)$, is

$$(x - 3/2)^2\pi + (x - 3/2)\pi + \pi .$$

We find there is no $x \in (1, 2)$ such that the above expression equals 0. Because the essential spectrum is invariant under finite dimensional perturbations we know that the only spectrum of $P$ that could be contained in the complement of $[1, 2]$ are isolated eigenvalues of finite multiplicity. We can apply the classical $W - A$ formula to show that no such eigenvalues appear. The $W - A$ determinant for real $x \notin [1, 2]$ is found to be $w(x) = (1/2 - x)^2 \ln[(2 - x)/(1 - x)] + (1/2 - x)$. The function $w(x)/(1/2 - x)$ is strictly negative and decreasing in the interval $(-\infty, 1)$; it is strictly positive and decreasing in the interval $(2, \infty)$.

The only remaining possibility for the perturbed spectral measure, $F(\cdot)$, to fail to be absolutely continuous is if $P$ has either 1 or 2 as an eigenvalue. This possibility can be easily eliminated by direct calculation and consequently we have shown that $F(\cdot)$ is absolutely continuous.

Although we are unable to make the above calculations in profound generality, it seems clear that the calculations can be completed in a large number of problems. Theorems 4 and 5 can be applied in several situations. By using the inversion formula for finite dimensional perturbations of the identity operator, which we introduced in the proof of Theorem 3, we can obtain a representation for the resolvent $S(z)$ which suffices in many situations. The resolvent for the perturbed operator can be obtained in some problems related to ordinary differential equations.

This leads us to remark that almost every result in this paper carries over to the case that $T$ is self adjoint and densely defined, although it is not bounded. This generalization only requires an occasional remark about the domains of composite operators.

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## Vol. 34, No. 3          July, 1970