ALGEBRAIC EQUIVALENCE OF LOCALLY NORMAL REPRESENTATIONS

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It will be shown that (i) the absolute value of every locally normal linear functional is again locally normal; (ii) two locally normal representations π₁ and π₂ of \( \mathcal{A} \) generate isomorphic von Neumann algebras \( \mathcal{M}(\pi₁) \) and \( \mathcal{M}(\pi₂) \) if and only if there exists an automorphism \( \sigma \) of \( \mathcal{A} \) such that \( \pi₁ \circ \sigma \) and \( \pi₂ \) are quasi-equivalent, provided that either \( \mathcal{M}(\pi₁) \) or \( \mathcal{M}(\pi₂) \) is \( \sigma \)-finite.

This paper is motivated by a recent work [6] of R. Haag, R. V. Kadison and D. Kastler. As they mentioned, the recent progress in mathematical physics has made a precise analysis of representations of a \( C^* \)-algebra furnished with a net of von Neumann algebras a growing necessity.

In the first half of this paper, we shall show that the space of all locally normal linear functionals of a \( C^* \)-algebra with a net of von Neumann algebras is a closed invariant subspace of the conjugate space in the sense of [14], which will imply that the absolute value of a locally normal linear functional is locally normal too.

The last half of this paper will be devoted to extending a result of Powers [11] for UHF algebra to a \( C^* \)-algebra \( \mathcal{A} \) with a proper sequential type \( I_{\infty} \) funnel. Namely it will be shown that two locally normal representations \( \pi₁ \) and \( \pi₂ \) of the \( C^* \)-algebra \( \mathcal{A} \) generate isomorphic von Neumann algebras if and only if they are connected by an automorphism of \( \mathcal{A} \). This is proven under the assumption that one of the generated von Neumann algebras is \( \sigma \)-finite.

2. The locally normal conjugate space of a \( C^* \)-algebra with a net of von Neumann algebras. Let \( \mathcal{A} \) be a \( C^* \)-algebra. Suppose a system \( \mathcal{F} = (\mathcal{A}_\alpha) \) of \( C^* \)-subalgebras of \( \mathcal{A} \) indexed by a directed set \( \{\alpha\} \) is given such that:
   (i) \( \mathcal{A}_\alpha \) is a von Neumann subalgebra of \( \mathcal{A} \) if \( \alpha \leq \beta \);
   (ii) \( \bigcup_\alpha \mathcal{A}_\alpha \) is dense in \( \mathcal{A} \) with respect to the norm topology.

The system \( \mathcal{F} = \{\mathcal{A}_\alpha\} \) is called a net (in \( \mathcal{A} \)) of von Neumann algebras and each \( \mathcal{A}_\alpha \) is called local subalgebra of \( \mathcal{A} \).

DEFINITION 1. A continuous linear functional \( \varphi \) (resp. representation \( \pi \)) of \( \mathcal{A} \) is said to be locally normal if \( \varphi \) (resp. \( \pi \)) is \( \sigma \)-weakly continuous on each local subalgebra \( \mathcal{A}_\alpha \).
PROPOSITION 2. Let $V$ be the set of all locally normal linear functionals on a C*-algebra $\mathcal{A}$ with a net $\mathcal{N} = \{\mathcal{A}_\alpha\}$ of von Neumann algebras. Then $V$ is a closed, invariant subspace of $\mathcal{A}^*$. Namely, if $\varphi \in \mathcal{A}^*$ is locally normal, then $a\varphi$ and $\varphi a$, $a \in \mathcal{A}$, are both locally normal, where $a\varphi$ and $\varphi a$ defined by $a\varphi(x) = \varphi(ax)$ and $\varphi a(x) = \varphi(ax)$, $x \in \mathcal{A}$.

Therefore, there exists a unique central projection $z_0$ of the universal enveloping von Neumann algebra $\mathcal{A}$ of $\mathcal{A}$, the second conjugate space of $\mathcal{A}$ as a Banach space, such that

$$z_0\mathcal{A}^* = V.$$  

Proof. Let $\{\varphi_n\}$ be a sequence in $V$ converging to $\varphi \in \mathcal{A}^*$ with respect to the norm topology. For each $\alpha$, we have

$$||\varphi|_{\mathcal{A}_\alpha} - \varphi_n|_{\mathcal{A}_\alpha}|| \leq ||\varphi - \varphi_n|| \to 0$$

as $n \to \infty$; hence $\{\varphi_n|_{\mathcal{A}_\alpha}\}$ converges to $\varphi|_{\mathcal{A}_\alpha}$. Since the predual $\mathcal{A}_\alpha^*$ of each $\mathcal{A}_\alpha$ is complete, $\varphi|_{\mathcal{A}_\alpha}$ belongs to $\mathcal{A}_\alpha$, so that $\varphi$ is locally normal. Hence $V$ is closed.

Take an arbitrary element $\varphi \in V$. Let $a$ be an element of $\mathcal{A}_\alpha$. For each $\beta$, there exists an index $\gamma$ such that $\alpha \leq \gamma, \beta \leq \gamma$. Since $\varphi|_{\mathcal{A}_\gamma}$ is normal and $a$ is in $\mathcal{A}_{\mathcal{A}_\gamma}$, $a\varphi|_{\mathcal{A}_\gamma}$ is normal, so that

$$a\varphi|_{\mathcal{A}_\beta} = (a\varphi|_{\mathcal{A}_\gamma})|_{\mathcal{A}_\beta}$$

is normal. Hence $a\varphi$ belongs to $V$. Therefore, if $a$ belongs to $\bigcup \mathcal{A}_\alpha$, then $a\varphi$ is locally normal. If $a$ is an arbitrary element of $\mathcal{A}$, then there exists a sequence $\{a_n\}$ in $\bigcup \mathcal{A}_\alpha$ such that

$$\lim_{n \to \infty} ||a - a_n|| = 0;$$

hence

$$\lim_{n \to \infty} ||a\varphi - a_n\varphi|| \leq \lim_{n \to \infty} ||a - a_n|| ||\varphi|| = 0.$$

Therefore $a\varphi$ belongs to $V$ since $V$ is closed. By symmetry, $\varphi a$ is also in $V$. Hence $V$ is invariant.

The last half of our assertion follows from the fact that $V$ is invariant as a subspace of $\tilde{\mathcal{A}}_*$ by [14]. This completes the proof.

As an immediate consequence of the above result, we get

COROLLARY 3. In the same situation as Proposition 1, if $\varphi \in \mathcal{A}^*$ is locally normal, then the absolute value $|\varphi|$ of $\varphi$, in the sense of the polar decomposition, is locally normal too. In particular, if $\varphi \in \mathcal{A}^*$ is locally normal and self-adjoint, then the positive part $\varphi^+$
and the negative part $\varphi^-$ of $\varphi$ are both locally normal.

This generalizes a result [6; Proposition 6] of Haag, Kadison and Kastler.

**Proposition 4.** In the same situation as before, $V$ is weak* sequentially complete. That is, if $\varphi$ is a weak* limit of a sequence $\{\varphi_n\}$ of locally normal linear functionals, then $\varphi$ is locally normal too.

This follows directly from the weak sequential completeness of the predual $\mathcal{A}_{sa}$ of each $\mathcal{A}_a$, see for example [12].

3. Algebraic equivalence of locally normal representations.

First of all, we recall the definition of algebraic equivalence of two representations given by Powers [11]:

**Definition 5.** Let $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ be two representations of a C*-algebra $\mathcal{A}$. If the von Neumann algebras $\mathcal{M}(\pi_1)$ and $\mathcal{M}(\pi_2)$ generated by $\pi_1(\mathcal{A})$ and $\pi_2(\mathcal{A})$ respectively are isomorphic, then $\pi_1$ and $\pi_2$ are said to be algebraically equivalent.

The following is a slight modification of a definition given by Haag, Kadison and Kastler [6].

**Definition 6.** A sequential type I$_n$ funnel $\{\mathcal{A}_n\}$ of a C*-algebra $\mathcal{A}$ is said to be proper if each relative commutant $\mathcal{A}_n' \cap \mathcal{A}_{n+1}$ of $\mathcal{A}_n$ in $\mathcal{A}_{n+1}$ is of type I$_n$.

The following lemma is a modification of Glimm and Kadison's result [3].

**Lemma 7.** Let $\mathcal{M}$ be a von Neumann algebra generated by an increasing sequence $\{\mathcal{A}_n\}$ of C*-algebras, each of which contains the identity 1 of $\mathcal{M}$. Let $\mathcal{U}(\mathcal{A}_n)$ denote the group of all unitary operators of $\mathcal{A}_n$. Then the union $\bigcup_{n=1}^\infty \mathcal{U}(\mathcal{A}_n)$ is strongly dense in the group $\mathcal{U}(\mathcal{M})$ of unitary operators of $\mathcal{M}$.

**Proof.** Take an arbitrary unitary operator $u \in \mathcal{U}(\mathcal{M})$. There exists a self-adjoint operator $h \in \mathcal{M}$ such that $u = \exp(2\pi ih)$ and $\|h\| \leq 1$. Since $\bigcup_{n=1}^\infty \mathcal{A}_n$ is a strongly dense *-subalgebra of $\mathcal{M}$, there exists, by Kaplansky's density theorem [2: Th. 3, p. 43], a net $\{h_j\}_{j \in J}$ of self-adjoint elements in $\bigcup_{n=1}^\infty \mathcal{A}_n$ such that $\{h_j\}_{j \in J}$ converges strongly to $h$ and $\|h_j\| \leq 1$. Put $u_j = \exp(2\pi ih_j)$, $j \in J$. Since each $h_j$...
belongs to some $\mathcal{A}$, $u_j$ belongs to $\bigcup_{n=1}^{\infty} \mathcal{U}(\mathcal{A})$. By the strong continuity of the functional calculus on the bounded set of self-adjoint elements (see [10]), the net $\{u_j\}$ converges strongly to $u$. This completes the proof.

**Lemma 8.** Let $\mathcal{M}$ be a $\sigma$-finite (countably decomposable) von Neumann algebra. Suppose $\mathcal{A}$ and $\mathcal{B}$ are type $I_\infty$ subfactors of $\mathcal{M}$ with properly infinite relative commutants $\mathcal{A}' \cap \mathcal{M}$ and $\mathcal{B}' \cap \mathcal{M}$. Then there exists a unitary operator $u$ in $\mathcal{M}$ such that $uBu^{-1} = \mathcal{A}$.

**Proof.** Let $\{u_{i,j}; i, j = 1, 2, \cdots\}$ and $\{v_{i,j}; i, j = 1, 2, \cdots\}$ be matrix units of $\mathcal{A}$ and $\mathcal{B}$ respectively. Put $e = u_{1,1}$ and $f = v_{1,1}$. Then $e$ and $f$ are minimal projections in $\mathcal{A}$ and $\mathcal{B}$ respectively. Since $\mathcal{A}' \cap \mathcal{M}$ is properly infinite, $\mathcal{A}' \cap \mathcal{M}$ contains an infinite sequence $\{p_n\}$ of equivalent orthogonal projections with $\sum p_n = 1$. For each index $n$, let $u_n$ be a partial isometry in $\mathcal{A}' \cap \mathcal{M}$ such that $u_n^*u_n = p_i$ and $u_nu_n^* = p_n$. Then we have

\[
(u_n e)(u_n e)^* = e u_n^* u_n e = e p_1 ; \\
(u_n e)(u_n e)^* = u_n e u_n^* = e p_n .
\]

Hence $\{e p_n\}$ is an infinite sequence of equivalent orthogonal projections with $\sum_{n=1}^{\infty} e p_n = e$, which means that $e$ is a properly infinite projection in $\mathcal{M}$. Similarly $f$ is a properly infinite projection in $\mathcal{M}$. Since $e$ and $f$ both have central support 1, they are equivalent in $\mathcal{M}$, that is, there exists a partial isometry $w \in \mathcal{M}$ with $w^* w = e$ and $w w^* = f$ because of the $\sigma$-finiteness of $\mathcal{M}$.

Put

\[
u = \sum_{i=1}^{\infty} v_{i,1} w u_{i,1} .
\]

Then we have

\[
u^* \nu = \sum_{i,j=1}^{\infty} u_{i,j} w^* v_{j,1} w u_{i,j}
\]

\[
= \sum_{i=1}^{\infty} u_{i,1} w^* w u_{i,1}
\]

\[
= \sum_{i=1}^{\infty} u_{i,1} u_{i,1} u_{i,1} = \sum_{i=1}^{\infty} u_{i,1}
\]

\[
= 1 ;
\]

similarly

\[
u \nu^* = 1 .
\]

Hence $\nu$ is a unitary operator in $\mathcal{M}$. By a straightforward calculation, we have

\[
u u_{i,j} u = v_{i,j} , \quad i, j = 1, 2, \cdots ;
\]
hence we have
\[ u\mathcal{A}u^* = \mathcal{B} . \]
This completes the proof.

**Lemma 9.** Let \( \mathcal{M} \) be a \( \sigma \)-finite von Neumann algebra generated by a \( C^* \)-algebra \( \mathcal{A} \) with a proper sequential type \( I_\infty \) funnel \( \{\mathcal{A}_n\} \), where we assume each \( \mathcal{A}_n \) to be a von Neumann subalgebra of \( \mathcal{M} \). Suppose \( \mathcal{B} \) is a type \( I_\infty \) factor contained in \( \mathcal{M} \) with properly infinite relative commutant \( \mathcal{B}' \cap \mathcal{M} \). For any \( \sigma \)-strong* neighborhood \( U \) of the identity 1 in \( \mathcal{M} \), there exists \( n \) and a unitary operator \( u \) in \( U \) such that
\[ u\mathcal{B}u^{-1} \subset \mathcal{A}_n . \]

**Proof.** By Lemma 8, there exists a unitary operator \( v \in \mathcal{M} \) such that \( v\mathcal{B}v^{-1} = \mathcal{A}_n \). Since \( v^{-1} \) is in \( \mathcal{M} \), it follows from Lemma 7 that there exists a unitary operator \( w \in \mathcal{A}_n \) such that \( w \in Uv^{-1} \). Put \( u = vw \). Then \( u \) belongs to \( U \) and
\[ u\mathcal{B}u^{-1} = vw\mathcal{B}v^{-1}w^{-1} = w\mathcal{A}_n w^{-1} \subset \mathcal{A}_n . \]
This completes the proof.

**Lemma 10.** Suppose \( \mathcal{M} \), \( \mathcal{A} \) and \( \{\mathcal{A}_n\} \) are as in Lemma 9. Suppose \( \mathcal{B} \) and \( \mathcal{B}' \) are both type \( I_\infty \) subfactors of \( \mathcal{M} \) such that \( \mathcal{B} \subset \mathcal{B}' \) and the relative commutant \( \mathcal{B}' \cap \mathcal{M} \) is properly infinite. Suppose \( u \) is a unitary operator in \( \mathcal{M} \) such that
\[ u\mathcal{B}u^{-1} \subset \mathcal{A}_n . \]
For any \( \sigma \)-strong* neighborhood \( U \) of 1 in \( \mathcal{M} \), there exist a unitary operator \( u \), and index \( n_1 > n_0 \) such that

1. \( u_1\mathcal{B}u^{-1}_1 \subset \mathcal{A}_{n_1} \);
2. \( u_2xu_2^{-1} = uxu^{-1} \) for every \( x \in \mathcal{B} \);
3. \( u_1u_2 \in U \).

**Proof.** Put \( \mathcal{C} = u\mathcal{B}u^{-1} \) and \( \mathcal{C}' = u_1\mathcal{B}u^{-1}_1 \). Then \( \mathcal{C} \) and \( \mathcal{C}' \) are both type \( I_\infty \) subfactors of \( \mathcal{M} \) with properly infinite relative commutant in \( \mathcal{M} \). Put \( \mathcal{N} = \mathcal{C}' \cap \mathcal{M} \). Since \( \mathcal{C} \) is a type \( I \) subfactor of \( \mathcal{M} \), \( \mathcal{M} \) is decomposed into the tensor product:

\[ \mathcal{M} = \mathcal{C} \otimes \mathcal{N} . \]

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\(^1\) The \( \sigma \)-strong* topology in a von Neumann algebra \( \mathcal{M} \) is defined as the locally convex topology induced by the family of seminorms: \( x \in \mathcal{M} \rightarrow p_\omega(x) = \omega(x^*x + xx^*)^{1/2} \), where \( \omega \) runs over all normal states of \( \mathcal{M} \). The \( \sigma \)-strong* topology agree with the strong operator topology on the unitary group of \( \mathcal{M} \), but their uniform structures are different.
If $\mathcal{A}_n \supset \mathcal{C}$, then $\mathcal{A}_n$ is also decomposed with respect to this tensor product:

$$\mathcal{A}_n \cong \mathcal{C} \otimes (\mathcal{C}' \cap \mathcal{M}) .$$

Since $\bigcup \mathcal{A}_n$ generates $\mathcal{M}$, $\bigcup_{n=1}^\infty (\mathcal{C}' \cap \mathcal{A}_n)$ generates $\mathcal{C}' \cap \mathcal{M} = \mathcal{N}$. Let $\mathcal{D}$ be the uniform closure of $\bigcup_{n=1}^\infty (\mathcal{C}' \cap \mathcal{A}_n)$. Then $\mathcal{D}$ has a sequential type I funnel $\{\mathcal{C}' \cap \mathcal{A}_n\}$. Since

$$\mathcal{C} = u\mathcal{B}u^{-1} \subset \mathcal{A}_0$$

by assumption, $\mathcal{C}' \cap \mathcal{A}_n$ is properly infinite for $n > n_0$ because

$$\mathcal{C}' \cap \mathcal{A}_n \supset \mathcal{A}_n \cap \mathcal{A}_0' .$$

Moreover, we have, for $n > n_0$,

$$(\mathcal{C}' \cap \mathcal{A}_n)' \cap (\mathcal{C}' \cap \mathcal{A}_{n+1}) \supset \mathcal{C}' \cap \mathcal{A}_n' \cap \mathcal{A}_{n+1}$$

$$\supset \mathcal{A}_n' \cap \mathcal{A}_n' \cap \mathcal{A}_{n+1} = \mathcal{A}_n' \cap \mathcal{A}_{n+1} ,$$

hence $(\mathcal{C}' \cap \mathcal{A}_n)' \cap (\mathcal{C}' \cap \mathcal{A}_{n+1})$ is properly infinite. Hence the type $L_\infty$ funnel $\{\mathcal{C}' \cap \mathcal{A}_n\}_{n>n_0}$ of $\mathcal{D}$ is proper. Put $\mathcal{D}_1 = \mathcal{C}' \cap \mathcal{C}_1$. Then $\mathcal{D}_1$ is a type $I_\infty$ subfactor of $\mathcal{N}$. Since

$$\mathcal{D}_1' \cap \mathcal{N} = (\mathcal{C}' \cap \mathcal{C}_1)' \cap (\mathcal{C}' \cap \mathcal{M})$$

$$\supset \mathcal{C}_1' \cap \mathcal{M} = u(\mathcal{B}_1' \cap \mathcal{M})u^{-1} ,$$

$\mathcal{D}_1$ has a properly infinite relative commutant $\mathcal{D}_1' \cap \mathcal{N}$. By Lemma 9, there exists a unitary operator $v \in U \cap \mathcal{N}$ and an index $n_1$ such that $v\mathcal{D}_1v^{-1} \subset (\mathcal{A}_{n_1} \cap \mathcal{C}') \subset \mathcal{A}_{n_1}$. Put $u_1 = vu$. Then $u_1$ is in $\mathcal{M}$ and $u_1u^{-1}$ is in $U$. For each $x \in \mathcal{B}$, $uxu^{-1}$ is in $\mathcal{C}$; hence it commutes with $v$, so that

$$u_1uxu^{-1} = v(uxu^{-1})v^{-1} = uxu^{-1} .$$

Since

$$u_1(\mathcal{B}' \cap \mathcal{B})u_1^{-1} = vu(\mathcal{B}' \cap \mathcal{B})u^{-1}v$$

$$= v(\mathcal{C}' \cap \mathcal{C}_1)v^{-1}$$

$$= v\mathcal{D}_1v^{-1} \subset \mathcal{A}_{n_1} ,$$

we have

$$u_1\mathcal{B}_1u_1^{-1} = u_1(\mathcal{B} \cup (\mathcal{B}' \cap \mathcal{B}_1))''u_1^{-1}$$

$$= (u_1\mathcal{B}u_1^{-1} \cup u_1(\mathcal{B}' \cap \mathcal{B}_1)u_1^{-1} )''$$

$$= (\mathcal{C} \cup v\mathcal{D}_1v^{-1})'' \subset (\mathcal{A}_{n_0} \cup \mathcal{A}_{n_1})'' = \mathcal{A}_{n_1} .$$

This completes the proof.
Lemma 11. Suppose $\mathcal{M}$, $\mathcal{A}$ and $\{\mathcal{A}_n\}$ are as in Lemma 9. Suppose $\mathcal{B}$ is another $\mathcal{C}^*$-subalgebra of $\mathcal{M}$ with a proper sequential type $I_\omega$ funnel $\{\mathcal{B}_n\}$, which is $\sigma$-weakly dense in $\mathcal{M}$. Then there exists a unitary operator $u \in \mathcal{M}$ such that

$$u\left(\bigcup_{n=1}^{\infty} \mathcal{A}_n \right) u^{-1} = \bigcup_{n=1}^{\infty} \mathcal{B}_n ;$$

hence

$$u_\mathcal{A} u^{-1} = \mathcal{B} .$$

Proof. By the $\sigma$-finiteness of $\mathcal{M}$, there exists a faithful normal state $\phi$ of $\mathcal{M}$. Define a distance function $d$ on $\mathcal{M}$ by:

$$d(x, y) = \{\phi((x - y)^* (x - y)) + \phi((x - y)(x - y)^*)\}^{1/2}, \quad x, y \in \mathcal{M} .$$

Then the topology induced by the metric $d$ coincides with the $\sigma$-strong* topology on the bounded part of $\mathcal{M}$. Furthermore, the group $\mathcal{U}$ of all unitary operators of $\mathcal{M}$ is complete with respect to this metric $d$.

By induction, we construct increasing sequences $\{\mathcal{C}_i\}, \{\mathcal{D}_i\}$ of type $I_\omega$ subfactors of $\mathcal{M}$, a sequence $\{u_i: i = 1, 2, \ldots\}$ of unitary operators in $\mathcal{M}$ and increasing sequences $\{n_i\}$ and $\{m_i\}$ of integers with the properties:

(i) $\mathcal{A}_{n_i+1} \subset \mathcal{C}_{i+1} \subset \mathcal{A}_{n_{i+1}}$;

$$\mathcal{B}_{m_{i+1}} \subset \mathcal{D}_i \subset \mathcal{B}_{m_{i+1}}$$

for $i = 1, 2, \ldots, k$;

(ii) $u_{k_{i-1}} \mathcal{B}_{m_{i+1}} u_{k_{i-1}}^{-1} = \mathcal{C}_i$,

$$u_{k_{2i}} \mathcal{A}_{n_i+1} u_{k_{2i}}^{-1} = \mathcal{D}_i$$

for $i = 1, 2, \ldots, k$;

(iii) $u_{k_{2i+1}} x u_{k_{2i+1}} = u_{k_{2i+1}}^{-1} x u_{k_{2i+1}}^{-1}$ for $x \in \mathcal{C}_i$,

$$u_{k_{2i+1}} x u_{k_{2i+1}}^{-1} = u_{k_{2i+1}} u_{k_{2i+1}}^{-1}$$

for $x \in \mathcal{D}_i$

for $i = 1, 2, \ldots, k$;

(iv) $d(u_i, u_{i+1}) < 1/2^i$

for $i = 1, 2, \ldots, 2k$.

For $k = 1$, we choose $m_i = 0, n_i = 1$ and $\mathcal{C}_1 = \mathcal{A}_1$. Then by Lemma 8, there exists a unitary operator $u_1$ such that

$$u_1 \mathcal{B}_1 u_1^{-1} = u_1 \mathcal{B}_{n_1+1} u_1^{-1} = \mathcal{C}_1 = \mathcal{A}_1 \subseteq \mathcal{A}_{n_1} .$$

Consider the triplet $(\mathcal{C}_1, \mathcal{A}_2, u_1)$ as $(\mathcal{B}_1, \mathcal{A}_1, u)$ in Lemma 10. Then
we can find a unitary operator $v \in \mathcal{M}$ and index $m_2$ such that
\[
v_{\mathcal{M}_{m_2}^+} v^{-1} = v_{\mathcal{M}_{m_2}^-} v^{-1} \subset \mathcal{B}_{m_2} ;
\]
$v v^{-1} = u_i u_i^{-1}$ for every $x \in C_i$ ;
\[
d(u_i, v) < \frac{1}{2} .
\]
Put $u_2 = v$ and $\mathcal{D}_k = v_{\mathcal{M}_{m_2}^+} v^{-1}$.

Suppose $\{n_i, \cdots, n_k\}, \{m_i, \cdots, m_k\}, \{C_i, \cdots, C_k\}, \{\mathcal{D}_i, \cdots, \mathcal{D}_k\}$ and $\{u_i, \cdots, u_{2k}\}$ have been chosen so that condition (i), (ii), (iii) and (iv) are satisfied. Applying Lemma 10 to $\{\mathcal{D}_k, \mathcal{M}_{m_{k+1}}, u_{2k}\}$, we can find an index $n_{k+1}$ and a unitary operator $u_{2k+1}$ such that
\[
u_{\mathcal{M}_{m_{k+1}}^+} u_{2k+1}^{-1} \subset \mathcal{M}_{m_{k+1}}^- ;
\]
$u_{2k+1} u_{2k+1} = u_{2k} u_{2k}^{-1}$ for $x \in \mathcal{D}_k$ ;
\[
d(u_{2k}, u_{2k+1}) < 1/2^{k+1} .
\]
Put $C_{k+1} = u_{2k+1} \mathcal{M}_{m_{k+1}} u_{2k+1}^{-1}$ Since $C_{k+1}$ contains $\mathcal{M}_{m_{k+1}}^-$. Now again applying Lemma 10 to the triplet $\{C_{k+1}, \mathcal{M}_{m_{k+1}^+}, u_{2k+1}^{-1}\}$, we can choose an index $m_{k+1}$ and a unitary operator $u_{2(k+1)}$ in $\mathcal{M}_{m_{k+1}}$ such that
\[
u_{\mathcal{M}_{m_{k+1}^+}} u_{2(k+1)}^{-1} \subset \mathcal{M}_{m_{k+1}^-} ;
\]
$u_{2(k+1)} x u_{2(k+1)}^{-1} = u_{2k+1} x u_{2k+1}$ for $x \in C_{k+1}$ ;
\[
d(u_{2k+1}, u_{2(k+1)}) < 1/2^{k+1} .
\]
Put $\mathcal{D}_{k+1} = u_{2(k+1)} \mathcal{M}_{m_{k+1}^+} u_{2(k+1)}$. Hence the existence of sequences $\{n_i\}, \{m_i\}, \{C_i\}, \{\mathcal{D}_i\}$ and $\{u_i\}$ has been established. From condition (i) it follows that
\[
\bigcup_{i=1}^{n_i} \mathcal{A}_i = \bigcup_{i=1}^{m_i} C_i \quad \text{and} \quad \bigcup_{i=1}^{n_i} \mathcal{D}_i = \bigcup_{i=1}^{m_i} \mathcal{D}_i .
\]
From condition (iv), $\{u_k\}$ is a Cauchy sequence of unitary operators with respect to the metric $d$. Hence $u_k$ converges $\sigma$-strongly* to a unitary operator $u$ of $\mathcal{M}$. By condition (iii), for every $x \in \mathcal{M}_{m_{k+1}}$, we have
\[
u_{\mathcal{M}_{m_{k+1}}^+} u_{2k+1}^{-1} = u_{2k+1} x u_{2k+1}^{-1}
\]
for each $k \geq 2i$. Hence we have
\[
u_{\mathcal{M}_{m_{i+1}}^+} u_{2i-1} = C_i .
\]
Thus we have
This completes the proof.

As an immediate consequence of Lemma 11, we have the following extension of a corresponding result of Powers for UHF-algebras in [11].

**Theorem 12.** Suppose $\mathcal{A}$ is a $C^*$-algebra with a proper sequential type I$_\infty$ funnel $\{\mathcal{A}_n\}$. Suppose $\{\pi_1, \mathcal{H}_1\}$ and $\{\pi_2, \mathcal{H}_2\}$ are two locally normal representations of $\mathcal{A}$. Suppose either the von Neumann algebra $\mathcal{M}(\pi_1)$ generated by $\pi_1$ or the one $\mathcal{M}(\pi_2)$ generated by $\pi_2$ is $\sigma$-finite. Then the representations $\pi_1$ and $\pi_2$ are algebraically equivalent if and only if there exists an automorphism $\sigma$ of $\mathcal{A}$ such that the representations $\pi_1$ and $\pi_2 \circ \sigma$ are quasi-equivalent. If this is the case, then $\sigma$ may be chosen such that $\sigma(\bigcup_{n=1}^\infty \mathcal{A}_n) = \bigcup_{n=1}^\infty \mathcal{A}_n$.

**Proof.** If there exists an automorphism $\sigma$ of $\mathcal{A}$ such that $\pi_1$ and $\pi_2 \circ \sigma$ are quasi-equivalent, then there exists an isomorphism $\rho$ of $\mathcal{M}(\pi_1)$ onto the von Neumann algebra $\mathcal{M}(\pi_2 \circ \sigma)$ generated by $\pi_2 \circ \sigma(\mathcal{A})$ such that $\rho \circ \pi_1 = \pi_2 \circ \sigma$. But it is clear that $\mathcal{M}(\pi_2 \circ \sigma) = \mathcal{M}(\pi_2)$. Hence $\rho$ implements the algebraic equivalence of $\pi_1$ and $\pi_2$.

Suppose $\mathcal{M}(\pi_1)$ is $\sigma$-finite. Since $\mathcal{A}$ is simple (see [6: Proposition 10]), $\pi_1$ is an isomorphism of $\mathcal{A}$ into $\mathcal{M}(\pi_1)$. Hence we may identify $\mathcal{A}$ with the subalgebra $\pi_2(\mathcal{A})$ of $\mathcal{M}(\pi_1)$ which generates $\mathcal{M}(\pi_1)$. Suppose $\rho$ is an isomorphism of $\mathcal{M}(\pi_1)$ onto $\mathcal{M}(\pi_2)$. Put $\mathcal{B} = \rho \circ \pi_2(\mathcal{A})$ and $\mathcal{B}_n = \rho \circ \pi_2(\mathcal{A}_n)$. Then $\mathcal{M}(\pi_1)$, $\mathcal{A}$, $\{\mathcal{A}_n\}$ and $\mathcal{B}$, $\{\mathcal{B}_n\}$ satisfy all the assumptions of Lemma 11. Hence there exists a unitary operator $u$ in $\mathcal{M}(\pi_1)$ such that

$$u \mathcal{A} u^{-1} = \mathcal{B}, \quad u \left( \bigcup_{n=1}^\infty \mathcal{A}_n \right) u^{-1} = \bigcup_{n=1}^\infty \mathcal{B}_n.$$ 

Define a map $\sigma$ of $\mathcal{A}$ into $\mathcal{A}$ by

$$\sigma(x) = u^{-1} \rho \circ \pi_2(x) u, \quad x \in \mathcal{A}.$$ 

Then we have, for $x \in \mathcal{A}$,

$$\pi_1 \circ \sigma(x) = \sigma(x) = u(\rho \circ \pi_2(x)) u^{-1};$$

hence $\pi_1 \circ \sigma$ is unitary equivalent to $\rho \circ \pi_2$ and $\rho \circ \pi_2$ is quasi-equivalent to $\pi_2$ by definition, so that $\pi_1 \circ \sigma$ and $\pi_2$ are quasi-equivalent. This completes the proof.
COROLLARY 13. If $\mathcal{A}$ is a $C^*$-algebra with a proper sequential type $I_\infty$ funnel, then the group of automorphisms of $\mathcal{A}$ acts transitively on the set of all locally normal pure states.

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