ON $S$-UNITS ALMOST GENERATED BY $S$-UNITS OF SUBFIELDS

JOHN H. SMITH
ON S-UNITS ALMOST GENERATED BY S-UNITS OF SUBFIELDS

JOHN H. SMITH

Let $K/k$ be a finite galois extension of number fields, $S$ a finite set of primes of $K$, and $\Phi$ a set of intermediate fields. We assume that $S$ and $\Phi$ are closed under the action of $G(K/k)$ and that $S$ contains all the archimedean primes. This paper determines conditions under which the $S$-units of fields of $\Phi$ “almost generate” those of $K$ (i.e., generate a subgroup of finite index).

Let $U$ be the $S$-units of $K$ and $U'$ the subgroup generated by $S$-units of fields in $\Phi$. For any subgroup, $H$, of $G$, let $\chi_H$ be the character of $G$ induced by the trivial character on $H$ and let $M_H$ be the corresponding $C[G]$-module.

**Theorem 1.** $U/U'$ is finite if and only if every irreducible $C[G]$-module, $M$, which occurs in some $M_{H(\psi)}$, $\psi \in S$ also occurs in some $M_{J(F)}$, $F \in \Phi$. (Here $H(\psi)$ denotes the splitting group of the prime $\psi$ and $J(F)$ the group of automorphisms fixing the elements of $F$).

**Proof.** $U/U'$ is finite if and only if $U \otimes C = U' \otimes C$. But we know the structure of $U \otimes C$ (see, e.g., p. 10 of [1]). If $\theta$ is the sum, over all conjugacy classes of primes of $S$, of $\chi_{H(\psi)}$, and $N = U \otimes C$ then the character of $N$ is $\theta - \chi_{G}$. Hence, except for components with character $\chi_{G}$, the components of $N$ are those and only those which occur in some $M_{H(\psi)}$, $\psi \in S$.

Now $U'$ is generated by those elements which are invariant under some $J(F)$, $F \in \Phi$. So $U/U'$ is finite if and only if $N$ is generated by such elements, which of course is the case if and only if each irreducible component is so generated. Such a component, $N'$, is so generated if and only if it has a nontrivial element fixed by some $J(F)$. By Frobenius reciprocity this is equivalent to saying that $N'$ occurs in some $M_{J(F)}$.

**Corollary 1.** If for every $\psi \in S$ there is an $F \in \Phi$ such that $\psi$ does not split at all from $F$ to $K$ then $U/U'$ is finite.

**Proof.** In this case each $H(\psi)$ contains some $J(F)$.

2. In this section we suppose that every irreducible character
of $G$ occurs in some $M_{H(\beta)}$, $\beta \in S$ (for example, if $k$ has a complex prime or $K$ has a real one.)

**Corollary 2.** Let $\Phi$ be the set of all proper subextensions. Then $U/U'$ is infinite if and only if $G$ admits a fixed point free (complex) representation (i.e., one in which only the identity has eigenvalue 1).

*Proof.* Clear.

**Remark.** Groups admitting such a representation are fairly special, the only familiar ones being the cyclic groups, certain metacyclic groups, and $\text{SL}(2, 5)$. A complete classification is given in [3].

**Corollary 3.** Let $G$ be abelian and let $\Phi$ consist of cyclic subextensions. Then $U/U'$ is finite if and only if every maximal cyclic subextension belongs to $\Phi$.

*Proof.* Clear.

**Theorem 2.** The $S$-units if $K$ of degree $\leq m$ over $k$ generate a subgroup of finite index in $U$ if and only if every irreducible (complex) representation of $G$ factors through a transitive permutation representation on at most $m$ symbols.

*Proof.* We let $\Phi$ be the set of all intermediate fields of degree $m$. Then the first condition is equivalent to the finiteness of $U/U'$, which is equivalent to the occurrence of each irreducible representation of $G$ in some $M_{J(F)}$, $F \in \Phi$.

Now the representation afforded by $M_{J(F)}$ factors through the action of $G$ on cosets of $J(F)$ by translation, hence any component of it factors through permutations on $[G: J(F)] = [F: k] \leq m$ elements. Conversely if a representation $\varphi$ factors through a transitive representation on a set $\Omega$, $|\Omega| \leq m$, let $J \subset G$ be the stabilizer of a point of $\Omega$. Then the action on $\Omega$ is equivalent to translation of the cosets of $J$, which gives rise to the character $\chi_j$. Since $\varphi$, restricted to $J$, has a fixed point, $\varphi$ occurs in $\chi_j$ by Frobenius reciprocity. Clearly the field $F$, corresponding to $J$, has degree $[G: J] \leq m$.

**Remark.** For $m = 2, 3, 4$ (but not higher) the above condition is easily seen to be equivalent to the assertion that every irreducible representation factors through $S_m$.

Since explicit algorithms are available for finding units (in fact fundamental units) in quadratic and cubic extensions of $Q$ (see [2]) we mention the following example.
THEOREM 3. The S-units of degrees 2 and 3 over k "almost generate" the S-units of K if and only if G is of one of the following forms:

1. G abelian of exponent 2 or 3
2. G has an abelian subgroup, A of exponent 3 such that A is of index 2 and G/A acts on A by inversion.

Proof. It is easy to check that all irreducible representations of groups of the above forms factor through $S_3$. Conversely suppose all the irreducible representations of G factor through $S_3$. Then the same is true of quotients of G, and, by Frobenius reciprocity, of subgroups. In particular all elements are of orders 1, 2 or 3. This takes care of the abelian case.

If G is not abelian, let $\varphi$ be any irreducible representation of $G'$. If $\psi$ is an irreducible component of the induced representation of G then the restriction of $\psi$ to $G'$ contains $\varphi$ by Frobenius reciprocity. Since $\psi$ factors through $S_3$, $\varphi$ factors through $S_3$. Hence $G'$ is abelian of exponent 3. Since $G/G'$ is abelian it is abelian of exponent 2 or 3. If it is a 3-group, so is G, but then, since all irreducible representations of G factor through the 3-Sylow subgroup of $S_3$, G itself would be abelian. Hence $G/G'$ is of exponent 2. The action of $G/G'$ on $G'$ gives an ordinary representation of $G/G'$, which can be diagonalized. If $G/G'$ had more than one generator some element of order 2 would commute with an element of order 3, giving an element of order 6 which is impossible. Hence $G/G' \cong \mathbb{Z}/2\mathbb{Z}$ and the action on $G'$ is inversion.

REFERENCES

1. E. Artin and J. Tate, Class field theory, Benjamin, 1968.

Received February 10, 1969.

BOSTON COLLEGE
Richard Hindman Bouldin, *The perturbation of the singular spectrum* ........ 569
Hugh D. Brunk and Søren Glud Johansen, *A generalized Radon-Nikodym derivative* ......................................................... 585
Esmond Ernest Devun, *Special semigroups on the two-cell* ............. 639
Murray Eisenberg and James Howard Hedlund, *Expansive automorphisms of Banach spaces* ................................................. 647
Frances F. Gulick, *Actions of functions in Banach algebras* .............. 657
Douglas Harris, *Regular-closed spaces and proximities* .................... 675
Norman Lloyd Johnson, *Derivable semi-translation planes* ............... 687
Donald E. Knuth, *Permutations, matrices, and generalized Young tableaux* ................................................................. 709
Herbert Frederick Kreimer, Jr., *On the Galois theory of separable algebras* ................................................................. 729
You-Feng Lin and David Alon Rose, *Ascoli’s theorem for spaces of multifunctions* ................................................................. 741
David London, *Rearrangement inequalities involving convex functions* .... 749
Louis Pigno, *A multiplier theorem* ............................................. 755
Helga Schirmer, *Coincidences and fixed points of multifunctions into trees* ................................................................. 759
Richard A. Scoville, *Some measure algebras on the integers* .............. 769
Ralph Edwin Showalter, *Local regularity of solutions of Sobolev-Galpern partial differential equations* ...................................... 781
Allan John Sieradski, *Twisted self-homotopy equivalences* ................. 789
John H. Smith, *On $S$-units almost generated by $S$-units of subfields* .... 803
Masamichi Takesaki, *Algebraic equivalence of locally normal representations* ............................................................... 807
Joseph Earl Valentine, *An analogue of Ptolemy’s theorem and its converse in hyperbolic geometry* ...................................... 817
David Lawrence Winter, *Solvability of certain $p$-solvable linear groups of finite order* ............................................................ 827