Pacific Journal of Mathematics

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Vol. 34, No. 3 July 1970

PACIFIC JOURNAL OF MATHEMATICS
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Let K/k be a finite galois extension of number fields, S a finite set of primes of K, and Φ a set of intermediate fields. We assume that S and Φ are closed under the action of $G(K/k)$ and that S contains all the archimedean primes. This paper determines conditions under which the S-units of fields of Φ "almost generate" those of K (i.e., generate a subgroup of finite index).

Let U be the S-units of K and U' the subgroup generated by S-units of fields in Φ. For any subgroup, H, of G, let χ_H be the character of G induced by the trivial character on H and let M_H be the corresponding $\boldsymbol{C}[G]$-module.

THEOREM 1. *U/U' is finite if and only if every irreducible $\boldsymbol{C}[G]$-module, M, which occurs in some $M_{H(\mathfrak{P})}$, $\mathfrak{P} \in S$ also occurs in some $M_{J(F)}$, $F \in \Phi$. (Here $H(\mathfrak{P})$ denotes the splitting group of the prime $\mathfrak{P}$ and $J(F)$ the group of automorphisms fixing the elements of F).*

Proof. U/U' is finite if and only if $U \otimes \boldsymbol{C} = U' \otimes \boldsymbol{C}$. But we know the structure of $U \otimes \boldsymbol{C}$ (see, e.g., p. 10 of [1]). If θ is the sum, over all conjugacy classes of primes of S, of $\chi_{H(\mathfrak{P})}$, and $N = U \otimes \boldsymbol{C}$ then the character of N is $\theta - \chi_G$. Hence, except for components with character χ_G, the components of N are those and only those which occur in some $M_{H(\mathfrak{P})}$, $\mathfrak{P} \in S$.

Now U' is generated by those elements which are invariant under some $J(F)$, $F \in \Phi$. So U/U' is finite if and only if N is generated by such elements, which of course is the case if and only if each irreducible component is so generated. Such a component, N', is so generated if and only if it has a nontrivial element fixed by some $J(F)$. By Frobenius reciprocity this is equivalent to saying that N' occurs in some $M_{J(F)}$.

COROLLARY 1. *If for every $\mathfrak{P} \in S$ there is an $F \in \Phi$ such that $\mathfrak{P}$ does not split at all from F to K then U/U' is finite.*

Proof. In this case each $H(\mathfrak{P})$ contains some $J(F)$.

2. In this section we suppose that every irreducible character

of G occurs in some $M_{H(\mathfrak{P})}$, $\mathfrak{P} \in S$ (for example, if k has a complex prime or K has a real one.)

COROLLARY 2. *Let Φ be the set of all proper subextensions. Then U/U' is infinite if and only if G admits a fixed point free (complex) representation (i.e., one in which only the identity has eigenvalue* 1).

Proof. Clear.

REMARK. Groups admitting such a representation are fairly special, the only familiar ones being the cyclic groups, certain metacyclic groups, and SL(2, 5). A complete classification is given in [3].

COROLLARY 3. *Let G be abelian and let Φ consist of cyclic subextensions. Then U/U' is finite if and only if every maximal cyclic subextension belongs to Φ.*

Proof. Clear.

THEOREM 2. *The S-units if K of degree $\leqq m$ over k generate a subgroup of finite index in U if and only if every irreducible (complex) representation of G factors through a transitive permutation representation on at most m symbols.*

Proof. We let Φ be the set of all intermediate fields of degree m. Then the first condition is equivalent to the finiteness of U/U', which is equivalent to the occurrence of each irreducible representation of G in some $M_{J(F)}$, $F \in \Phi$.

Now the representation afforded by $M_{J(F)}$ factors through the action of G on cosets of $J(F)$ by translation, hence any component of it factors through permutations on $[G\colon J(F)] = [F\colon k] \leqq m$ elements. Conversely if a representation φ factors through a transitive representation on a set Ω, $|\Omega| \leqq m$, let $J \subset G$ be the stabilizer of a point of Ω. Then the action on Ω is equivalent to translation of the cosets of J, which gives rise to the character χ_J. Since φ, restricted to J, has a fixed point, φ occurs in χ_J by Frobenius reciprocity. Clearly the field F, corresponding to J, has degree $[G\colon J] \leqq m$.

REMARK. For $m = 2, 3, 4$ (but not higher) the above condition is easily seen to be equivalent to the assertion that every irreducible representation factors through S_m.

Since explicit algorithms are available for finding units (in fact fundamental units) in quadratic and cubic extensions of Q (see [2]) we mention the following example.

THEOREM 3. *The S-units of degrees 2 and 3 over k "almost generate" the S-units of K if and only if G is of one of the following forms*:

(1) *G abelian of exponent 2 or 3*

(2) *G has an abelian subgroup, A of exponent 3 such that A is of index 2 and G/A acts on A by inversion.*

Proof. It is easy to check that all irreducible representations of groups of the above forms factor through S_3. Conversely suppose all the irreducible representations of G factor through S_3. Then the same is true of quotients of G, and, by Frobenius reciprocity, of subgroups. In particular all elements are of orders 1, 2 or 3. This takes care of the abelian case.

If G is not abelian, let φ be any irreducible representation of G'. If ψ is an irreducible component of the induced representation of G then the restriction of ψ to G' contains φ by Frobenius reciprocity. Since ψ factors through S_3, φ factors through S_3'. Hence G' is abelian of exponent 3. Since G/G' is abelian it is abelian of exponent 2 or 3. If it is a 3-group, so is G, but then, since all irreducible representations of G factor through the 3-Sylow subgroup of S_3, G itself would be abelian. Hence G/G' is of exponent 2. The action of G/G' on G' gives an ordinary representation of G/G', which can be diagonalized. If G/G' had more than one generator some element of order 2 would commute with an element of order 3, giving an element of order 6 which is impossible. Hence $G/G' \cong Z/2Z$ and the action on G' is inversion.

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Received February 10, 1969.

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The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

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