SPECIAL SEMIGROUPS ON THE TWO-CELL

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A commutative semigroup $S$ has property $(\alpha)$ if (1) $S$ is topologically a two-cell, (2) $S$ has no zero divisors, and (3) the boundary of $S$ is the union of two unit intervals with the usual multiplication. A characterization of semigroups having property $(\alpha)$ will be given. Let $(I, \cdot)$ denote the closed unit interval with the usual multiplication. Let $M$ be a closed ideal of $(I, \cdot) \times (I, \cdot)$ such that $M$ contains $(I \times \{0\}) \cup ([0] \times I)$, and $M \cap (I \times \{1\}) = \{(0, 1)\}$ or $M \cap ([1] \times I) = \{(1, 0)\}$. For each $a, b \in (0, 1)$ define a relation $R(a, b; M)$ on $(I, \cdot) \times (I, \cdot)$ by $(x, y) \in R(a, b; M)$ if (1) $x = y$ or (2) $x, y \in (I \times \{0\}) \cup ([0] \times I)$, or (3) there exists an $s \in (0, \infty)$ such that $x$ and $y$ are in the same component of $M \cap \{(a^s, b^{s-1}); 0 \leq t \leq 1\}$.

**Lemma.** The relation $R(a, b; M)$ is a closed congruence.

**Theorem.** A semigroup $S$ has property $(\alpha)$ if and only if there exists $a, b, M$ such that $(I, \cdot) \times (I, \cdot)/R(a, b; M)$ is isomorphic to $S$.

A central problem in the theory of topological semigroups is to characterize those semigroups whose underlying space is fixed. In general this problem is much too difficult; however, in some special cases considerable progress has been made. For example semigroups on the unit interval with identities are completely classified in [3], [4], and [7]. Some special cases on the two-cell have also been investigated [1], [2], [5], [6] and [7].

In this paper we are concerned with commutative semigroups having property $(\alpha)$. A semigroup $S$ has property $(\alpha)$ if (1) $S$ is topologically a two-cell, (2) $S$ has no zero divisors, and (3) the boundary of $S$ is the union of two unit intervals with the usual multiplication. A description of commutative semigroups satisfying property $(\alpha)$ will be given.

We begin by giving a method of constructing commutative semigroups having property $(\alpha)$. We will show later that this method yields all commutative semigroups having property $(\alpha)$.

Let $(I, \cdot)$ denote the closed unit interval with the usual multiplication. Let $M$ be a closed ideal of $(I, \cdot) \times (I, \cdot)$ such that $M$ contains $(I \times \{0\}) \cup ([0] \times I)$ and $M \cap (I \times \{1\}) = \{(0, 1)\}$ or $M \cap ([1] \times I) = \{(1, 0)\}$. For $a, b$ contained in the open interval $(0, 1)$ define the relation $R(a, b; M)$ on $(I, \cdot) \times (I, \cdot)$ by $(x, y) \in R(a, b; M)$ if (1) $x = y$ or (2) $x, y \in (I \times \{0\}) \cup ([0] \times I)$ or (3) there exists an $s$ contained in the positive reals such that $x$ and $y$ are in the same component of...
LEMMA 1. The relation $R(a, b; M)$ is a closed congruence, and hence $(I, \cdot) \times (I, \cdot)/R(a, b; M)$ is a semigroup.

Proof. We will first show $R(a, b; M)$ is closed. Let $(\hat{r}_n, \hat{s}_n) \in R(a, b; M)$ for $n = 1, 2, 3, \ldots$, with $(\hat{r}_n, \hat{s}_n) \rightarrow (\hat{r}, \hat{s})$. If an infinite number of the elements of the sequence satisfy (1) or (2), then $(\hat{r}, \hat{s}) \in R(a, b; M)$. Hence we can assume all of the elements of the sequence satisfy (3). This implies there exist sequences $w_n, c_n, d_n$ such that $\hat{r}_n = (a w_n c_n, b w_n^{-w_n c_n})$ and $\hat{s}_n = (a w_n d_n, b w_n^{-w_n d_n})$ where $w_n$ is a positive real number and $c_n, d_n \in [0, 1]$. Since $\hat{r}_n \rightarrow \hat{r}$ and $\hat{s}_n \rightarrow \hat{s}$, we have either (a) $w_n \rightarrow \infty$ or (b) $w_n \rightarrow w \in (0, \infty)$, $c_n \rightarrow c$ and $d_n \rightarrow d$. If (a) holds we have $a^{w_n c_n} \rightarrow 0$ or $b^{w_n^{-w_n c_n}} \rightarrow 0$, and $a^{w_n d_n} \rightarrow 0$ or $b^{w_n^{-w_n d_n}} \rightarrow 0$, hence $\hat{r}, \hat{s} \in \{(0) \times I\} \cup (I \times \{0\})$ and $(\hat{r}, \hat{s}) \in R(a, b; M)$. If (b) holds we use the fact that $(a^{w_n c_n}, b^{w_n^{-w_n c_n}}) \in M$ for any $c_n$ satisfying $\min(c_n, d_n) \leq e \leq \max(c_n, d_n)$. Let it be the case that $\min(c, d) \leq e \leq \max(c, d)$. Then there exists a sequence such that $\min(c_n, d_n) \leq e_n \leq \max(c_n, d_n)$ and $e_n \rightarrow e$. Since $(a^{w_n c_n}, b^{w_n^{-w_n c_n}}) \in M$ and $M$ is closed we obtain $(a^{w_n c_n}, b^{w_n^{-w_n c_n}}) \rightarrow (a^e, b^{-w e}) \in M$. Hence $\hat{r}$ and $\hat{s}$ are in the same component of $(M \cap \{(a^t, b^{w-t}): 0 \leq t \leq 1\})$, which implies $(\hat{r}, \hat{s}) \in R(a, b; M)$.

To show that $R(a, b; M)$ is a congruence, after a moments reflection, it becomes clear that we need only show $((x, 1)\hat{r}, (x, 1)\hat{s})$ satisfies property (3) whenever $(\hat{r}, \hat{s})$ satisfies property (3) and $0 < x < 1$. Let $\hat{r} = (a^{w e}, b^{-w e})$ and $\hat{s} = (a^{w d}, b^{-w d})$ with $c \leq d$. Also $\{(a^e, b^{-w e}): c \leq e \leq d\} \subseteq M$. Since $0 < x < 1$, there exist a $q \in (0, \infty)$ such that $(a^q, 1) = (x, 1)$. Using the fact that $M$ is an ideal of $(I, \cdot) \times (I, \cdot)$ we see that

$$(x, 1)(a^e, b^{-w e}) = (a^q, 1)(a^{w e}, b^{-w e}) = (a^{q+w e}, b^{-w e}) = (a^m f, b^{-m-f}) \in M$$

for $m = q + w$ and $f = ew + q/w + q$ and $c \leq e \leq d$. This completes the proof.

One can observe that the map $\varphi: (I, \cdot) \times (I, \cdot) \rightarrow (I, \cdot) \times (I, \cdot)/R(a, b; M)$ which sends elements to their equivalence classes is a monotone map, and no equivalence class of $R(a, b; M)$ separates $(I, \cdot) \times (I, \cdot)$. A theorem of Whyburn [8] reveals that $(I, \cdot) \times (I, \cdot)/R(a, b; M)$ is a two-cell. Also since $(I, \cdot) \times (I, \cdot)/R(a, b; M)$ is the homomorphic image of $(I, \cdot) \times (I, \cdot)$ which is commutative, it is commutative. Furthermore, the boundary of $(I, \cdot) \times (I, \cdot)/R(a, b; M)$ equals $\varphi((I, \cdot) \times \{1\}) \cup \varphi((I) \times (I, \cdot))$, and hence is the union two unit intervals with usual multiplication. Finally since $(I \times \{0\}) \cup (\{0\} \times I)$ is a completely prime
ideal of \( (I, \cdot) \times (I, \cdot), (I, \cdot) \times (I, \cdot)/R(a, b; M) \) has no zero divisors. Thus we have proved the following:

**Theorem A.** \( (I, \cdot) \times (I, \cdot)/R(a, b; M) \) is a commutative semigroup satisfying property (a).

Now we will take a commutative semigroup \( S \) satisfying property (a) and find \( a, b \in (0, 1) \) and an ideal \( M \) such that \( (I, \cdot) \times (I, \cdot)/R(a, b; M) \) is isomorphic to \( S \).

We begin this section by letting the boundary of \( S \) equal \( U \cup V \) where \( U \) and \( V \) are unit intervals with the usual multiplication.

Without much difficulty it can be shown that \( S = U \cup V \) and \( U \cap V = \{z, i\} \) where \( z \) is the zero for \( S \) and \( i \) is the identity for \( S \). Letting \( f: (I, \cdot) \to U \) and \( g(I, \cdot) \to V \) be isomorphisms and defining \( h: (I, \cdot) \times (I, \cdot) \to S \) by \( h(x, y) = f(x)g(y) \), we see that \( h \) is a continuous homomorphism from \( (I, \cdot) \times (I, \cdot) \) onto \( S \).

**Lemma 2.** If \( h(x_1, y_1) = h(x_2, y_2) = z \), then one and only one of the following holds:

1. \( x_1 = x_2 \) and \( y_1 = y_2 \)
2. \( (x_1 - x_2)(y_1 - y_2) < 0 \).

**Proof.** Let \( h(x_1, 1) = u_i \) and \( h(1, y_1) = v_j, j = 1, 2 \). If (1) is not true, then suppose \( x_1 > x_2 \). This is the case if and only if there exist \( u \in U, u \neq i \) such that \( uu_i = u_2 \). Now \( y_1 \geq y_2 \) if and only if there exist \( v \) such that \( vv_i = v \). Since \( h(x_1, y_1) = h(x_2, y_2) \) we have \( u_i v_i = u_2 v_2 \) or \( u_i v_i = (u_i v_i)u(uv) \) which implies \( u_i v_i = (u_i v_i)\cdot u^n \cdot v^n \) for \( n = 1, 2, 3, \ldots \). Hence, \( u_i v_i = (u_i v_i)\cdot u^n \cdot v^n = z \). This is a contradiction.

Note for \( x \neq 0 \), \( \{h(x, y) : 0 \leq y \leq 1\} \) is an arc in \( S \).

**Lemma 3.** If \( s \in S \setminus \{z\} \), then there exist \( (x_1, y_1), (x_2, y_2) \in h^{-1}(s) \) such that for all \( (x, y) \in h^{-1}(s) \) we have \( x_1 \geq x \geq x_2 \) and \( y_1 \geq y \geq y_2 \).

**Proof.** Set \( x_1 = \sup \{x : h(x, y) = s\} \). Construct a sequence \( (q_n, r_n) \in h^{-1}(s) \) with \( q_n \geq q_n \geq q_n \) such that \( \lim q_n = x_1 \). Noting that \( r_{n+1} \leq r_n \), set \( y_1 = \lim r_n \). Since \( s = h(q_n, r_n) \) and \( (q_n, r_n) \to (x_1, y_1) \) we have \( h(x_1, y_1) = \lim h(q_n, r_n) = s \). This implies \( x_1 \) is the maximum \( x \) and \( y_1 \) is the minimum \( y \) such that \( h(x, y) = s \). A similar argument yields an \( (x_2, y_2) \in h^{-1}(s) \).

**Lemma 4.** If \( s \in S \setminus \{z\} \), then \( \pi(h^{-1}(s)) \) is connected.

**Proof.** Let \( x_1 < x < x_2 \) with \( (x_1, y_1), (x_2, y_2) \in h^{-1}(s) \). We will show
there exist a $\bar{y}$ such that $h(x, \bar{y}) = s$. The arc \{h(x, y): 0 \leq y \leq 1\} must intersect one of the two arcs \{h(x_1, y): y_1 \leq y \leq 1\} and \{h(x_2, y): y_2 \leq y \leq 1\}. Suppose it intersects the latter, then there exist $y, y'$ such that $h(x, y') = h(x_2, y)$. Hence, if one chooses $\bar{y} = y'y''$ where $yy'' = y_2$, then $h(x, \bar{y}) = h(x, y'y'') = h(x, y'') = h(x_2, y) = h(x_2, yy'') = h(x_2, y_2) = s$. This completes the proof.

**Remark 1.** By using Lemma 2 we note that the $\bar{y}$ obtained in the proof above is unique.

**Lemma 5.** If $s \in S \setminus \{z\}$, then for all $(x_1, y_1), (x_2, y_2) \in h^{-1}(s)$ we have $(\sqrt{x_1x_2}, \sqrt{y_1y_2}) \in h^{-1}(s)$.

**Proof.** Suppose $x_2 > x_1$, then $x_1 < \sqrt{x_1x_2} < x_2$, and there exist a unique $y$ such that $h(\sqrt{x_1x_2}, y) = s$. Now $s^2 \neq z$, and $h(x_2, y_1y_2) = s^2 = h(x_2, y')$. Hence $y = \sqrt{y_1y_2}$.

**Remark 2.** Note that $h^{-1}(z) = I \times \{0\} \cup \{0\} \times I$.

**Lemma 6.** If $s \in S \setminus \{z\}$, then there exist $(x_1, y_1), (x_2, y_2) \in h^{-1}(s)$ such that $h^{-1}(s) = \{(x_1^{1-t}, y_1^{1-t}): 0 \leq t \leq 1\}$.

**Proof.** Let $(x_1, y_1), (x_2, y_2)$ be the ordered pairs obtained in Lemma 3. By inducting on the previous lemma we see \{(x_1^{d}, y_1^{d-1}): 0 \leq d \leq 1, d \text{ a dyadic rational}\} \subset h^{-1}(s)$. Taking the closure of this set we get \{(x_1^{1-t}, y_1^{1-t}): 0 \leq t \leq 1\} \supset h^{-1}(s)$. Since $h^{-1}(s)$ cannot properly include this set, they are equal.

Let $J = \{s: s \in S$ and $h^{-1}(s)$ is not a point$\}$. Note that $J$ is an ideal of $S$, and hence $h^{-1}(J)$ and $h^{-1}(J^*)$ are ideals of $(I, \cdot) \times (I, \cdot)$.

**Lemma 7.** If $s \in J \setminus \{z\}$, then there exist $a, b \in (0, 1)$ such that $h^{-1}(s) \subset \{(a^t, b^{1-t}): 0 \leq t \leq 1\}$.

**Proof.** Let $(x_1, y_1), (x_2, y_2)$ be the ordered pairs obtained in Lemma 3. We know $x_1 > x_2 > 0$ and $y_2 > y_1 > 0$. Both $x_1$ and $y_1$ cannot be equal to 1 for if both were we would have $h(1, y_1) = h(x_2, 1)$ contradicting the fact that $U \cap V = \{z, i\}$. We shall assume $y_2 \neq 1$, hence there exist $\beta$ such that $0 < \beta < 1$ and $y_2^{1-\beta} = y_2$, also $0 < x_1 \leq 1$ and hence there exist $\gamma$ such that $0 \leq \gamma < 1$ and $x_2^{1-\gamma} = x_2$. Setting $a = (x_1x_2^{\gamma-1})^{1/\beta}$ and $b = (y_2^{1-\gamma})^{1/(1-\gamma)}$, it can be shown by simple algebraic manipulation that $a, b \in (0, 1)$ and $h^{-1}(s) \subset \{(a^t, b^{1-t}): 0 \leq t \leq 1\}$.

Note that there exist $t_1$ and $t_2$ such that $h^{-1}(s) \subset \{(a^t, b^{1-t}): 0 \leq t_1 \leq t \leq t_2 \leq 1\}$.
We will now show that the \( \alpha, b \in (0,1) \) obtained in the previous theorem is somewhat unique.

**Lemma 8.** If \( s, s' \in J \setminus \{z\} \) and suppose \( h^{-1}(s) = \{(a^t, b^{t-i}) : t_1 \leq t \leq t_2\} \), then there exists \( w \in (0, \infty) \) such that \( h^{-1}(s') = \{(a^w, b^{w-i-w}) : t'_1 \leq t \leq t'_2\} \).

**Proof.** Let \( h(x_1, y_1) = s \) and \( h(x_2, y_2) = s' \). From the previous lemma we know there exist \( c, d \in (0,1) \) such that \( h^c(s') = \{(c, d^c) : t_1 \leq t \leq t_2\} \). For \( (x, y) \in h^{-1}(s) \cap (x_2, y_2) \) we have \( h(x, y) = ss' \), also for \( (x', y') \in (x_1, y_1)h^{-1}(s') \) we have \( h(x', y') = ss' \). But \( h^{-1}(s) \cdot (x_2, y_2) = \{(a^x, b^{x-i}) : \delta_1 \leq \delta \leq \delta_2\} \) and \( (x_1, y_1)h^{-1}(s') = \{(c^y, d^{c-y}) : \eta_1 \leq \eta \leq \eta_2\} \). However, there exist \( p, q \in (0,1) \) such that \( h^p(ss') = \{(p, q^p) : \lambda_1 \leq \lambda \leq \lambda_2\} \). This implies \( a^w = p = c^v \), \( b^u = q = c^v \) or \( c = a^{v/v}, d = b^{v/v} \).

**Notation.** Let \( \text{Comp} (a^w, b^{i-w}) \) be the component of \( h^{-1}(J) \cap \{(a^t, b^{t-1}) : 0 \leq t \leq 1\} \) containing \( (a^{x_i}, b^{i-1}) \).

**Lemma 9.** If \( s \in J \setminus \{z\} \) and \( \{a^t, b^{t-1}) : t_1 \neq t_2 \) and \( t_1 \leq t \leq t_2 \) \( \subset h^{-1}(s) \), then \( h^{-1}(s) = \text{Comp} (a^w, b^{i-w}) \).

**Proof.** Let \( a^w, b^{i-w} \in \text{Comp} (a^t, b^{i-t}) \) and suppose \( w < t_1 \) and \( h(a^w, b^{i-w}) = s' \neq s \). Now \( \{h(a^t, b^{t-1}) : w \leq t \leq t_1\} \) is a curve in \( J \) containing \( s \) and \( s' \). Also for each \( q \in [w, t] \) there exist \( \beta_q, \gamma_q \) such that \( \beta_q < \gamma_q \) and \( h^{-1}(a^q, b^{i-q}) = \{(a^q, b^{i-q}) : \beta_q \leq t \leq \gamma_q\} \). Moreover, for \( s_i, s_2 \in s \) and \( s_1 \neq s_2 \) we have \( h^{-1}(s_i) \cap h^{-1}(s_2) = \emptyset \). Hence \( \{h^{-1}(s) : s \in h(a^t, b^{t-1}) : w \leq t \leq t_1\} \) is an uncountable collection of disjoint closed intervals contained in the interval \( \{(a^t, b^{t-1}) : 0 \leq t \leq 1\} \). This is impossible.

**Lemma 10.** If \( s \in J \), then \( sS = sU = sV \).

**Proof.** If \( s = z \), then \( zS = zU = zV = \{z\} \). Let \( s \neq z \) and \( h(x, y) = s = h(x', y') \) with \( x > x' \) and \( y' > y \). Choose \( x'', y'' \) such that \( xx'' = x' \) and \( y'y'' = y \). Let \( (\bar{x}, 1) = \{(t, 1) : x'' \leq t \leq 1\} \). We will show there exists \( (1, \bar{y}) \in (1, s) : y'' \leq s \leq 1 \) such that \( s \cdot h(\bar{x}, 1) = s \cdot h(1, \bar{y}) \). Now \( s \cdot (\bar{x}, 1) = h(x, y) \cdot h(\bar{x}, 1) = h(x\bar{x}, y) \) and \( x \leq x\bar{x} \leq x' \). Hence there exists a unique \( \bar{y} \) such that \( y \leq \bar{y} \leq y' \) and \( h(x\bar{x}, \bar{y}) = s \). Choose \( \bar{y} \) such that \( \bar{y}\bar{y} = y \). We see \( y'' \leq \bar{y} \leq 1 \), and

\[
s \cdot h(1, \bar{y}) = h(x\bar{x}, \bar{y}) \cdot h(1, \bar{y}) = h(x\bar{x}, \bar{y}\bar{y}) = h(x\bar{x}, y)
\]

\[
= h(x, y) \cdot h(\bar{x}, 1) = s \cdot h(\bar{x}, 1) \cdot .
\]

The same method yields for each \( (1, \bar{y}) \in (1, s) : y'' \leq s \leq 1 \) an \( (\bar{x}, 1) \in (t, 1): x'' \leq t \leq 1 \) such that \( s \cdot h(1, \bar{y}) = s \cdot h(\bar{x}, 1) \). Let \( s' \in S \). Then there exist \( m, n \) positive integers and \( x_0, y_0 \) such that \( x'' \leq x_0 \leq 1, y'' \leq y_0 \leq 1 \) and

\[
s \cdot h(1, \bar{y}) = h(x\bar{x}, \bar{y}) \cdot h(1, \bar{y}) = h(x\bar{x}, \bar{y}\bar{y}) = h(x\bar{x}, y)
\]

\[
= h(x, y) \cdot h(\bar{x}, 1) = s \cdot h(\bar{x}, 1) \cdot .
\]
such that $h(x^0, y^0) = s'$. Hence $s \cdot s' = s \cdot h(x^0, y^0) = s \cdot h(x^0, 1)^m h(1, y_0)^m$. But there exist $x, y$ such that $x'' \leq \hat{x} \leq 1$ and $y'' \leq \hat{y} \leq 1$ and

$$s \cdot h(x, 1)^m \cdot h(\hat{x}, 1)^m = s \cdot h(x, 1)^m \cdot h(l, y_0)^m = s \cdot h(1, \hat{y})^n \cdot h(1, y_0)^m.$$

That is $s \cdot U = s \cdot S = s \cdot V$.

**Lemma 11.** $h^{-1}(J) \cap (\{1\} \times I) = \{(1, 0)\}$ or $h^{-1}(J) \cap (I \times \{1\}) = \{(0, 1)\}$.

**Proof.** Suppose this is false. Then there exist $(x, 1), (1, y) \in h^{-1}(J)$ and $0 < x < 1$ and $0 < y < 1$. From the previous theorem, letting $h(x, 1)$ represent the element $s$, we obtain $x' \neq 0$ such that $h(x, 1) h(1, y) = h(x, 1) h(x', 1) = h(x x', 1)$. Also letting $h(1, y)$ represent the element $s$, we get $y' \neq 0$ such that $h(x, 1) h(1, y) = h(1, y') h(1, y) = h(1, yy')$. So $h(x x', 1) = h(1, yy')$. But this contradicts the assumption that $U \cap V = \{z, \emptyset\}$.

**Lemma 12.** If $(1, d) \in h^{-1}(J)^\circ$, then $(1, c) \in h^{-1}(J)$ for $0 \leq c \leq d$.

**Proof.** Let $(1, d) \in h^{-1}(J)^\circ$. One sees immediately that $\{(x, y) : 0 \leq x < 1, 0 \leq y < d\} \subset h^{-1}(J)$. Let $a, b \in (0, 1)$ be as in Lemma 7. For $0 < c < d$ we have $(1, c) = (1, b^c)$, and hence there exists $t$, such that $\{(a^t w, b w^{-t} w^t) : 0 < t < t\} \subset h^{-1}(J)$. From Lemma 9 there exists an $s \in S$ such that $h(a^t w, b w^{-t} w^t) = s$ for $t \in (0, t)$. Using the continuity of $h$ we get $\lim_{t \to 0} h(a^t w, b w^{-t} w^t) = h(1, b^c) = s$. That is $(1, c) \in h^{-1}(J)$. For $c = 0$, $h(1, c) = h(1, 0) = z$ which is always in $J$.

The same method of proof also shows that if $(d, 1) \in h^{-1}(J)^\circ$, then $(c, 1) \in h^{-1}(J)$ for $0 \leq c < d$.

**Corollary 13.** If $(x, 1), (1, y) \in h^{-1}(J)^\circ$, then $x = 0$ or $y = 0$.

Let $S$ be a commutative semigroup satisfying property $(\alpha)$. If $J \neq \{z\}$, then there exist $a, b \in (0, 1)$ which satisfies the conditions of Lemma 7. If $J = \{z\}$, let $a = 1/2, b = 1/2$. From Theorem A we see that $(I, \cdot) \times (I, \cdot)/R(a, b, h^{-1}(J)^\circ)$ is a commutative semigroup satisfying property $(\alpha)$. Moreover, the following theorem holds.

**Theorem B.** The semigroups $S$ and $(I, \cdot) \times (I, \cdot)/R(a, b, h^{-1}(J)^\circ)$ are isomorphic.

**Proof.** Consider the diagram

$$
\begin{array}{ccc}
(I, \cdot) \times (I, \cdot) & \xrightarrow{h} & S \\
\varphi \downarrow & & \downarrow h_{\varphi^{-1}} \\
(I, \cdot) \times (I, \cdot)/R(a, b, h^{-1}(J)^\circ) & & \end{array}
$$
when $h$ and $\varphi$ are the maps described earlier. We will show the relation $h\varphi^{-1}$ is an isomorphism. To prove this we need only show that for $(x, y) \in (I, \cdot) \times (I, \cdot)$, $\varphi^{-1}\varphi(x, y) = h^{-1}h(x, y)$. Let $(x, y) \in (I, \cdot) \times (I, \cdot)$. If $x = 0$ or $y = 0$ then $\varphi^{-1}\varphi(x, y) = ([0] \times I) \cup (I \times [0]) = h^{-1}h(x, y)$. Also if $\varphi^{-1}\varphi(x, y) = \{(x, y)\}$, then $h^{-1}h(x, y) = \{(x, y)\}$. Suppose $\varphi^{-1}\varphi(x, y)$ is not a point and $\varphi^{-1}\varphi(x, y)$ is the component of $(h^{-1}(J)^* \cap \{(a^{w_t}, b^{w-t w}): 0 \leq t \leq 1\})$ containing $(x, y) = \{(a^{w_t}, b^{w-t w}): t_1 \leq t \leq t_2, t_1 \neq t_2\}$. Hence $\{(c, d): c < a^{w_t}, d < b^{w-t w}, \text{for some } t \in [t_1, t_2]\} \subset h^{-1}(J)$. Let $w_n = w^{1+(1/n)}, n = 1, 2, 3, \ldots$, then $a^{w_{n t}} < a^{w_t}$ and $b^{w_{n-t w}} < b^{w-t w}$ for $t_1 \leq t \leq t_2$. This implies $\{(a^{w_{n t}}, b^{w_{n-t w}}): t_1 \leq t \leq t_2\} \subset h^{-1}(J)$. Using Lemma 9 we see

$$h(a^{w_{n t} t_1}, b^{w_{n-t w} t_1}) = h(a^{w_{n t}}, b^{w_{n-t w} t}) = h(a^{w_{n t} t_2}, b^{w_{n-t w} t_2})$$

for $t_1 \leq t \leq t_2$. Also $\lim h(a^{w_{n t}}, b^{w_{n-t w} t}) = h(a^{w_t}, b^{w-t w}) = h(x, y)$ for $t_1 \leq t \leq t_2$. And we have $h^{-1}h(x, y) = \varphi^{-1}\varphi(x, y)$. The induced homomorphism theorem implies $h\varphi^{-1}$ is an isomorphism.

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