ALGEBRAIC EQUIVALENCE OF LOCALLY NORMAL REPRESENTATIONS

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It will be shown that (i) the absolute value of every locally normal linear functional is again locally normal; (ii) two locally normal representations π_1 and π_2 of $\mathscr{A}$ generate isomorphic von Neumann algebras $\mathscr{M}(\pi_1)$ and $\mathscr{M}(\pi_2)$ if and only if there exists an automorphism σ of $\mathscr{A}$ such that $\pi_1 \circ \sigma$ and π_2 are quasi-equivalent, provided that either $\mathscr{M}(\pi_1)$ or $\mathscr{M}(\pi_2)$ is σ-finite.

This paper is motivated by a recent work [6] of R. Haag. R. V. Kadison and D. Kastler. As they mentioned, the recent progress in mathematical physics has made a precise analysis of representations of a C^*-algebra furnished with a net of von Neumann algebras a growing necessity.

In the first half of this paper, we shall show that the space of all locally normal linear functionals of a C^*-algebra with a net of von Neumann algebras is a closed invariant subspace of the conjugate space in the sense of [14], which will imply that the absolute value of a locally normal linear functional is locally normal too.

The last half of this paper will be devoted to extending a result of Powers [11] for UHF algebra to a C^*-algebra $\mathscr{A}$ with a proper sequential type I_∞ funnel. Namely it will be shown that two locally normal representations π_1 and π_2 of the C^*-algebra $\mathscr{A}$ generate isomorphic von Neumann algebras if and only if they are connected by an automorphism of $\mathscr{A}$. This is proven under the assumption that one of the generated von Neumann algebras is σ-finite.

2. **The locally normal conjugate space of a C^*-algebra with a net of von Neumann algebras.** Let $\mathscr{A}$ be a C^*-algebra. Suppose a system $\mathfrak{F} = (\mathscr{A}_\alpha)$ of C^*-subalgebras of $\mathscr{A}$ indexed by a directed set $\{\alpha\}$ is given such that:

(i) $\mathscr{A}_\alpha$ is a von Neumann subalgebra of $\mathscr{A}_\beta$ if $\alpha \leq \beta$;

(ii) $\bigcup_\alpha \mathscr{A}_\alpha$ is dense in $\mathscr{A}$ with respect to the norm topology. The system $\mathfrak{F} = \{\mathscr{A}_\alpha\}$ is called a *net* (in $\mathscr{A}$) of von Neumann algebras and each $\mathscr{A}_\alpha$ is called *local subalgebra* of $\mathscr{A}$

DEFINITION 1. A continuous linear functional φ (resp. representation π) of $\mathscr{A}$ is said to be *locally normal* if φ (resp. π) is σ-weakly continuous on each local subalgebra $\mathscr{A}_\alpha$.

PROPOSITION 2. *Let V be the set of all locally normal linear functionals on a C*-algebra $\mathscr{A}$ with a net $\mathfrak{F} = \{\mathscr{A}_\alpha\}$ of von Neumann algebras. Then V is a closed, invariant subspace of $\mathscr{A}^*$. Namely, if $\varphi \in \mathscr{A}^*$ is locally normal, then $a\varphi$ and φa, $a \in \mathscr{A}$, are both locally normal, where $a\varphi$ and φa defined by $a\varphi(x) = \varphi(xa)$ and $\varphi a(x) = \varphi(ax)$, $x \in \mathscr{A}$.*

Therefore, there exists a unique central projection z_0 of the universal enveloping von Neumann algebra $\widetilde{\mathscr{A}}$ of $\mathscr{A}$, the second conjugate space of $\mathscr{A}$ as a Banach space, such that

$$z_0 \mathscr{A}^* = V .$$

Proof. Let $\{\varphi_n\}$ be a sequence in V converging to $\varphi \in \mathscr{A}^*$ with respect to the norm topology. For each α, we have

$$\| \varphi|_{\mathscr{A}_\alpha} - \varphi_n|_{\mathscr{A}_\alpha} \| \leqq \| \varphi - \varphi_n \| \to 0$$

as $n \to \infty$; hence $\{\varphi_n|_{\mathscr{A}_\alpha}\}$ converges to $\varphi|_{\mathscr{A}_\alpha}$. Since the predual $\mathscr{A}_{\alpha*}$ of each $\mathscr{A}_\alpha$ is complete, $\varphi|_{\mathscr{A}_\alpha}$ belongs to $\mathscr{A}_{\alpha*}$, so that φ is locally normal. Hence V is closed.

Take an arbitrary element $\varphi \in V$. Let a be an element of $\mathscr{A}_\alpha$. For each β, there exists an index γ such that $\alpha \leqq \gamma$, $\beta \leqq \gamma$. Since $\varphi|_{\mathscr{A}_\gamma}$ is normal and a is in $\mathscr{A}_\gamma$, $a\varphi|_{\mathscr{A}_\gamma}$ is normal, so that

$$a\varphi|_{\mathscr{A}_\beta} = (a\varphi|_{\mathscr{A}_\gamma})|_{\mathscr{A}_\beta}$$

is normal. Hence $a\varphi$ belongs to V. Therefore, if a belongs to $\bigcup \mathscr{A}_\alpha$, then $a\varphi$ is locally normal. If a is an arbitrary element of $\mathscr{A}$, then there exists a sequence $\{a_n\}$ in $\bigcup \mathscr{A}_\alpha$ such that

$$\lim_{n \to \infty} \| a - a_n \| = 0 ;$$

hence

$$\lim_{n \to \infty} \| a\varphi - a_n\varphi \| \leqq \lim_{n \to \infty} \| a - a_n \| \, \| \varphi \| = 0 .$$

Therefore $a\varphi$ belongs to V since V is closed. By symmetry, φa is also in V. Hence V is invariant.

The last half of our assertion follows from the fact that V is invariant as a subspace of $\widetilde{\mathscr{A}_*}$ by [14]. This completes that proof.

As an immediate consequence of the above result, we get

COROLLARY 3. *In the same situation as Proposition 1, if $\varphi \in \mathscr{A}^*$ is locally normal, then the absolute value $|\varphi|$ of φ, in the sense of the polar decomposition, is locally normal too. In particular, if $\varphi \in \mathscr{A}^*$ is locally normal and self-adjoint, then the positive part φ^+*

and the negative part φ^- of φ are both locally normal.

This generalizes a result [6; Proposition 6] of Haag, Kadison and Kastler.

PROPOSITION 4. *In the same situation as before, V is weak* sequencially complete. That is, if φ is a weak* limit of a sequence $\{\varphi_n\}$ of locally normal linear functionals, then φ is locally normal too.*

This follows directly from the weak sequential completeness of the predual $\mathscr{A}_{\alpha*}$ of each $\mathscr{A}_\alpha$, see for example [12].

3. Algebraic equivalence of locally normal representations. First of all, we recall the definition of algebraic equivalence of two representations given by Powers [11]:

DEFINITION 5. Let $(\pi_1, \mathscr{H}_1)$ and $(\pi_2, \mathscr{H}_2)$ be two representations of a C^*-algebra $\mathscr{A}$. If the von Neumann algebras $\mathscr{M}(\pi_1)$ and $\mathscr{M}(\pi_2)$ generated by $\pi_1(\mathscr{A})$ and $\pi_2(\mathscr{A})$ respectively are isomorphic, then π_1 and π_2 are said to be *algebraically equivalent*.

The following is a slight modification of a definition given by Haag, Kadison and Kastler [6].

DEFINITION 6. A sequential type I_∞ funnel $\{\mathscr{A}_n\}$ of a C^*-algebra $\mathscr{A}$ is said to be *proper* if each relative commutant $\mathscr{A}_n' \cap \mathscr{A}_{n+1}$ of $\mathscr{A}_n$ in $\mathscr{A}_{n+1}$ is of type I_∞.

The following lemma is a modification of Glimm and Kadison's result [3].

LEMMA 7. *Let $\mathscr{M}$ be a von Neumann algebra generated by an increasing sequence $\{\mathscr{A}_n\}$ of C^*-algebras, each of which contains the identity 1 of $\mathscr{M}$. Let $\mathscr{U}(\mathscr{A}_n)$ denote the group of all unitary operators of $\mathscr{A}_n$. Then the union $\bigcup_{n=1}^\infty \mathscr{U}(\mathscr{A}_n)$ is strongly dense in the group $\mathscr{U}(\mathscr{M})$ of unitary operators of $\mathscr{M}$.*

Proof. Take an arbitrary unitary operator $u \in \mathscr{U}(M)$. There exists a self-adjoint operator $h \in \mathscr{M}$ such that $u = \exp(2\pi i h)$ and $\|h\| \leq 1$. Since $\bigcup_{n=1}^\infty \mathscr{A}_n$ is a strongly dense *-subalgebra of $\mathscr{M}$, there exists, by Kaplansky's density theorem [2: Th. 3, p. 43], a net $\{h_j\}_{j \in J}$ of self-adjoint elements in $\bigcup_{n=1}^\infty \mathscr{A}_n$ such that $\{h_j\}_{j \in J}$ converges strongly to h and $\|h_j\| \leq 1$. Put $u_j = \exp(2\pi i h_j)$, $j \in J$. Since each h_j

belongs to some $\mathscr{A}_n$, u_j belongs to $\bigcup_{n=1}^{\infty} \mathscr{U}(\mathscr{A})$. By the strong con-
tinuity of the functional calculus on the bounded set of self-adjoint
elements (see [10]), the net $\{u_j\}$ converges strongly to u. This com-
pletes the proof.

LEMMA 8. *Let $\mathscr{M}$ be a σ-finite (countably decomposable) von
Neumann algebra. Suppose $\mathscr{A}$ and $\mathscr{B}$ are type I_∞ subfactors of
$\mathscr{M}$ with properly infinite relative commutants $\mathscr{A}' \cap \mathscr{M}$ and $\mathscr{B}' \cap \mathscr{M}$.
Then there exists a unitary operator u in $\mathscr{M}$ such that $u\mathscr{B}u^{-1} = \mathscr{A}$.*

Proof. Let $\{u_{i,j} : i, j = 1, 2, \cdots\}$ and $\{v_{i,j} : i, j = 1, 2, \cdots\}$ be matrix
units of $\mathscr{A}$ and $\mathscr{B}$ respectively. Put $e = u_{1,1}$ and $f = v_{1,1}$. Then e
and f are minimal projections in $\mathscr{A}$ and $\mathscr{B}$ respectively. Since
$\mathscr{A}' \cap \mathscr{M}$ is properly infinite, $\mathscr{A}' \cap \mathscr{M}$ contains an infinite sequence
$\{p_n\}$ of equivalent orthogonal projections with $\sum p_n = 1$. For each
index n, let u_n be a partial isometry in $\mathscr{A}' \cap \mathscr{M}$ such that $u_n^* u_n = p_1$
and $u_n u_n^* = p_n$. Then we have

$$(u_n e)^*(u_n e) = e u_n^* u_n e = e p_1 \; ;$$
$$(u_n e)(u_n e)^* = u_n e u_n^* = e p_n \; .$$

Hence $\{e p_n\}$ is an infinite sequence of equivalent orthogonal projections
with $\sum_{n=1}^{\infty} e p_n = e$, which means that e is a properly infinite projection
in $\mathscr{M}$. Similarly f is a properly infinite projection in $\mathscr{M}$. Since e
and f both have central support 1, they are equivalent in $\mathscr{M}$, that
is, there exists a partial isometry $w \in \mathscr{M}$ with $w^* w = e$ and $ww^* = f$
because of the σ-finiteness of $\mathscr{M}$. Put

$$u = \sum_{i=1}^{\infty} v_{i,1} w u_{1,i} \; .$$

Then we have

$$\begin{aligned}
u^* u &= \sum_{i,j=1}^{\infty} u_{i,1} w^* v_{1,i} v_{j,1} w u_{1,j} \\
&= \sum_{i=1}^{\infty} u_{i,1} w^* w u_{1,i} \\
&= \sum_{i=1}^{\infty} u_{i,1} u_{1,1} u_{1,i} = \sum_{i=1}^{\infty} u_{i,i} \\
&= 1 \; ;
\end{aligned}$$

similarly

$$uu^* = 1 \; .$$

Hence u is a unitary operator in $\mathscr{M}$. By a straightforward calcula-
tion, we have

$$u u_{i,j} u^* = v_{i,j}, \qquad\qquad i, j = 1, 2, \cdots ;$$

hence we have

$$u \mathscr{A} u^* = \mathscr{B} \; .$$

This completes the proof.

LEMMA 9. *Let $\mathscr{M}$ be a σ-finite von Neumann algebra generated by a C^*-algebra $\mathscr{A}$ with a proper sequential type I_∞ funnel $\{\mathscr{A}_n\}$, where we assume each $\mathscr{A}_n$ to be a von Neumann subalgebra of $\mathscr{M}$. Suppose $\mathscr{B}$ is a type I_∞ factor contained in $\mathscr{M}$ with properly infinite relative commutant $\mathscr{B}' \cap \mathscr{M}$. For any σ-strong* neighborhood[1] U of the identity 1 in $\mathscr{M}$, there exists n and a unitary operator u in U such that*

$$u \mathscr{B} u^{-1} \subset \mathscr{A}_n \; .$$

Proof. By Lemma 8, there exists a unitary operator $v \in \mathscr{M}$ such that $v \mathscr{B} v^{-1} = \mathscr{A}_1$. Since v^{-1} is in $\mathscr{M}$, it follows from Lemma 7 that there exists a unitary operator $w \in \mathscr{A}_n$ such that $w \in U v^{-1}$. Put $u = wv$. Then u belongs to U and

$$u \mathscr{B} u^{-1} = wv \mathscr{B} v^{-1} w^{-1} = w \mathscr{A}_1 w^{-1} \subset \mathscr{A}_n \; .$$

This completes the proof.

LEMMA 10. *Suppose $\mathscr{M}$, $\mathscr{A}$ and $\{\mathscr{A}_n\}$ are as in Lemma 9. Suppose $\mathscr{B}$ and $\mathscr{B}_1$ are both type I_∞ subfactors of M such that $\mathscr{B} \subset \mathscr{B}_1$ and the relative commutant $\mathscr{B}_1' \cap \mathscr{M}$ is properly infinite. Suppose u is a unitary operator in $\mathscr{M}$ such that*

$$u \mathscr{B} u^{-1} \subset \mathscr{A}_{n_0} \; .$$

For any σ-strong* neighborhood U of 1 in $\mathscr{M}$, there exist a unitary operator u_1 and index $n_1 > n_0$ such that
   (i) $u_1 \mathscr{B} u_1^{-1} \subset \mathscr{A}_{n_1}$;
   (ii) $u_1 x u_1^{-1} = u x u^{-1}$ *for every* $x \in \mathscr{B}$;
   (iii) $u_1 u^{-1} \in U$.

Proof. Put $\mathscr{C} = u \mathscr{B} u^{-1}$ and $\mathscr{C}_1 = u \mathscr{B}_1 u^{-1}$. Then $\mathscr{C}$ and $\mathscr{C}_1$ are both type I_∞ subfactors of $\mathscr{M}$ with properly infinite relative commutant in $\mathscr{M}$. Put $\mathscr{N} = \mathscr{C}' \cap \mathscr{M}$. Since $\mathscr{C}$ is a type I subfactor of $\mathscr{M}$, $\mathscr{M}$ is decomposed into the tensor product:

$$\mathscr{M} \cong \mathscr{C} \otimes (\mathscr{C}' \cap \mathscr{M}) \, .$$

If $\mathscr{A}_n \supset \mathscr{C}$, then $\mathscr{A}_n$ is also decomposed with respect to this tensor product:

$$\mathscr{A}_n \cong \mathscr{C} \otimes (\mathscr{C}' \cap \mathscr{A}_n) \, .$$

Since $\bigcup \mathscr{A}_n$ generates $\mathscr{M}$, $\bigcup_{n=1}^{\infty} (\mathscr{C}' \cap \mathscr{A}_n)$ generates $\mathscr{C}' \cap \mathscr{M} = \mathscr{N}$. Let $\mathscr{D}$ be the uniform closure of $\bigcup_{n=1}^{\infty} (\mathscr{C}' \cap \mathscr{A}_n)$. Then $\mathscr{D}$ has a sequential type I funnel $\{\mathscr{C}' \cap \mathscr{A}_n\}$. Since

$$\mathscr{C} = u\mathscr{B}u^{-1} \subset \mathscr{A}_{n_0}$$

by assumption, $\mathscr{C}' \cap \mathscr{A}_n$ is properly infinite for $n > n_0$ because

$$\mathscr{C}' \cap \mathscr{A}_n \supset \mathscr{A}_n \cap \mathscr{A}'_{n_0} \, .$$

Moreover, we have, for $n > n_0$,

$$(\mathscr{C}' \cap \mathscr{A}_n)' \cap (\mathscr{C}' \cap \mathscr{A}_{n+1}) \supset \mathscr{C}' \cap \mathscr{A}'_n \cap \mathscr{A}_{n+1}$$
$$\supset \mathscr{A}'_{n_0} \cap \mathscr{A}'_n \cap \mathscr{A}_{n+1} = \mathscr{A}'_n \cap \mathscr{A}_{n+1} \, ,$$

hence $(\mathscr{C}' \cap \mathscr{A}_n)' \cap (\mathscr{C}' \cap \mathscr{A}_{n+1})$ is properly infinite. Hence the type I_∞ funnel $\{\mathscr{C}' \cap \mathscr{A}_n\}_{n > n_0}$ of $\mathscr{D}$ is proper. Put $\mathscr{D}_1 = \mathscr{C}' \cap \mathscr{C}_1$. Then $\mathscr{D}_1$ is a type I_∞ subfactor of $\mathscr{N}$. Since

$$\mathscr{D}'_1 \cap \mathscr{N} = (\mathscr{C}' \cap \mathscr{C}_1)' \cap (\mathscr{C}' \cap \mathscr{M})$$
$$\cong \mathscr{C}'_1 \cap \mathscr{M} = u(\mathscr{B}'_1 \cap \mathscr{M})u^{-1} \, ,$$

$\mathscr{D}_1$ has a properly infinite relative commutant $\mathscr{D}'_1 \cap \mathscr{N}$. By Lemma 9, there exists a unitary operator $v \in U \cap \mathscr{N}$ and an index n_1 such that $v\mathscr{D}_1 v^{-1} \subset (\mathscr{A}_{n_1} \cap \mathscr{C}') \subset \mathscr{A}_{n_1}$. Put $u_1 = vu$. Then u_1 is in $\mathscr{M}$ and $u_1 u^{-1}$ is in U. For each $x \in \mathscr{B}$, uxu^{-1} is in $\mathscr{C}$; hence it commutes with v, so that

$$u_1 x u_1^{-1} = v(uxu^{-1})v^{-1} = uxu^{-1} \, .$$

Since

$$u_1(\mathscr{B}' \cap \mathscr{B}_1)u_1^{-1} = vu(\mathscr{B}' \cap \mathscr{B}_1)u^{-1}v$$
$$= v(\mathscr{C}' \cap \mathscr{C}_1)v^{-1}$$
$$= v\mathscr{D}v^{-1} \subset \mathscr{A}_{n_1}$$

we have

$$u_1\mathscr{B}_1 u_1^{-1} = u_1(\mathscr{B} \cup (\mathscr{B}' \cap \mathscr{B}_1))''u_1^{-1}$$
$$= (u_1\mathscr{B}u_1^{-1} \cup u_1(\mathscr{B}' \cap \mathscr{B}_1)u_1^{-1})''$$
$$= (\mathscr{C} \cup v\mathscr{D}v^{-1})'' \subset (\mathscr{A}_{n_0} \cup \mathscr{A}_{n_1})'' = \mathscr{A}_{n_1} \, .$$

This completes the proof.

LEMMA 11.   *Suppose $\mathcal{M}$, $\mathcal{A}$ and $\{\mathcal{A}_n\}$ are as in Lemma 9.   Suppose $\mathcal{B}$ is another C*-subalgebra of $\mathcal{M}$ with a proper sequential type I_∞ funnel $\{\mathcal{B}_n\}$, which is σ-weakly dense in $\mathcal{M}$.   Then there exists a unitary operator $u \in \mathcal{M}$ such that*

$$u\left(\bigcup_{n=1}^{\infty} \mathcal{A}_n\right)u^{-1} = \bigcup_{n=1}^{\infty} \mathcal{B}_n \, ;$$

hence

$$u\mathcal{A}u^{-1} = \mathcal{B} \, .$$

Proof.   By the σ-finiteness of $\mathcal{M}$, there exists a faithful normal state φ of $\mathcal{M}$.   Define a distance function d on $\mathcal{M}$ by:

$$d(x, y) = \{\varphi((x - y)^*(x - y)) + \varphi((x - y)(x - y)^*)\}\}^{1/2} \, , \qquad x, y \in \mathcal{M} \, .$$

Then the topology induced by the metric d coincides with the σ-strong* topology on the bounded part of $\mathcal{M}$.   Furthermore, the group $\mathcal{U}$ of all unitary operators of $\mathcal{M}$ is complete with respect to this metric d.

By induction, we construct increasing sequences $\{\mathcal{C}_i\}$, $\{\mathcal{D}_i\}$ of type I_∞ subfactors of $\mathcal{M}$, a sequence $\{u_i : i = 1, 2, \cdots\}$ of unitary operators in $\mathcal{M}$ and increasing sequences $\{n_i\}$ and $\{m_i\}$ of integers with the properties:

(i)
$$\mathcal{A}_{n_i+1} \subset \mathcal{C}_{i+1} \subset \mathcal{A}_{n_{i+1}} \, ;$$
$$\mathcal{B}_{m_i+1} \subset \mathcal{D}_i \subset \mathcal{B}_{m_{i+1}}$$

for $i = 1, 2, \cdots, k$;

(ii)
$$u_{2i-1}\mathcal{B}_{m_j+1}u_{2i-1}^{-1} = \mathcal{C}_i \, ,$$
$$uu_{2i}^{-1}\mathcal{A}_{n_i+1}u_{2i} = \mathcal{D}_i$$

for $i = 1, 2, \cdots, k$;

(iii)
$$u_{2i}^{-1}xu_{2i} = u_{2i-1}^{-1}xu_{2i-1} \text{ for } x \in \mathcal{C}_i \, ,$$
$$u_{2i+1}xu_{2i+1}^{-1} = u_{2i}xu_{2i}^{-1} \text{ for } x \in \mathcal{D}_i$$

for $i = 1, 2, \cdots, k$;

(iv)
$$d(u_i, u_{i+1}) < 1/2^i$$

for $i = 1, 2, \cdots, 2k$.

For $k = 1$, we choose $m_1 = 0$, $n_1 = 1$ and $\mathcal{C}_1 = \mathcal{A}_1$.   Then by Lemma 8, there exists a unitary operator u_1 such that

$$u_1\mathcal{B}_1u_1^{-1} = u_1\mathcal{B}_{m_1+1}u_1^{-1} = \mathcal{C}_1 = \mathcal{A}_1 \subseteqq \mathcal{A}_{n_1} \, .$$

Consider the triplet $(\mathcal{C}_1, \mathcal{A}_2, u_1^{-1})$ as $(\mathcal{B}, \mathcal{B}_1, u)$ in Lemma 10.   Then

we can find a unitary operator $v \in \mathscr{M}$ and index m_2 such that

$$v\mathscr{A}_{n_1+1}v^{-1} = v\mathscr{A}_2v^{-1} \subset \mathscr{B}_{m_2} ;$$

$$vxv^{-1} = u_1^{-1}xu_1 \text{ for every } x \in \mathscr{C}_1 ;$$

$$d(u_1^{-1}, v) < \tfrac{1}{2} .$$

Put $u_2 = v$ and $\mathscr{D}_1 = v\mathscr{A}_2v^{-1}$.

Suppose $\{n_1, \cdots, n_k\}, \{m_1, \cdots, m_k\}, \{\mathscr{C}_1, \cdots, \mathscr{C}_k\}, \{\mathscr{D}_1, \cdots, \mathscr{D}_k\}$ and $\{u_1, \cdots, u_{2k}\}$ have been chosen so that condition (i), (ii), (iii) and (iv) are satisfied. Applying Lemma 10 to $\{\mathscr{D}_k, \mathscr{B}_{m_k+1}, u_{2k}\}$, we can find an index n_{k+1} and a unitary operator u_{2k+1} such that

$$u_{2k+1}\mathscr{B}_{m_k+1}u_{2k+1}^{-1} \subset \mathscr{A}_{n_{k+1}} ;$$

$$u_{2k+1}xu_{2k+1}^{-1} = u_{2k}xu_{2k}^{-1} \text{ for } x \in \mathscr{D}_k ;$$

$$d(u_{2k}, u_{2k+1}) < 1/2^{2k+1} .$$

Put $\mathscr{C}_{k+1} = u_{2k+1}\mathscr{B}_{m_k+1}u_{2k+1}^{-1}$. Since

$$u_{2k}\mathscr{D}_ku_{2k}^{-1} = \mathscr{A}_{n_k+1} ,$$

$\mathscr{C}_{k+1}$ contains $\mathscr{A}_{n_k+1}$. Now again applying Lemma 10 to the triplet $\{\mathscr{C}_{k+1}, \mathscr{A}_{n_{k+1}+1}, u_{2k+1}^{-1}\}$, we can choose an index m_{k+1} and a unitary operator $u_{2(k+1)}$ in $\mathscr{M}$ such that

$$u_{2(k+1)}^{-1}\mathscr{A}_{n_{k+1}+1}u_{2(k+1)} \subset \mathscr{B}_{m_{k+1}} ;$$

$$u_{2(k+1)}^{-1}xu_{2(k+1)} = u_{2k+1}^{-1}xu_{2k+1} \text{ for } x \in \mathscr{C}_{k+1} ;$$

$$d(u_{2k+1}, u_{2(k+1)}) < 1/2^{2(k+1)} .$$

Put $\mathscr{D}_{k+1} = u_{2(k+1)}^{-1}\mathscr{A}_{n_{k+1}+1}u_{2(k+1)}$.

Hence the existence of sequences $\{m_i\}, \{n_i\}, \{\mathscr{C}_i\}, \{\mathscr{D}_i\}$ and $\{u_i\}$ has been established. From condition (i) it follows that

$$\overset{\infty}{\underset{i=1}{\cup}} \mathscr{A}_i = \overset{\infty}{\underset{i=1}{\cup}} \mathscr{C}_i \text{ and } \overset{\infty}{\underset{i=1}{\cup}} \mathscr{B}_i = \overset{\infty}{\underset{i=1}{\cup}} \mathscr{D}_i .$$

From condition (iv), $\{u_k\}$ is a Cauchy sequence of unitary operators with respect to the metric d. Hence u_k converges σ-strongly* to a unitary operator u of $\mathscr{M}$. By condition (iii), for every $x \in \mathscr{B}_{m_i+1}$, we have

$$u_kvu_k^{-1} = u_{2i-1}xu_{2i-1}^{-1}$$

for each $k \geq 2i$. Hence we have

$$u\mathscr{B}_{m_i+1}u^{-1} = \mathscr{C}_i .$$

Thus we have

$$u\Big(\bigcup_{k=1}^{\infty}\mathscr{B}_k\Big)u^{-1} = u\Big(\bigcup_{j=1}^{\infty}\mathscr{B}_{m_j+1}\Big)u^{-1}$$
$$= \bigcup_{i=1}^{\infty}\mathscr{C}_i = \bigcup_{i=1}^{\infty}\mathscr{A}_i .$$

This completes the proof.

As an immediate consequence of Lemma 11, we have the following extension of a corresponding result of Powers for UHF-algebras in [11].

THEOREM 12. *Suppose $\mathscr{A}$ is a C*-algebra with a proper sequencial type I_{∞} funnel $\{\mathscr{A}_n\}$. Suppose $\{\pi_1, \mathscr{H}_1\}$ and $\{\pi_2, \mathscr{H}_2\}$ are two locally normall representations of $\mathscr{A}$. Suppose either the von Neumann algebra $\mathscr{M}(\pi_1)$ generated by $\pi_1(\mathscr{A})$ or the one $\mathscr{M}(\pi_2)$ generated by $\pi_2(\mathscr{A})$ is σ-finite. Then the representations π_1 an π_2 are algebraically equivalent if and only if there exists an automorphism σ of $\mathscr{A}$ such that the representations π_1 and $\pi_2 \cdot \sigma$ are quasi-equivalent. If this is the case, then σ may be chosen such that $\sigma(\bigcup_{n=1}^{\infty}\mathscr{A}_n) = \bigcup_{n=1}^{\infty}\mathscr{A}_n$.*

Proof. If there exists an automorphism σ of $\mathscr{A}$ such that π_1 and $\pi_2 \circ \sigma$ are quasi-equivalent, then there exists an isomorphism ρ of $\mathscr{M}(\pi_1)$ onto the von Neumann algebra $\mathscr{M}(\pi_2 \circ \sigma)$ generated by $\pi_2 \circ \sigma(\mathscr{A})$ such that $\rho \circ \pi_1 = \pi_2 \circ \sigma$. But it is clear that $\mathscr{M}(\pi_2 \circ \sigma) = \mathscr{M}(\pi_2)$. Hence ρ implements the algebraic equivalence of π_1 and π_2.

Suppose $\mathscr{M}(\pi_1)$ is σ-finite. Since $\mathscr{A}$ is simple (see [6: Proposition 10]), π_1 is an isomorphism of $\mathscr{A}$ into $\mathscr{M}(\pi_1)$. Hence we may identity $\mathscr{A}$ with the subalgebra $\pi_1(\mathscr{A})$ of $\mathscr{M}(\pi_1)$ which generates $\mathscr{M}(\pi_1)$. Suppose ρ is an isomorphism of $\mathscr{M}(\pi_2)$ onto $\mathscr{M}(\pi_1)$. Put $\mathscr{B} = \rho \circ \pi_2(\mathscr{A})$ and $\mathscr{B}_n = \rho \circ \pi_2(\mathscr{A}_n)$. Then $\mathscr{M}(\pi_1)$, $\mathscr{A}$, $\{\mathscr{A}_n\}$ and $\mathscr{B}$, $\{\mathscr{B}_n\}$ satisfy all the assumptions of Lemma 11. Hence there exists a unitary operator u in $\mathscr{M}(\pi_1)$ such that

$$u\mathscr{A}u^{-1} = \mathscr{B}, \; u\Big(\bigcup_{n=1}^{\infty}\mathscr{A}_n\Big)u^{-1} = \bigcup_{n=1}^{\infty}\mathscr{B}_n .$$

Define a map σ of $\mathscr{A}$ into $\mathscr{A}$ by

$$\sigma(x) = u^{-1}\rho \circ \pi_2(x)u , \qquad\qquad x \in \mathscr{A} .$$

Then we have, for $x \in \mathscr{A}$,

$$\pi_1 \circ \sigma(x) = \sigma(x) = u(\rho \circ \pi_2(x))u^{-1} ;$$

hence $\pi_1 \circ \sigma$ is unitary equivalent to $\rho \circ \pi_2$ and $\rho \circ \pi_2$ is quasi-equivalent to π_2 by definition, so that $\pi_1 \circ \sigma$ and π_2 are quasi-equivalent. This completes the proof.

COROLLARY 13. *If $\mathscr{A}$ is a C*-algebra with a proper sequential type I_∞ funnel, then the group of automorphisms of $\mathscr{A}$ acts transitively on the set of all locally normal pure states.*

REFERENCES

1. J. Dixmier, *Les C*-Algèbres et leurs représentations*, Gauthier-Villars, Paris, 1964.
2. ————, *Les algebres d'opérateurs dans l'espace Hilbertien*, 2é edition, Gauthier-Villars, Paris, 1968.
3. J. Glimm and R. V. Kadison, *Unitary operators in C*-algebras*, Pacific J. Math. **10** (1960), 547–556.
4. A. Guichardet, *Algebres d'observables associes aux relations de commutation* (to appear)
5. ————, *Tensor products of C*-algebras, Part II Infinite tensor product*, Lecture notes, Aahus Univ., No. 12, 1969.
6. R. Haag, R. V. Kadison, and D. Kastler, *Nets of C*-algebras and classification of states* (to appear)
7. R. Haag and D. Kastler, *An algebraic approach to field theory*, J. Math. Phys. **5** (1964), 848–861.
8. N. M. Hugenholtz, M. Takesaki, and M. Winnink, *Local normality of the KMS-states in quantum statistical mechanics* (in preparation)
9. N. M. Hugenholtz and J. D. Wieringa, *On locally normal states in quantum statistical mechanics*, Comm. Math. Phys. **11** (1969), 183–197.
10. I. Kaplansky, *A theorem on rings of operators*, Pacific J. Math. **1** (1951), 227–232.
11. R. T. Powers, *Representations of uniformly hyperfinite algebras and their associated von Neumann rings*, Ann. of Math. **86** (1967), 138–171.
12. S. Sakai, *The theory of W*-algebras*, mimeographed notes, Yale Univ., 1962.
13. Z. Takeda, *Inductive limit and infinite direct product of operator algebras*, Tôhoku Math. J. **7** (1955), 67–86.
14. M. Takesaki, *On the conjugate space of operator algebra*, Tôhoku Math. J. **10** (1958), 194–203.

UNIVERSITY OF CALIFORNIA, LOS ANGELES, AND
TÔHOKU UNIVERSITY, SENDAI

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