

# Pacific Journal of Mathematics

**A MULTIPLIER THEOREM**

LOUIS PIGNO

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**Let  $G$  be a locally compact abelian group and  $\varphi$  a complex-valued function defined on the dual  $\Gamma$ . In this paper we prove that  $\varphi$  is a multiplier of type  $(L^1 \cap L^\infty, L^1 \cap C)$  if and only if  $\varphi = \hat{f}$  for some  $f \in L^1(G)$ .**

Throughout the paper  $M(G)$  denotes the measure algebra of the locally compact group  $G$ ,  $L^p(G)$  ( $1 \leq p \leq \infty$ ) the usual Lebesgue space of index  $p$  formed with respect to left Haar measure on  $G$ ,  $C(G)$  the set of all bounded continuous complex-valued functions on  $G$  and  $C_0(G)$ , the set of all  $f \in C(G)$  which vanish at infinity.

For a locally compact abelian group  $G$  with dual  $\Gamma$  the Fourier transform  $\hat{f}$  of a function  $f \in L^1(G)$  is defined by

$$\hat{f}(\gamma) = \int_G f(x) (-x, \gamma) dx \quad (\gamma \in \Gamma).$$

The Fourier-Stieltjes transform  $\hat{\mu}$  of a measure  $\mu \in M(G)$  is defined by

$$\hat{\mu}(\gamma) = \int_G (-x, \gamma) d\mu(x) \quad (\gamma \in \Gamma).$$

For  $y \in G$ , the translate  $f_y$  of the function  $f$  is defined by

$$f_y(t) = f(t - y) \quad (t \in G).$$

The translate  $\mu_y$  of the measure  $\mu \in M(G)$  is defined by

$$\mu_y(E) = \mu(E - y)$$

where  $E$  is any Borel set in  $G$ .

A complex-valued function  $\varphi$  defined on  $\Gamma$  is said to be a multiplier of type  $(L^1 \cap L^\infty, L^1 \cap C)$  if given  $f \in L^1(G) \cap L^\infty(G)$  there corresponds a  $g \in L^1(G) \cap C(G)$  such that  $\varphi \hat{f} = \hat{g}$ . The set of all multipliers of type  $(L^1 \cap L^\infty, L^1 \cap C)$  will be denoted by  $(L^1 \cap L^\infty, L^1 \cap C)$ . The multiplier problem  $(L^1 \cap L^\infty, L^1 \cap C)$  is then the determination of necessary and sufficient conditions which insure that  $\varphi \in (L^1 \cap L^\infty, L^1 \cap C)$ . The multiplier problems,  $(L^1 \cap L^\infty, L^1 \cap C_0)$ ,  $(L^1 \cap C_0, L^1)$ , etc., are defined similarly.

For the classical groups  $\mathbf{T}$  and  $\mathbf{R}$ , the multiplier problem  $(L^1 \cap L^\infty, L^1 \cap C_0)$  has been solved by Zygmund [9] and Doss [1, p. 191], respectively. The solution for  $G = \mathbf{T}$  has also been given by Verblunsky [8, p. 303]. Edwards [3, pp. 376-378] has solved the

problem for compact groups satisfying the first axiom of countability. Hewitt and Ross have recently solved the problem (to appear in [5]) for all compact groups. We prove for arbitrary *LCA* groups the following theorem :

**THEOREM 1.**  $(L^1 \cap L^\infty, L^1 \cap C) = (L^1 \cap L^\infty, L^1 \cap C_0) = L^1(G)^\wedge$ .

*Proof.* By  $L^1(G)^\wedge$  we mean the set of  $\hat{f}$  on  $\Gamma$  which are Fourier transforms of functions  $f \in L^1(G)$ . Suppose  $\varphi = \hat{f}$  for some  $f \in L^1(G)$ . If  $g \in L^1(G) \cap L^\infty(G)$  then, by [6, p. 4], the convolution  $f * g \in L^1(G) \cap C_0(G)$ . Thus  $\varphi \in (L^1 \cap L^\infty, L^1 \cap C)$  and  $(L^1 \cap L^\infty, L^1 \cap C_0)$ .

Next suppose  $\varphi \in (L^1 \cap L^\infty, L^1 \cap C)$ . We first show that  $\varphi = \hat{\mu}$  for some  $\mu \in M(G)$ . Assume temporarily that  $G$  is compact. Since  $\varphi \in (L^\infty, C)$  we have  $\varphi \in (L^\infty, L^\infty)$ . By a result of Edwards [3, p. 374]  $\varphi = \hat{\mu}$  for some  $\mu \in M(G)$ .

If  $G$  is a noncompact *LCA* group we proceed as follows.  $\varphi \in (L^1 \cap L^\infty, L^1 \cap C)$  implies  $\varphi \in (L^1 \cap C_0, L^1)$ . Doss [1, p. 189] has proved that, for  $G = \mathbf{R}$ ,  $\varphi \in (L^1 \cap C_0, L^1)$  if and only if  $\varphi = \hat{\mu}$  for some  $\mu \in M(\mathbf{R})$ . We have been able to generalize his proof to noncompact *LCA* groups, but the proof is rather lengthy. Frank Forelli has recently given a simple proof that  $(L^1 \cap C_0, L^1) = M(G)^\wedge$ . (See Theorem 3.2 of [4].)

So for  $f \in L^1(G) \cap L^\infty(G)$

$$g(x) = \int_G f(x - t) d\mu(t) \text{ a.e.}$$

where  $\hat{g} = \hat{\mu}\hat{f}$  and  $g \in L^1(G) \cap C(G)$ . We now show that  $\mu$  is absolutely continuous with respect to Haar measure. Let  $A$  be any relatively compact Borel subset of  $G$  and  $\psi$  the characteristic function of  $A$ . Then the convolution  $\psi * \mu$  is equal a.e. to a continuous function. Thus for each relatively compact Borel subset  $A$  of  $G$  the function

$$x \longrightarrow \mu(x + A)$$

is equal a.e. to a continuous function. This implies by the following theorem that  $d\mu(x) = f(x) dx$  for some  $f \in L^1(G)$  and hence concludes the proof.

**THEOREM 2.** *Let  $G$  be a locally compact group and  $\mu \in M(G)$  such that for each relatively compact Borel subset  $A$  of  $G$ , the function  $x \rightarrow \mu(x + A)$  is equal locally a.e. to a continuous function on  $G$ . Then  $d\mu(x) = f(x) dx$  for some  $f \in L^1(G)$ .*

Compare this with Theorem 2 of [2, p. 407], where  $\mu$  can be any Radon measure but where  $G$  is assumed to be a first countable *LCA*

group. In this connection see also 1.6 of [7, p. 230]. The proof of the present theorem may be obtained by simple modifications of the proof of Theorem (35.13) of [5], which we omit.

*Remark.* Let  $G$  be a noncompact LCA group. Since

$$(L^1 \cap C_o, L^1) = M(G)^\wedge$$

we have that

$$\begin{aligned} M(G)^\wedge &= (L^1 \cap C, L^1) \\ &= (L^1 \cap L^p, L^1) \\ &= (L^1 \cap L^p, L^1 \cap L^p) \\ &= (L^1 \cap C, L^1 \cap L^p) \\ &= (L^1 \cap C_o, L^1 \cap L^p) \\ &= (L^1 \cap C_o, L^1 \cap C_o) \\ &= (L^1 \cap C_o, L^1 \cap C) \\ &= (L^1 \cap C, L^1 \cap C) \end{aligned}$$

where  $M(G)^\wedge$  is the set of  $\hat{\mu}$  on  $\Gamma$  which are the Fourier-Stieltjes transforms of measures  $\mu \in M(G)$ . For infinite compact groups it is false that  $(C, L^1) = M(G)^\wedge$  since  $(L^2, L^2) = L^\infty(\Gamma)$ .

The author wishes to thank the referee for bringing to his attention theorem (35.13) of [5] and for the reference to [4]. Theorem (35.13) enabled the removal of the (unnecessary) hypothesis that  $G$  be first countable in Theorem 1.

#### REFERENCES

1. R. Doss, *On the multipliers of some classes of Fourier transforms*, Proc. London Math. Soc. (2) **50** (1949), 169-195.
2. R. E. Edwards, *Translates of  $L^\infty$  functions and of bounded measures*, J. Austr. Math. Soc. **4** (1964), 403-409.
3. ———, *On factor functions*, Pacific J. Math. **5** (1955), 367-378.
4. A. Figà-Talamanca and G. I. Gaudry, *Multipliers and sets of uniqueness of  $L^p$*  (to appear Michigan Math. J. **17** (1970))
5. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, Vol. II, Springer-Verlag, Heidelberg and New York, 1970.
6. W. Rudin, *Fourier analysis on groups*, Interscience, New York, 1962.
7. ———, *Measure algebras on abelian groups*, Bull. Amer. Math. Soc. **65** (1959), 227-247.
8. S. Verblunsky, *On some classes of Fourier series*, Proc. London Math. Soc. (2) **33** (1932), 287-327.
9. A. Zygmund, *Remarque sur un théorème de M. Fekete*, Bull. Acad. Polonaise Sci. Lett. (1927), 343-347.

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Richard Hindman Bouldin, <i>The perturbation of the singular spectrum</i> . . . . .	569
Hugh D. Brunk and Søren Glud Johansen, <i>A generalized Radon-Nikodym derivative</i> . . . . .	585
Henry Werner Davis, F. J. Murray and J. K. Weber, <i>Families of <math>L_p</math>-spaces with inductive and projective topologies</i> . . . . .	619
Esmond Ernest Devun, <i>Special semigroups on the two-cell</i> . . . . .	639
Murray Eisenberg and James Howard Hedlund, <i>Expansive automorphisms of Banach spaces</i> . . . . .	647
Frances F. Gulick, <i>Actions of functions in Banach algebras</i> . . . . .	657
Douglas Harris, <i>Regular-closed spaces and proximities</i> . . . . .	675
Norman Lloyd Johnson, <i>Derivable semi-translation planes</i> . . . . .	687
Donald E. Knuth, <i>Permutations, matrices, and generalized Young tableaux</i> . . . . .	709
Herbert Frederick Kreimer, Jr., <i>On the Galois theory of separable algebras</i> . . . . .	729
You-Feng Lin and David Alon Rose, <i>Ascoli's theorem for spaces of multifunctions</i> . . . . .	741
David London, <i>Rearrangement inequalities involving convex functions</i> . . . . .	749
Louis Pigno, <i>A multiplier theorem</i> . . . . .	755
Helga Schirmer, <i>Coincidences and fixed points of multifunctions into trees</i> . . . . .	759
Richard A. Scoville, <i>Some measure algebras on the integers</i> . . . . .	769
Ralph Edwin Showalter, <i>Local regularity of solutions of Sobolev-Galpern partial differential equations</i> . . . . .	781
Allan John Sieradski, <i>Twisted self-homotopy equivalences</i> . . . . .	789
John H. Smith, <i>On <math>S</math>-units almost generated by <math>S</math>-units of subfields</i> . . . . .	803
Masamichi Takesaki, <i>Algebraic equivalence of locally normal representations</i> . . . . .	807
Joseph Earl Valentine, <i>An analogue of Ptolemy's theorem and its converse in hyperbolic geometry</i> . . . . .	817
David Lawrence Winter, <i>Solvability of certain <math>p</math>-solvable linear groups of finite order</i> . . . . .	827