Pacific Journal of Mathematics

EXTENSIONS OF CONTINUOUS AFFINE FUNCTIONS

LEONARD ASIMOW

Vol. 35, No. 1 September 1970

PACIFIC JOURNAL OF MATHEMATICS
Vol. 35, No. 1, 1970

EXTENSIONS OF CONTINUOUS AFFINE FUNCTIONS

LEONARD ASIMOW

Conditions are given for a closed face F of a compact convex set X to have the property that if $f \in A(F), g_1, \ldots, g_m \in A(X)$, and f dominates each g_i on F then f can be extended to $g \in A(X)$ where g dominates each g_i on X.

Let X be a compact convex set in a Hausdorff locally convex space. We identify X in the standard fashion with the set of positive elements of norm one in $A(X)^*$ (weak*-topology), where $A(X)$ is the ordered Banach space (sup-norm) of continuous affine functions on X. A face of X is a convex subset which contains the endpoints of every open line segment in X which it intersects. It is known (for example [2]) that every continuous affine function on a closed face F of X admits a continuous affine extension to all of X if and only if the linear span, $\langle F \rangle$, of F is weak* closed in $A(X)^*$. If additional conditions of a geometric nature on F and X are made then much more can be said about the type of extensions which are possible. For example if X is a Choquet simplex (in which case $\langle F \rangle$ is weak* closed whenever F is), a theorem of Edwards [3] states that
(*) if $\{f_i\}_{i=1}^m, \{g_j\}_{j=1}^n \in A(X)$ and $f \in A(F)$ such that

$$f_i|_F \leqq f \leqq g_j|_F \qquad (i = 1, \cdots, m; j = 1, \cdots n)$$

then there is an extension $g \in A(X)$ of f such that

$$f_i \leqq g \leqq g_j \qquad (i = 1, \cdots, m; j = 1, \cdots, n) .$$

This extension property is quite strong in the sense that it in fact characterizes simplexes among the compact convex sets.

One can ask under what conditions on F and X the following weaker extension property holds:
(**) if $\{f_i\}_{i=1}^m \in A(X)$ and $f \in A(F)$ such that

$$f_i|_F \leqq f \qquad (i = 1, \cdots, m)$$

then there is an extension $g \in A(X)$ of f such that

$$f_i \leqq g .$$

Closed faces which possess property (**) are termed *strongly archimedean* by Alfsen [1] (see also Størmer [5] for the origin of the terminology). In [2] we give conditions on F such that (**) holds for functions f_i, f identically zero on F. This implies in particular that F is (within a G_δ set) a peak-face of X. We give here a some-what strengthened form of these conditions which guarantees an extension can be found such that (**) holds in general.

We shall say X is *decomposable* at the closed face F (under f) if there exists a bounded linear functional f on $A(X)^*$ such that f is identically zero on $\langle F\rangle^-$ (weak* closure) and $X = \mathrm{conv}\,(K \cup F)$, where $K = \{x \in X: 1 \leqq f(x) \leqq \|f\|\}$. If X is decomposable at F and the linear span of F is weak* closed, then we show (Theorem 2.6 and Corollary 2.7) that the extension property (**) holds.

The closed faces of α-polytopes (see Phelps [4]), for example, satisfy these conditions and hence are strongly archimedean. We show also that if F is a closed face *complemented* in X then a weak version of the extension property (*) can be obtained. As a corollary we obtain (*) as stated for simplexes.

1. Preliminaries.

Let h be any real-valued function on the compact convex set X. We call the set of ordered pairs $(x, r) \in X \times R \subset A(X)^* \times R$ such that $r \geqq h(x)$ the *upper-graph* of h. The lower-graph is defined analogously. We note that h is convex if and only if upper-graph (h) is convex and h is concave if and only if lower-graph (h) is convex. Also h is lower-semi-continuous if and only if upper-graph (h) is closed and upper-semi-continuous if and only if lower-graph (h) is closed.

If $x \in X$ we define the gage functional p_x on X by

$$p_x(y) = \inf\{r \geqq 0: y \in x + r(X - x)\}\ .$$

Then p_x is lower-semi-continuous, convex and affine along any line segment in X with one endpoint x. Also for each $y \in X$ there is a $z \in X$ such that

$$y = p_x(y)z + (1 - p_x(y))x\ .$$

Let F be a closed face of X.

PROPOSITION 1.1. *For each $y \in X$ the function $x \to p_x(y)$ is lower-semi-continuous on F.*

Proof. Let $\varphi: F \times X \times [0, 1] \to X$ be defined by

$$\varphi(x, z, \lambda) = \lambda z + (1 - \lambda)x\ .$$

Then for $x \in F$, $p_x(y) \leqq r$ if and only if there is a $z \in X$ and $s \in [0, r]$ such that

$$y = sz + (1 - s)x\ .$$

Thus $\{x \in F: p_x(y) \leqq r\}$ is exactly the natural projection of $\varphi^{-1}(y) \cap$

$(F \times X \times [0, r])$ into F which is clearly closed.

We define $p_F: X \to [0, 1]$ by

$$p_F(y) = \inf \{p_x(y): x \in F\} .$$

By the proposition the infimum is actually attained and so each $y \in X$ can be written as

$$y = p_F(y)z + (1 - p_F(y))x ; \qquad z \in X \text{ and } x \in F .$$

It also follows that p_F is affine along the line segment $[x, z]$ and thus $p_F(z) = 1$. In addition upper-graph (p_F)

$$= \{(x, r): x \in X, r \geqq 1\} \cup \operatorname{conv} [(X \times \{1\}) \cup (F \times \{0\})] ,$$

and hence is a closed convex set. Consequently p_F is lower-semi-continuous and convex.

In the sequel it is necessary to assume that the closed face F of X is self-determining, that is, if N is the weak* closure of $\langle F \rangle$ in $A(X)^*$ then $N \cap X = F$. If this is the case then let $q: A(X)^* \to A(X)^*/N$ be the quotient map. The quotient space can be identified with the dual of the space of continuous affine functions on qX vanishing at 0[2]. We then define the semi-norm p_N on $A(X)^*$ by

$$p_N(x) = \|qx\| .$$

It follows that p_N is weak* lower-semi-continuous and sub-additive on $A(X)^*$.

2. Decomposable faces. As our first step we give conditions which assure that if $h \in A(X)$, $h|_F \geqq 0$ then there is a $g \in A(X)$ such that $g|_F = h|_F$ and $g \geqq 0$ on X.

LEMMA 2.1. *Let F be a subset of X and assume there exists M and β ($M \geqq 0$ and $0 \leqq \beta < 1$) such that if $h \in A(X)$, $h|_F \geqq 0$, $h + a \geqq 0$ on X ($a \geqq 0$) there is a $g_1 \in A(X)$, $g_1|_F = h|_F$, $g_1 + \beta a \geqq 0$ on X and $\|g_1 - h\| \leqq Ma$.*

Then for each $h \in A(X)$, $h|_F \geqq 0$, there is a $g \in A(X)$, $g|_F = h|_F$, $g \geqq 0$ on X and $\|g - h\| \leqq \|h\|/(1 - \beta)M$.

Proof. Let $h \in A(X)$ such that $h|_F \geqq 0$ be given. Then $h + \|h\| \geqq 0$ so there is $g_1 \in A(X)$, $g_1|_F = h|_F$, $g_1 + \beta\|h\| \geqq 0$ and $\|g_1 - h\| \leqq M\|h\|$. Now apply the hypothesis to g_1 with $a = \beta\|h\|$ and get $g_2 \in A(X)$ such that $g_2|_F = h|_F$, $g_2 + \beta^2\|h\| \geqq 0$ and $\|g_2 - g_1\| \leqq \beta M\|h\|$. Continuing by induction we get a sequence $\{g_n\}_{n=1}^{\infty}$ such that $g_n|_F = h|_F$, $g_n + \beta^n\|h\| \geqq 0$ and $\|g_{n+1} - g_n\| \leqq \beta^n M\|h\|$. Thus $\{g_n\}_{n=1}^{\infty}$ converges uniformly to $g \in A(X)$ such that $g|_F = h|_F$, $g \geqq 0$ on X and

$$||g - h|| \leqq \sum_{n=1}^{\infty} ||g_{n+1} - g_n|| + ||g_1 - h|| \leqq \frac{M}{1-\beta} ||h|| .$$

THEOREM 2.2. *Let F be a self-determining face of X and assume that $p_F \leqq \alpha p_N$. Then if $r > \alpha$ and $h \in A(X)$, $h|_F \geqq 0$, there exists $g \in A(X)$, $g|_F = h|_F$, $g \geqq 0$ on X and $||h - g|| \leqq 2r\,||h||$.*

Proof. It suffices to show that the hypotheses of the preceding lemma is satisfied with $M = 2$ and $\beta = 1 - 1/r$. Since $p_N \leqq p_F\,\alpha$ must be greater than or equal to one and hence $0 \leqq \beta < 1$. Given $h \in A(X)$, $h|_F \geqq 0$ and $h + a \geqq 0$ on $X(a > 0)$ define $\bar{h}$ on $A(X)^*$ by

$$\bar{h} = h + ap_N .$$

Then $\bar{h}$ is sub-additive and weak* lower-semi-continuous. If $y \in X$ then since $y = p_F(y)z + (1 - p_F(y))x$ with $x \in F$ and $p_F(z) = 1$ we have

$$\begin{aligned}\bar{h}(y) &= h(y) + ap_N(y) = p_F(y)(h(z) + ap_N(z)) + (1 - p_F(y))h(x) \\ &\geqq p_F(y)\Big(h(z) + \frac{a}{\alpha}\Big) + (1 - p_F(y))h(x) \\ &= p_F(y)(h(z) + a) + (1 - p_F(y))h(x) - ap_F(y)\Big(1 - \frac{1}{\alpha}\Big) .\end{aligned}$$

Thus since $h(z) + a$, $h(x) \geqq 0$ and $\alpha < r$

$$(*) \qquad \bar{h}(y) + a\beta > 0 \quad \text{for all} \quad y \in X.$$

Let $Y = \text{conv}(\{0\} \cup X)$ in $A(X)^*$. Then $(*)$ continues to hold on Y. Thus $\{(x, r) \in A(X)^* \times R \colon r \geqq \bar{h}(x) + a\beta\}$ is a closed convex set disjoint from $Y \times \{0\}$. A weak* closed separating hyperplaneyields exactly the graph of a weak* continuous affine function f on $A(X)^*$ such that $f > 0$ on Y, $f < \bar{h} + a\beta$ on $A(X)^*$ and $0 < f(0) < a\beta$. On N we have $\bar{h} = h$ and hence $f < h + a\beta$ there. Since N is a subspace and h is linear we have $f = h + f(0)$ on N. Let $g = f - f(0)$. Then $g \in A(X)$, $g|_F = h|_F$ and $g + a\beta > g + f(0) > 0$ on X. Also

$$g < f < \bar{h} + a\beta \leqq h + a(p_N + \beta) .$$

Since $p_N \leqq 1$ on $\text{conv}(X \cup -X)$ we have

$$g < h + 2a \quad \text{on} \quad X \text{ and } -X.$$

Thus $||h - g|| \leqq 2a$ and the proof is complete.

In [2] we define the self-determining face F to be *conical* in X (under f) if f is a bounded linear functional on $A(X)^*$ such that $f \equiv 0$ on $N = \langle F \rangle^-$, $f \geqq 0$ on X and $x \in f(x)X + N$ for all $x \in X + N$.

We will say that X is *decomposable* at F (under f) if f is a bounded linear functional on $A(X)^*$ such that $f \equiv 0$ on N and

$$X = \operatorname{conv}(F \cup \{x \in X: 1 \leqq f(x) \leqq \|f\|\}) .$$

Note that if X is decomposable at F then F is automatically self-determining. Also, as noted in [2], if X is decomposable at F under f then F is a conical face of X under f.

PROPOSITION 2.3. *If X is decomposable at F under f then $p_F \leqq \|f\| p_N$.*

Proof. If $y \in X$ then $y = \lambda z + (1 - \lambda)x (0 < \lambda < 1)$ with $x \in F$, $z \in X$ and $f(z) \geqq 1$. Thus $p_F(y) \leqq \lambda$ and $f(y) = \lambda f(z) \geqq \lambda$ and hence

(1) $$p_F \leqq f .$$

If $q: A(X)^* \rightarrow A(X)^*/N$ is the quotient map we can define $\bar{f}$ on $A(X)^*/N$ by $f = \bar{f} \cdot q$ (since $f \equiv 0$ on N). Also $\|f\| = \sup f(X) = \sup \bar{f}(qX) = \|\bar{f}\|$ since the unit ball in $A(X)^*/N$ is $\operatorname{conv}(qX \cup -qX)$. Hence for $y \in X$

$$f(y) = \bar{f} \circ q(y) \leqq \|\bar{f}\| \|qy\| = \|\bar{f}\| p_N(y) .$$

Thus

(2) $$f \leqq \|f\| p_N .$$

Combining (1) and (2), $P_F \leqq \|f\| p_N$.

COROLLARY 2.4. *If X is decomposable at F under f then for any $h \in A(X)$, $h|_F \geqq 0$ and for any $r > \|f\|$ there is a $g \in A(X)$, $g|_F = h|_F$, $g \geqq 0$ on X and $\|g - h\| \leqq 2r \|h\|$.*

From [2; §2], we have the following theorem.

THEOREM 2.5. *If X is decomposable at F under f then given $g_1, \cdots, g_n \in A(X)$, each $g_i \equiv 0$ on F, then there is an $h \in A(X)$, $h \equiv 0$ on F such that $h \geqq g_1, \cdots, g_n$ on X. Furthermore, if $r > \|f\|$, h can be chosen such that $\|h\| \leqq r$.*

By combining Theorem 2.5 and Corollary 2.4 we obtain a result of the type mentioned in the introduction.

THEOREM 2.6. *If X is decomposable at F and $f_1, \cdots, f_n, h \in A(X)$ are given such that each $f_i|_F \leqq h|_F$ there is a $g \in A(X)$ such that $g|_F = h|_F$ and each $f_i \leqq g$ on X.*

Proof. Since $(h - f_i)|_F \geqq 0$ there is $g_i \in A(X)$, $g_i|_F = (h - f_i)|_F$ and

$g_i \geqq 0$ on X. Let $g_i' = g_i + f_i$. Then $g_i'|_F = h|_F$ and $g_i' \geqq f_i$ on X. Now each $g_i' - h$ is identically zero on F. Hence there is $g' \in A(X)$, $g' \equiv 0$ on F and $g' \geqq g_i' - h$ on X. Let $g = g' + h$. Then $g|_F = h|_F$ and

$$g = g' + h \geqq (g_i' - h) + h = g_i' \geqq f_i \qquad (i = 1, \cdots, n)$$

on X.

COROLLARY 2.7. *If X is decomposable at the closed face F and the linear span of $\langle F \rangle$ in $A(X)^*$ is weak* (or equivalently norm) closed, then for any $f_1, \cdots, f_n \in A(X)$ and $h \in A(F)$ such that each $f_i|_F \leqq h$ there is an extension $g \in A(X)$ of h such that each $f_i \leqq g$ on X.*

Proof. Since $\langle F \rangle$ is closed we can find some extension $h' \in A(X)$ of h (see, for example [2], Th. 3.1). Thus, Theorem 2.6 applies with $f_1, \cdots, f_n$ and h'.

3. Complemented faces. We shall say the closed face F of X is *complemented* in X (by F') if there is a disjoint face F' (not necessarily closed) in X such that each $y \in X$ has a *unique* representation of the form

$$y = \lambda x + (1 - \lambda) z; \qquad x \in F, z \in F' .$$

This implies in particular that $\langle F \rangle$ and $\langle F' \rangle$ are complemented subspaces in $A(X)^*$.

For a complemented face F in X we obtain a stronger extension property. We establish a preliminary result first.

LEMMA 3.1. *Let F be a closed face in X complemented by F'. Let $f \in A(F)$ and $g \in A(X)$ such that $f \leqq g|_F$. Then the function h defined by*

$$h(y) = \lambda f(x) + (1 - \lambda) g(z); \qquad y = \lambda x + (1 - \lambda) z ,$$
$$0 \leqq \lambda \leqq 1, x \in F, z \in F'$$

is affine and lower-semi-continuous on X.

Proof. The fact that h is affine follows directly from the definition of complemented faces. If $(y, h(y)) \in \text{graph}\,(h)$ then

$$(y, h(y)) = \lambda(x, f(x)) + (1 - \lambda)(z, g(z)) \in \text{conv}\,(\text{graph}\,(f) \cup \text{graph}\,(g)) .$$

If $(w, s) \in \text{conv}\,(\text{graph}\,(f) \cup \text{graph}\,(g))$ then

$$(w, s) = \alpha(x, f(x)) + \beta(x', g(x')) + \gamma(z, g(z)) ;$$
$$\alpha + \beta + \gamma = 1, x, x' \in F, z \in F' .$$

Since $g(x') \geqq f(x')$, $s \geqq \alpha f(x) + \beta f(x') + \gamma g(z) = h(w)$ and hence $(w, s) \in$ upper-graph (h). Thus

$$\text{upper-graph } (h) = [\text{conv } (\text{graph } (f) \cup \text{graph } (g))] \cup \text{upper-graph } (g)$$

which is closed. Hence h is lower-semi-continuous.

THEOREM 3.2. *Let F be a closed face complemented by F' in X. Let $g_i, h_j \in A(X)$ such that there is $f \in A(F)$ and f' affine on F' with $g_i|_F \leqq f \leqq h_j|_F$ and $g_i|_{F'} \leqq f' \leqq h_j|_{F'}$ $(i = 1, \cdots, m; j = 1, \cdots, n)$. Then for any $\varepsilon > 0$ there is an extension $k \in A(X)$ of f such that*

$$g_i \leqq k \leqq h_j + \varepsilon \qquad (i = 1, \cdots, m; j = 1, \cdots, n) .$$

Proof. If $y \in X$ with $y = \lambda x + (1 - \lambda)z$ ($x \in F$, $z \in F'$ and $0 \leqq \lambda \leqq 1$) define $k_0(y) = \lambda f(x) + (1 - \lambda)f'(z)$. Then k_0 is affine and

$$g_i \leqq k_0 \leqq h_j \qquad (i = 1, \cdots, m; j = 1, \cdots, n) .$$

Let $G = \text{conv } (\bigcup_{i=1}^m \text{graph } (g_i) \cup \text{graph } (f))$. Since k_0 is affine and $k_0|_F = f$, graph (g_i), graph $(f) \subset$ lower-graph (k_0), a convex set. Hence

$$G \subset \text{lower-graph } (k_0) .$$

Similarly

$$H \equiv \text{conv } (\bigcup_{j=1}^{m} \text{graph } (h_j) \cup \text{graph } (f)) \subset \text{upper-graph } (k_0) .$$

Thus G can be separated from $H + (0, \varepsilon/2)$ by a hyperplane yielding $k_1 \in A(X)$ such that

$$g_i \leqq k_1 \leqq h_j + \frac{\varepsilon}{2} \qquad (i = 1, \cdots, m; j = 1, \cdots, n)$$
$$f \leqq k_1|_F \leqq f + \frac{\varepsilon}{2} .$$

Now define $\bar{k}_1(y) = \lambda f(x) + (1 - \lambda)k_1(z)$ and as in the lemma $\bar{k}_1$ is affine lower-semi-continuous. Since $g_i|_F \leqq f$ and $\bar{k}_1|_F = f$, $G \subset$ lower-graph $(\bar{k}_1)$. Also $k_1 - \varepsilon/2 \leqq \bar{k}_1$ implies that conv $(G \cup \text{graph } (k_1 - \varepsilon/2)) \subset$ lower-graph $(\bar{k}_1)$. Since upper-graph $(\bar{k}_1) + (0, \varepsilon/4)$ is closed, another separation yields k_2 such that

$$k_1 - \frac{\varepsilon}{2}, g_i \leqq k_2 \leqq \bar{k}_1 + \frac{\varepsilon}{4} \leqq k_1 + \frac{\varepsilon}{4} \leqq h_j + \frac{\varepsilon}{2} + \frac{\varepsilon}{4}$$
$$(i = 1, \cdots, m; j = 1, \cdots, n) .$$

and

$$f \leqq k_2|_F \leqq f + \frac{\varepsilon}{4} .$$

In particular $||k_2 - k_1|| \leqq \varepsilon/2$. Continuing inductively we get a sequence $\{k_r\}_{r=1}^{\infty}$ such that

$$g_i \leqq k_r \leqq h_j + \sum_{s=1}^{r} \frac{\varepsilon}{2^s} \qquad (i = 1, \cdots, m; j = 1, \cdots, n)$$

$$f \leqq k_r|_F \leqq f + \frac{\varepsilon}{2^r}$$

and

$$||k_{r+1} - k_r|| \leqq \frac{\varepsilon}{2^r} .$$

Hence $\{k_r\}_{r=1}^{\infty}$ converges to $k \in A(X)$ such that

$$g_i \leqq k \leqq h_j + \varepsilon \qquad (i = 1, \cdots, m; j = 1, \cdots, n)$$

and

$$k|_F = f .$$

COROLLARY 3.3. *If F is a closed face complemented in X then for any $\varepsilon > 0$ each $f \in A(F)$ has an extension $g \in A(X)$ such that*

$$||g|| \leqq (1 + \varepsilon) ||f|| .$$

COROLLARY 3.4. (*D. A. Edwards* [3]). *If X is a simplex and F is a closed face of X with $g_i, h_j \in A(X)$ such that $g_i \leqq h_j$ $(i = 1, \cdots, m; j = 1, \cdots, n)$ and $f \in A(F)$ such that $g_i|_F \leqq f \leqq h_j|_F$ then f can be extended to $k \in A(X)$ such that $g_i \leqq k \leqq h_j$ $(i = 1, \cdots, m; j = 1, \cdots, n)$.*

Proof. Since X is a simplex F has a complementary face F'. Also the upper envelope g' of $g_1 \vee \cdots \vee g_m$ is an affine function on X such that

$$g_i \leqq g' \leqq h_j \qquad (i = 1, \cdots, m; j = 1, \cdots, n) .$$

Hence the theorem applies with $f' = g'|_{F'}$ yielding k_1 such that $k_1|_F = f$ and

$$g_i \leqq k_1 \leqq h_j + \frac{1}{2} \qquad (i = 1, \cdots, m; j = 1, \cdots, n) .$$

Since $k_1 - 1/2, g_i \leqq h_j, k_1$ $(i = 1, \cdots, m; j = 1, \cdots, n)$ and $(k_1 - 1/2)|_F \leqq f = k_1|_F$ the theorem applies again yielding k_2 such that

$$||k_2 - k_1|| \leqq \frac{1}{2}$$
$$k_2|_F = f$$
$$g_i \leqq k_2 \leqq h_j + \frac{1}{4} \qquad (i = 1, \cdots, m; j = 1, \cdots, n) .$$

Continuing by induction we get a sequence $\{k_r\}_{r=1}^{\infty}$ such that

$$||k_{r+1} - k_r|| \leqq \frac{1}{2^r}$$
$$k_r|_F = f$$
$$g_i \leqq k_r \leqq h_j + \frac{1}{2^r} .$$

Hence $\{k_r\}_{r=1}^{\infty}$ converges to the desired k.

4. **α-polytopes.** It was shown in [2] that the α-polytopes [4] are conical at each closed face. We will show next that they are in fact decomposable at each closed face. Thus the (strongly) archimedean extension property holds at each face. This is a consequence of the fact that simplexes are decomposable at each closed face and the following theorem (see [2], Th. 3.7).

THEOREM 4.1. *Let X and Y be compact convex sets whose closed faces span closed subspaces and let $\varphi: X \to Y$ be a continuous affine surjection. Let $\bar{\varphi}: A(X)^* \to A(Y)^*$ be the natural extension of φ and suppose* $\dim(\ker \bar{\varphi}) < \infty$. *If F' is a closed face of Y such that X is decomposable at $\varphi^{-1}(F')$ then Y is decomposable at F'.*

Proof. It is sufficient to consider the case where $\ker \bar{\varphi} = Rx_0$. Let $N' = \langle F' \rangle$ and $F = \varphi^{-1}(F')$. Then $N \equiv \langle F \rangle = \bar{\varphi}^{-1}(N') + Rx_0$. Let $f \in A(X)^{**}$ such that $f \equiv 0$ on N and

$$X = \text{conv}\,(F \cup \{x \in X: f(x) \geqq 1\}) .$$

If $f(x_0) = 0$ then $f \cdot \bar{\varphi}^{-1}$ well-defines a decomposing functional for Y at F'. Suppose $f(x_0) = 1$. Then $(x_0 + N) \cap X = \phi$ since if $x_0 + n = y \in X$ then $\varphi y = \bar{\varphi} n \in (\bar{\varphi} N) \cap Y = N' \cap Y = F'$ and hence $y \in \varphi^{-1}(F') = F$. But then $f(y) = f(n) = 0$ contradicting $f(x_0) = 1$. Thus there exists $r > 0$ such that

$$(1) \qquad ||y - x_0 + n|| \geqq r; \quad \text{for all} \quad n \in N, x \in X .$$

We define the bounded projection $p: A(X)^* \to f^{-1}(0)$ by $p(x) = x - f(x)x_0$. If $B_r = \{x \in A(X)^*: ||x|| < r\}$ then

$$(2) \qquad (N + B_r) \cap p(\{x \in X: f(x) \geqq 1\}) = \phi .$$

If not there is $z \in X, f(z) \geqq 1$ and $n \in N$ such that $\|p(z) - n\| < r$. Let $x \in F$ and let $y \in X$ be given by

$$y = (1/f(z))z + (1 - 1/f(z))x .$$

Then

$$\begin{aligned} p(z) - n &= z - f(z)x_0 - n = f(z)y - (f(z) - 1)x - f(z)x_0 - n \\ &= f(z)(y - x_0 + n') . \end{aligned}$$

Thus $\|y - x_0 + n'\| < r/f(z) \leqq r$ contradicting (1).

Applying the separation theorem to (2) we obtain a bounded linear functional g on $f^{-1}(0)$ such that $\|g\| \leqq 1/r$; $g \equiv 0$ on N and $g \geqq 1$ on $p(\{x \in X: f(x) \geqq 1\})$. Thus pX is decomposable at $pF = F$ under g and $\bar{g} = g \cdot p \cdot \bar{\varphi}^{-1}$ decomposes Y at F'.

5. **Examples.** We now give some elementary examples indicating the relationships between conical, decomposing and archimedean faces. Let X be the closed convex set in the plane consisting of the unit square together with the disk $(x - 1/2)^2 + y^2 \leqq 1/4$. Let F be the face consisting of the line segment from $(0, 0)$ to $(0, 1)$. Then X is not decomposable at F since the only possible decomposing functional $(a, b) \rightarrow a$ does not work. Also F is not an archimedean face of X since the functional $(0, r) \rightarrow r$ on F cannot be extended nonnegatively to X. On the other hand X is conical at F under the functional $(a, b) \rightarrow a$.

It is possible for F to be archimedean without being a decomposing face. For example let X be the intersection in the plane of $(x - 1/2)^2 + y^2 \leqq 1/4$ and $x^2 + (y - 1/2)^2 \leqq 1/4$ and let F be the extreme point $\{(0, 0)\}$. Then X is not decomposable at $\{(0, 0)\}$ or even conical there since these notations coincide for F an extreme point. However X is archimedean at $\{(0, 0)\}$.

We next give an example of a closed face which is not self-determining. Let S be the set of nonnegative sequences in l^1 with norm less than or equal to one (weak* topology as dual of c_0). Let N be the subspace of sequences whose sum is zero and let F be a norm compact convex subset of the unit ball (containing 0) such that the norm-closed linear span of F is N. (Since N is separable there is a sequence (x_n) in N such that $\|x_n\| \rightarrow 0$ and $N = \langle x_n \rangle^-$. Let $F =$ norm cl-conv(x_n). Then F is norm compact by Krein's Theorem.) Let $X = \text{conv}(F \cup S)$. Since $S \subset X$, $A(X)$ consists exactly of all sequences in c_0 and their translates. But then F is a closed face of X whose linear span N is weak* dense in l^1. Hence $\langle F \rangle^- \cap X = X$. However X is nearly decomposable at F in the sense that the bounded linear function f on l^1 defined by $f(x) = \sum_{n=1}^{\infty} x_n$ is identically zero on F and

$$X = \text{conv}\,(F \cup \{x \in X : f(x) = 1\}) .$$

REFERENCES

1. E. M. Alfsen, *Facial structure of compact convex sets*, Proc. London Math. Soc. (3) **18** (1968), 385-404.

2. L. Asimow, *Directed Banach spaces of affine functions*, Trans. Amer. Math. Soc. **143** (1969), 117-132.

3. D. A. Edwards, *Séparation des fonctions réelles définies sur un simplexe de Choquet*, Comptes Rendus **261** (1965), 2798-2800.

4. R. R. Phelps, *Infinite dimensional compact convex polytopes*, Math. Scand. (to appear)

5. E. Størmer, *On partially ordered vector spaces and their duals, with applications to simplexes and C*-algebras*, Proc. London Math. Soc. (3) **18** (1968), 245-265.

Received August 25, 1969. This work was partially supported by NSF Grant GP-11475.

UNIVERSITY OF CALIFORNIA, LOS ANGELES

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 35, No. 1 September, 1970