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**LOCAL REGULARITY OF SOLUTIONS OF
SOBOLEV-GALPERN PARTIAL DIFFERENTIAL EQUATIONS**

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R. E. SHOWALTER

Let M and L be elliptic differential operators of orders $2m$ and 2ℓ , respectively, with $m \leq \ell$. The existence and uniqueness of a solution to the abstract mixed initial and boundary value problem

$$Mu'(t) + Lu(t) = 0, \quad u(0) = u_0$$

was established for u_0 given in the domain of the infinitesimal generator of a strongly-continuous semi-group. The purpose of this paper is to show that this semi-group is holomorphic and then obtain differentiability results for the solution and convergence of this solution to the initial function u_0 as $t \downarrow 0$.

Let G be a bounded open domain of R^n whose boundary ∂G is an $(n - 1)$ -dimensional manifold with G lying on one side of ∂G . $H^k = H^k(G)$ is the Hilbert space (of equivalence classes) of functions whose distributional derivatives through order k belong to $L^2(G)$ with the usual inner-product and norm,

$$(f, g)_k = \sum \left\{ \int_G D^\alpha f \overline{D^\alpha g} dx : |\alpha| \leq k \right\}$$

and

$$\|f\|_k = \sqrt{(f, f)_k}.$$

$H_0^k = H_0^k(G)$ is the closure in H^k of $C_0^\infty(G)$, the space of infinitely differentiable functions with compact support in G .

We specify the problem by means of the bilinear forms

$$B_M(\phi, \psi) = \sum \{(m^{\rho\sigma} D^\rho \phi, D^\sigma \psi)_0 : |\rho|, |\sigma| \leq m\}$$

and

$$B_L(\phi, \psi) = \sum \{(l^{\rho\sigma} D^\rho \phi, D^\sigma \psi)_0 : |\rho|, |\sigma| \leq l\},$$

defined initially for ϕ and ψ in $C_0^\infty(G)$. Furthermore, we require the following:

- P_1 : The coefficients $m^{\rho\sigma}, l^{\rho\sigma}$ are bounded and measurable.
- P_2 : $\operatorname{Re} B_M(\phi, \phi) \geq k_m \|\phi\|_m^2, k_m > 0$
 $\operatorname{Re} B_L(\phi, \phi) \geq k_l \|\phi\|_l^2, k_l > 0$
 for all ϕ in $C_0^\infty(G)$.
- P_3 : M is symmetric; that is $m^{\rho\sigma} = \overline{m^{\sigma\rho}}$ for all ρ, σ , (hence $B_M(\phi, \phi)$ is real for all ϕ in C_0^∞).

From the assumptions P_1 and P_2 and the general theory of elliptic operators, [1, 6, 7, 11, 12, 13], there are two operators, M_0 and L_0 , which are topological isomorphisms of H_0^m onto $H^{-m} = (H_0^m)'$ and H_0^l onto $H^{-l} = (H_0^l)'$ (where “'” denotes the continuous linear dual), and these are determined by the respective identities

$$B_M(\phi, \psi) = \langle M_0\phi, \bar{\psi} \rangle$$

and

$$B_L(\phi, \psi) = \langle L_0\phi, \bar{\psi} \rangle$$

on H_0^m and H_0^l , respectively, where “ \langle, \rangle ” denotes $\mathcal{D} - \mathcal{D}'$ duality, \mathcal{D}' being the space of distributions over G .

Since $l \geq m$ we have a topological inclusion $H_0^l \subset H_0^m$, hence, by duality, $H^{-m} \subset H^{-l}$. Thus the mapping $L_0^{-1}M_0$ is continuous from H_0^m into H_0^l and is a topological isomorphism only if $l = m$. Letting $D = L_0^{-1}M_0(H_0^m) \equiv L_0^{-1}(H^{-m})$, we have an unbounded operator $A = M_0^{-1}L_0$ on H_0^m with domain D dense in H_0^l . In [16] we showed that A is the infinitesimal generator of an equicontinuous semi-group of bounded operators [6, 9, 11] on H_0^m , denoted by $\{S(t): t \geq 0\}$. We shall prove that this semi-group is holomorphic.

We have already shown that the nonnegative real axis belongs to the resolvent set of A and, in fact,

$$(1) \quad |R(\lambda, A)|_M = |(\lambda - A)^{-1}|_M \leq (\operatorname{Re}(\lambda))^{-1}$$

for all real $\lambda \geq 0$, where the norm $|\cdot|_M$ defined by

$$|\phi|_M = \sqrt{B_M(\phi, \phi)}$$

on H_0^m is equivalent to $\|\cdot\|_m$ by P_1 and P_2 . Actually the whole right half of the complex plane belongs to the resolvent set of A , and (1) is true there. This can be shown by noting that for $\lambda = \sigma + i\tau$ we have

$$B_M((A - \lambda)\phi, \phi) = B_M((A - \sigma)\phi, \phi) - i\tau B_M(\phi, \phi)$$

and hence

$$\operatorname{Re} B_M((A - \lambda)\phi, \phi) = \operatorname{Re} B_M((A - \sigma)\phi, \phi)$$

in the argument leading to (1) for λ real. See [16] for details.

2. Our goal is to improve the estimate (1) to show that the family $\{\lambda R(\lambda, A)\}$ is uniformly bounded in $\mathcal{L}(H_0^m)$ for $\operatorname{Re}(\lambda) > 0$. First let ϕ be in D ; then

$$B_M((\lambda - A)\phi, \phi) = (\sigma + i\tau)B_M(\phi, \phi) + B_L(\phi, \phi).$$

Since M is symmetric it follows that $B_M(\phi, \phi)$ is real, so we obtain

$$(2) \quad \operatorname{Re} B_M((\lambda - A)\phi, \phi) = \sigma B_M(\phi, \phi) + \operatorname{Re} B_L(\phi, \phi) \geq k_l \|\phi\|_i^2,$$

since $\sigma > 0$. Similarly, from

$$\operatorname{Im} B_M((\lambda - A)\phi, \phi) = \tau B_M(\phi, \phi) + \operatorname{Im} B_L(\phi, \phi)$$

we obtain the estimate

$$(3) \quad |\operatorname{Im} B_M((\lambda - A)\phi, \phi)| \geq |\tau| |\phi|_M^2 - K_l \|\phi\|_i^2.$$

From (2) and (3) we conclude that either

$$(4) \quad |\operatorname{Im} B_M((\lambda - A)\phi, \phi)| \geq \frac{|\tau|}{2} |\phi|_M^2$$

or

$$(5) \quad |\operatorname{Re} B_M((\lambda - A)\phi, \phi)| \geq \frac{k_l}{2K_l} |\tau| |\phi|_M^2,$$

for if (4) is not true then by (3)

$$|\tau| |\phi|_M^2 - K_l \|\phi\|_i^2 \leq \frac{|\tau|}{2} |\phi|_M^2,$$

hence

$$\frac{|\tau|}{2} |\phi|_M^2 \leq K_l \|\phi\|_i^2,$$

which with (2) implies (5). From (4) and (5) we obtain the estimate

$$(6) \quad |B_M((\lambda - A)\phi, \phi)| \geq \frac{k_l}{2K_l} |\tau| |\phi|_M^2$$

for all ϕ in D , and this in turn yields

$$(7) \quad |R(\lambda, A)|_M \leq \frac{2K_l}{k_l} \frac{1}{|\tau|},$$

whenever $\operatorname{Re}(\lambda) > 0$. The calculation is as follows:

$$\frac{k_l}{2K_l} |\tau| |\phi|_M^2 \leq |B_M((\lambda - A)\phi, \phi)| \leq |(\lambda - A)\phi|_M |\phi|_M$$

implies

$$|(\lambda - A)\phi|_M \geq |\tau| \frac{k_l}{2K_l} |\phi|_M$$

for all ϕ in D , the domain of A , so (7) follows. The estimates (1) and (7) imply that

$$|\lambda R(\lambda, A)|_M \leq \frac{|\tau|}{\sigma} + 1$$

when $\sigma > 0$ and, respectively, that

$$|\lambda R(\lambda, A)|_M \leq \frac{2K_l}{k_l} \left(\frac{\sigma}{|\tau|} + 1 \right)$$

whenever $|\tau| \neq 0$, where $\lambda = \sigma + i\tau$. By considering the two cases, $|\tau| \geq \sigma$ and $|\tau| < \sigma$, we obtain, finally,

$$(8) \quad |\lambda R(\lambda, A)|_M \leq \frac{4K_l}{k_l}$$

for all λ in the right half of the complex plane. The estimate (8) yields the following result.

PROPOSITION [22]. *The semi-group $\{S(t): t \geq 0\}$ has a holomorphic extension into a sector of the complex plane. Furthermore, $S(t)$ maps H_0^m into D whenever $t > 0$, so $S(t)$ is infinitely differentiable and $S^{(p)}(t) = A^p S(t)$ for any integer $p \geq 1$.*

The significance of this result for our problem is that, for each $t > 0$, $S(t)$ maps H_0^m into the domain of A^p for an arbitrary integer $p \geq 1$.

3. The differentiability of the semi-group yields differentiability of the solution to the problem being considered; the latter is obtained by means of the following.

Let H_{loc}^k denote those (equivalence classes of) functions on G which are locally in H^k ; that is,

$$H_{loc}^k = \{f: f \in H^k(K) \text{ for each compact subset } K \text{ of } G\}.$$

The following result on the local regularity of solutions of elliptic equations is well known.

THEOREM [1, 4, 5, 7, 12, 13, 14]. *Let p be an integer $\geq -l$ for which $l^{\rho\sigma}$ is $\max\{1, |\rho| + p\}$ times continuously differentiable in G whenever $|\rho|$ and $|\sigma|$ are $\leq l$. If u belongs to H_0^l , and if $L_0 u$ is in H_{loc}^p , then u belongs to H_{loc}^{2l+p} . That is, L_0 is a topological isomorphism of $H_0^l \cap H_{loc}^{2l+p}$ onto $H^{-l} \cap H_{loc}^p$.*

Let k be a nonnegative integer and assume that we have

$P(k)$: $m^{\rho\sigma}$ and $l^{\rho\sigma}$ are $\max\{1, |\rho| - m + k\}$ times continuously differentiable in G .

From the above theorem it follows that M_0 is a bijection of $H_0^m \cap H_{loc}^{m+k}$ onto $H^{-m} \cap H_{loc}^{k-m}$. Also L_0^{-1} is a bijection of $H^{-l} \cap H_{loc}^{k-m}$ onto $H_0^l \cap H_{loc}^{2l-m+k}$. Since $H^{-m} \subset H^{-l}$, it follows that $A^{-1} = -L_0^{-1}M_0$ maps $H_0^m \cap H_{loc}^{m+k}$ into $H_0^l \cap H_{loc}^{2l-m+k}$.

COROLLARY. $P(2(p - 1)(l - m))$ implies that the domain of A^p is contained in $H_0^l \cap H_{loc}^{m+2p(l-m)}$ for $p \geq 1$.

From § 2 we know that $u(t)$ is in the domain of A^p for all $t > 0$ and $p > 1$. The corollary thus yields the following results.

THEOREM. Assume P_1, P_2 and P_3 of § 2. Let the coefficients in M and L satisfy $P(2(p - 1)(l - m))$ for some integer $p \geq 1$. Then $u(t) = S(t)u_0$ belongs to $H_0^l \cap H_{loc}^{m+2p(l-m)}$ for each $t > 0$, where u_0 is any element of H_0^m .

If p is sufficiently large we obtain pointwise-solutions by Sobolev's Lemma [17]:

If m is an integer $> (n/2)$, then H_{loc}^m is imbedded in $C^j(G)$, $j = m - [n/2] - 1$, and the injection is continuous when the range space is given the topology of uniform convergence in all derivatives of order $\leq j$ on compact of subsets of G .

COROLLARY. Assume the hypotheses of the above theorem hold with $m + 2p(l - m) - [n/2] - 1 = j \geq 0$. Then, for $t > 0$, $u(t)$ has j continuous derivatives in G and, for each point x in G , the function $t \rightarrow u(x, t)$ is infinitely differentiable.

Proof. Choose t' such that $t > t' > 0$. Since $u(t) = S(t)u_0$ belongs to $D(A^p)$, the semi-group property yields

$$\delta^{-1}[u(t + \delta) - u(t)] = A^{-p}\delta^{-1}[S(t + \delta - t') - S(t - t')]A^p u(t')$$

for δ sufficiently small. Since $A^p u(t')$ belongs to $D = D(A)$, the function to the right of A^{-p} has a limit in H_0^m as $\delta \rightarrow 0$, so the function $\delta^{-1}[u(t + \delta) - u(t)]$ has a limit in $H^{m+2p(l-m)}(K)$, where K is any compact subset of G . By Sobolev's Lemma, the function

$$\delta \rightarrow \delta^{-1}[u(x, t + \delta) - u(x, t)]$$

has a limit as $\delta \rightarrow 0$, so $u(x, t)$ is differentiable. A repetition of this argument shows that $u(x, t)$ is infinitely differentiable in t without any further assumptions on the coefficients.

All of the above results have been obtained for a solution with initial value u_0 in H_0^m . We note further that if u_0 is sufficiently

smooth then $u(t) \rightarrow u_0$ pointwise. (It is always true that $u(t) \rightarrow u_0$ in H_0^m .)

COROLLARY. *Assume the hypotheses of the above corollary and that u_0 belongs to the domain of A^p . Then each $u(t)$, $t \geq 0$ is a continuous function on G , and for each point x in G , $u(x, t) \rightarrow u_0(x) = u(x, 0)$ as $t \rightarrow 0$.*

Proof. This follows by an argument similar to the proof of the preceding corollary applied to the equation

$$u(t) - u_0 = A^{-p}(S(t) - I)(A^p u_0) .$$

We note that a sufficient condition for u_0 to be in $D = D(A)$ is that u_0 be in $H_0^l \cap H^{2l-m}$. Also if the initial function and all coefficients in M and L are infinitely differentiable, then the solution is infinitely differentiable.

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