DECOMPOSABLE SYMMETRIC TENSORS

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A $k$-field is a field over which every polynomial of degree less than or equal to $k$ splits completely. The main theorem characterizes the maximal decomposable subspaces of the $k$th symmetric space $V_k \times V$, where $V$ is finite-dimensional vector space over an infinite $k$-field. They come in three forms:

1. $\{x_1 \vee \cdots \vee x_k : x_i \in V\}, x_1, \cdots, x_{k-1}$ fixed;
2. $\langle a, b \rangle_k = \{x_1 \vee \cdots \vee x_k : x_i \in \langle a, b \rangle\}$; and
3. $\{x_1 \vee \cdots \vee x_{k-r} \vee \langle a, b \rangle_{r+1}, x_1, \cdots, x_{k-r}$ fixed;

where $a$ and $b$ are linearly independent vectors in $V$ and $\langle a, b \rangle$ is the subspace spanned by $a$ and $b$.

We consider symmetric tensor products of vector spaces and the problem of characterizing their maximal decomposable subspaces. This problem has been resolved in the skew-symmetric case by Westwick [4] using results due to Wei-Liang Chow [1, Lemma 5] when the underlying field is algebraically closed with characteristic zero.

A $k$-field is a field $F$ over which every polynomial of degree at most $k$ splits completely. In this paper we determine the maximal decomposable subspaces in the symmetric case when the underlying vector space is finite-dimensional over an infinite $k$-field whose characteristic (if any) exceeds the length of the product.

1. Let $F$ be a field and $V$ a vector space over $F$. The $k$-fold Cartesian product of $V$ will be denoted by $V^k$ where $1 < k$. A rank $k$ symmetric tensor space is a vector space together with a $k$-multi-linear symmetric mapping $\sigma$ which is universal for $k$-multilinear symmetric maps of $V^k$ and is spanned by $\sigma(V^k)$. We will use the notation $V_k V$ for this space. (The anti-symmetric or Grassman space is usually denoted by $\Lambda^k V$.)

If $V_k V$ with $\sigma: V_k \rightarrow V_k V$ is a symmetric tensor space, the decomposable symmetric tensors or "symmetric products" are those elements of $V_k V$ in the set $\sigma(V^k)$. We will denote $\sigma(x_1, \cdots, x_k)$ by $x_1 \vee \cdots \vee x_k$. A subspace $S$ of $V_k V$ is decomposable if $S \subseteq \sigma(V^k)$. Trivial decomposable subspaces are the zero subspace and those consisting of scalar multiples of a single product. The factors of the product $x_1 \vee \cdots \vee x_k$ are the 1-dimensional subspaces $\langle x_i \rangle, \cdots, \langle x_k \rangle$ of $V$.

If $V$ is $n$-dimensional, it is well-known that $V_k V$ is vector space isomorphic to the space of homogeneous polynomials of degree $k$ over $F$ [3, p. 428]. Any linear mapping $f: V \rightarrow V$ induces a unique linear mapping $V_k f: V_k V \rightarrow V_k V$ obtained by extending the mapping
$f^k : V^k \rightarrow V_k V$ defined by $f^k(x_1, \ldots, x_k) = f(x_1) \vee \cdots \vee f(x_k)$. This mapping will be denoted by simply $V_f$ when the length of the product is not in question.

**Proposition 1.** If $x$ and $y$ are decomposable symmetric tensors with $k-1$ common factors (counting repetitions), then $x + y$ is decomposable.

*Proof.* The mapping $\sigma$ is multilinear.

If $U$ is any subspaces of $V$ and $x_1, \ldots, x_k$ vectors of $V$ then 
\{ $x_1 \vee \cdots \vee x_k \vee u \mid u \in U$ \} is a decomposable subspace of $V_{k+1} V$ and will be denoted by $x_1 \vee \cdots \vee x_k \vee U$. Clearly,

$$x_1 \vee \cdots \vee x_k \vee U \subseteq x_1 \vee \cdots \vee x_k \vee V.$$ 

Decomposable subspaces of the form $x_1 \vee \cdots \vee x_{k-1} \vee V$ will be called type 1 subspaces.

2. Let $x$ be a product $x_1 \vee \cdots \vee x_k$ in $\sigma(V^k)$. If $w \in V$ then $w \vee x$ denotes the product $w \vee x_1 \vee \cdots \vee x_k$ in $\sigma(V^{k+1})$.

**Proposition 2.** If $D$ is a decomposable subspace of $V_k V$ then $w \vee D$ is a decomposable subspace of $V_{k+1} V$.

*Proof.* We will show that if $x + y = z \in \sigma(V^k)$ and $w \in V$ then $w \vee x + w \vee y = w \vee z$.

Define an injection $i : V^k \rightarrow V^{k+1}$ by 
$$i_w(v_1, \ldots, v_k) = (w, v_1, \ldots, v_k).$$ 

The universal property of $V_k V$ implies there is a unique linear $f : V_k V \rightarrow V_{k+1} V$ such that 
$$f(x_1 \vee \cdots \vee x_k) = w \vee x_1 \vee \cdots \vee x_k.$$ 

The desired result follows because $f$ is linear.

Clearly $f$ is injective. Moreover the image of a decomposable subspace of $V_k V$ under $f$ is decomposable.
Proposition 3. \( x_1 \lor \cdots \lor x_k = 0 \) if and only if some \( x_i = 0 \).

Proof. Suppose \( x_1, \ldots, x_k \) are nonzero vectors. Choose any basis \((e_i)_{i \in I}\) of \( V \) over a field \( F \). For each \( x_i \) assume the \( p_i^{th} \) coordinate to be nonzero. Let \( p = (p_1, \ldots, p_k) \). Define a multilinear and symmetric mapping \( f_p: V^k \to F \) by

\[
f_p(x_1, \ldots, x_k) = \alpha(1, p) \cdots \alpha(k, p_k)
\]

where each vector \( x_i \) has coordinates \( (\alpha(i, j))_{j \in J} \). Then \( f_p(x_1, \ldots, x_k) \) is nonzero and since \( f_p = \sigma \circ \overline{f}_p \), where \( \overline{f}_p \) is the extension of \( f_p \) to \( V^k \), \( x_1 \lor \cdots \lor x_k \) could not be zero.

Since \( \sigma \) is multilinear \( x_i = 0 \) for some \( i = 1, \ldots, k \) implies \( x_1 \lor \cdots \lor x_k = 0 \).

\( S_k \) denote the set of \( k! \) permutations of \( \{1, \ldots, k\} \).

Proposition 4. Let \( V \) be an \( n \)-dimensional vector space. The identity

\[
x_1 \lor \cdots \lor x_k = y_1 \lor \cdots \lor y_k \neq 0
\]

holds if and only if there is a \( \pi \in S_k \) and scalars \( \lambda_1, \ldots, \lambda_k \) such that

\[
\lambda_1 \lambda_2 \cdots \lambda_k = 1
\]

and

\[
x_i = \lambda_i y_{\pi(i)} \quad i = 1, \ldots, k.
\]

Proof. This is a result of the fact that the rank \( k \) symmetric tensor space is isomorphic to the \( k^{th} \) component of the polynomial algebra in \( n \) indeterminants over \( F \) [3, p. 428]. The latter is a unique factorization domain.

In what follows we will suppose \( x = x_1 \lor \cdots \lor x_k \) and \( y = y_1 \lor \cdots \lor y_k \) are independent products such that \( x + y \) is decomposable, say \( x + y = z_1 \lor \cdots \lor z_k \). We will often use the assumption that \( x \) and \( y \) are nonzero products without explicit mention. The subspace of \( V \) spanned by the vectors \( x_1, \ldots, x_k \) will be denoted \( [x] \) and its dimension by \( |x| \). For notational convenience we set

\[
x \cap y = [x] \cap [y]
\]

\[
x \cup y = [x] + [y].
\]

If \( S \) is a subspace of \( V \) then \( S_{(x)} \) is the set \( \{x_1 \lor \cdots \lor x_k | x_i \in S\} \). In general \( S_{(x)} \) is not a subspace. If \( U \) is a subspace of \( V \) then the one-dimensional subspace \( \langle v \rangle \) of \( V \) is a factor of \( U \) if

\[
U \subseteq v \lor V \lor \cdots \lor V.
\]
We will frequently denote a repeated product \( U \lor \cdots \lor U \) by \( U_{(b)} \).

**Remark.** If \( x + y = z \) it is always true that \( [z] \subseteq x \lor y \). For, if some \( z \in x \lor y \) and \( B \) is a basis of \( x \lor y \) we may choose \( f \in L(V, V) \) so that

\[
\begin{align*}
f(x_i) &= 0 \\
f(b) &= b \quad b \in B.
\end{align*}
\]

Then, \( x + y = (\lor f) z = 0 \), contradicting our standing assumption that \( x \) and \( y \) are independent.

**Proposition 5.** If \( B \) is a basis of \([y]\) and there are \( i, j \) such that \( B \cup \{x_i, z_j\} \) is an independent set then \( x \) and \( y \) have a common factor.

**Proof.** Choose \( f \in L(V, V) \) so that

\[
\begin{align*}
f(x_i) &= x_i \\
f(z_j) &= 0 \\
f(b) &= b \quad b \in B.
\end{align*}
\]

Then,

\[
f(x_i) \lor \cdots \lor x_i \lor \cdots \lor f(x_k) = - y_i \lor \cdots \lor y_k.
\]

Proposition 4 now implies \( \langle x_i \rangle \) is also a factor of \( y \).

**Proposition 6.** If \( x \) and \( y \) have no common factors and \([y] \not\subseteq [x]\) then for all \( i = 1, \ldots, k \)

\[
y_i \in [x] \text{ and } z_i \in [x].
\]

**Proof.** Let \( y_j \in [x] \). If \( B \) is any basis of \([x]\) we may complete the independent set \( B \cup \{y_j\} \) to a basis of \( V \). Consequently there is \( f \in L(V, V) \) such that

\[
\begin{align*}
f(y_j) &= 0 \\
f(b) &= b \quad b \in B.
\end{align*}
\]

If some \( z_i \in [x] \) we have

\[
x_i \lor \cdots \lor x_k = f(z_i) \lor \cdots \lor z_i \lor \cdots \lor f(z_k).
\]

Proposition 4 implies \( \langle z_i \rangle \) is then a factor of \( x \). The choice of any \( g \in L(V, V) \) with \( \ker g = \langle z_i \rangle \) together with Proposition 4 shows \( \langle z_i \rangle \) is also a factor of \( y \). We have shown that if \( x \) and \( y \) have no common factors then no \( z_i \in [x] \).
Choose some $z_i$ and complete the independent set $B \cup \{z_i\}$ to a basis. Define $h \in L(V, V)$ by
$$h(z_i) = 0$$
$$h(b) = b \quad b \in B.$$ 
Then
$$x_1 \lor \cdots \lor x_k = - h(y_1) \lor \cdots \lor h(y_k)$$
and we obtain a common factor whenever some $y_i \in [x]$ since then $h(y_i) = y_i$.

**Proposition 7.** If $B$ is any basis of $[y]$ and for some $i$ and $j$ $B \cup \{x_i, x_j\}$ is an independent set then $x$ and $y$ have a common factor.

**Proof.** Choose $f \in L(V, V)$ such that either $f(x_i) = 0$ or $f(x_j) = 0$ and $f(b) = b$ for every $b \in B$. Then
$$y_1 \lor \cdots \lor y_k = f(z_i) \lor \cdots \lor f(z_k).$$
If some $z_i \in [y]$ then it is a common factor. Assume no $z_i \in [y]$. We claim one of the following is the zero subspace:
$$[y] \cap \langle x_i, z_i \rangle$$
$$[y] \cap \langle x_j, z_i \rangle.$$ 
For, if both are nonzero there are scalars $\alpha, \beta$ such that
$$z_i = \alpha x_i + y' = \beta x_j + y'' \quad \text{where } y', y'' \in [y].$$
Hence,
$$\alpha x_i - \beta x_j \in [y].$$
Since $z_i \in [y]$, both $\alpha$ and $\beta$ are nonzero. But this violates the hypothesis. If $[y] \cap \langle x_i, z_i \rangle = 0$ we apply Proposition 5 to $B \cup \{x_i, z_i\}$ and conclude $x$ and $y$ have a common factor.

3. $F$ is a $k$-field if every polynomial over $F$ of degree at most $k$ splits completely over $F$. Let $L_k$ denote $\{x \in V : x = 1\}$. $L_k$ is composed of all products $\alpha x_1 \lor \cdots \lor x_i$ where $\alpha \in F$, $x_i \in V$. If $F$ is a $k$-field then in particular
$$\alpha x_1 \lor \cdots \lor x_i = (\alpha^{1/k} x_1) \lor \cdots \lor (\alpha^{1/k} x_i).$$
However $L_k$ need not be a subspace unless $k = p^r$ where $r$ is a positive
integer and \( p \) is the prime characteristic of \( F \). That it is a subspace in this case is apparent because \( \binom{p^k}{m} \) for \( m = 1, \ldots, p^k - 1 \) and so

\[
x_1 \lor \cdots \lor x_1 + y_1 \lor \cdots \lor y_1 = (x_1 + y_1) \lor \cdots \lor (x_1 + y_1).
\]

**Proposition 8.** If \( F \) has prime characteristic \( p \) and \( k = p^r, r \) a positive integer, then \( \dim L_k = \dim V \).

*Proof.* Under these conditions it is not difficult to show that \( x_1, \ldots, x_m \) are linearly independent in \( V \) if and only if \( x_1 \lor \cdots \lor x_1, \ldots, x_m \lor \cdots \lor x_m \) are linearly independent in \( L_k \).

**Proposition 9.** \( L_k \) is a decomposable subspace if and only if \( F \) has characteristic \( p \) and \( k = p^m, m \) a positive integer.

*Proof.* We have seen that this condition is sufficient. If \( u, v \) are independent vectors in \( V \) then \( u_{(k)} = u \lor \cdots \lor u, v_{(k)} = v \lor \cdots \lor v \) are in \( L_k \) and part of a basis for \( V_k V \) by Proposition 8. Since \( L_k \) is decomposable there is a nonzero scalar \( \gamma \) and vector \( w \) such that

\[
u_{(k)} + v_{(k)} = \gamma w_{(k)}.
\]

The remark preceding Proposition 5 implies there are scalars \( \alpha, \beta \) such that \( w = \alpha u + \beta v \). By induction,

\[
w_{(k)} = \alpha^k u_{(k)} + \binom{k}{1} \alpha^{k-1} u_{(k-1)} \lor v + \cdots
\]

\[
+ \binom{k}{r} \alpha^{k-r} \beta^r u_{(k-r)} \lor v_{(r)} + \cdots
\]

\[
+ \beta^k v_{(k)}.
\]

Since the products \( u_{(k-r)} \lor v_{(r)} \) are part of a basis of \( V_k V \) we obtain

\[
\gamma \alpha^k = \gamma \beta^k = 1
\]

\[
\gamma \binom{k}{r} \alpha^{k-r} \beta^r = 0 \quad r = 1, \ldots, k-1.
\]

Because both \( \alpha \) and \( \beta \) are nonzero \( \alpha^{k-r} \beta^r \) is and so

\[
\binom{k}{r} \cdot 1 = 0 \quad r = 1, \ldots, k-1.
\]

Hence \( F \) has characteristic \( p \) and

\[
p \left| \binom{k}{r} \right| \quad r = 1, \ldots, k-1.
\]
It is not difficult to show that this implies \( k \) is a power of \( p \).

4. If \( a \) and \( b \) are two independent vectors in \( V \) then the set

\[ \{ x_1 \lor \cdots \lor x_k \mid x_i \in \langle a, b \rangle \} \]

is denoted by \( \langle a, b \rangle_{(k)} \). Let \( F[\alpha] \) denote the polynomial algebra in one variable over \( F \) and define a linear mapping \( g : \langle a, b \rangle \to F[\alpha] \) by \( g(a) = \alpha, \ g(b) = 1 \). If \( f : V \to \langle a, b \rangle \) is a projection on \( \langle a, b \rangle \) then \( V_k g \circ f : V_k V \to F[\alpha] \) is a linear mapping obtained by extending \((g \circ f)^k : V^k \to F[\alpha] \) defined by

\[
(g \circ f)^k (v_1, \cdots, v_k) = \prod_{i=1}^{k} g \circ f(v_i). \quad v_i \in V.
\]

If

\[
t = \prod_{i=0}^{k} \gamma_i a_{(k-i)} \lor b_i \quad \gamma_i \in F
\]

is any element of \( \langle a, b \rangle_{(k)} \) then

\[
(\mathbf{V}_k g \circ f) t = n_0 + n_1 \alpha + \cdots + n_k \alpha^k.
\]

The equality (2) implies that the restriction of \( \mathbf{V}_k g \circ f \) to \( \langle a, b \rangle_{(r)} \) is a linear isomorphism onto \( F[\alpha] \) which preserves “products”, i.e., a decomposable tensor corresponds to a product of \( k \) linear polynomials.

**Proposition 10.** \( F \) is a \( k \)-field if and only if each \( \langle a, b \rangle_{(k)} \) is a decomposable subspace of \( \mathbf{V}_k V \).

**Proof.** Assume \( F \) is a \( k \)-field. If \( x \) and \( y \) are products in \( \langle a, b \rangle_{(k)} \) let \( P(\alpha) = (\mathbf{V}_k g \circ f) (x + y) \). There are elements \( r_i \) in \( F \) such that \( P(\alpha) = r_0 (\alpha - r_1) \cdots (\alpha - r_k) \). Consider

\[
z = r_0 (a - r_1 b) \lor \cdots \lor (a - r_k b) \in \langle a, b \rangle_{(k)}.
\]

Clearly, \( P(\alpha) = \mathbf{V}_k(g \circ f)z \) which implies \( x + y = z \) because the restriction of \( \mathbf{V}_k g \circ f \) to \( \langle a, b \rangle_{(k)} \) is injective. Therefore \( \langle a, b \rangle_{(k)} \) is decomposable.

Conversely if \( \langle a, b \rangle_{(k)} \) is decomposable and

\[
P(\alpha) = \gamma_0 + \gamma_1 \alpha + \cdots + \gamma_k \alpha^k \in F[\alpha]
\]

then (2) implies \( P(\alpha) = (\mathbf{V}_k g \circ f) t \) for some \( t \in \langle a, b \rangle_{(k)} \).

But \( t \) is a product, say

\[
t = (\lambda_1 a + \mu_1 b) \lor \cdots \lor (\lambda_k a + \mu_k b).
\]

Hence

\[
P(\alpha) = (\lambda_1 + \mu_1 \alpha) \cdots (\lambda_k + \mu_k \alpha).
\]
Lemma 11. If $F$ is infinite and $\langle x, y \rangle \subseteq \sigma (V^k)$ then $|x| > 2$ implies $x$ and $y$ a common factor.

Proof. Assume $x_1, x_2, x_3$ are independent and are contained in a basis $B$ of $V$. For every $\lambda \in F$ there is a product $z(\lambda) = z_1(\lambda) \lor \cdots \lor z_k(\lambda)$ such that $x + \lambda y = z(\lambda)$. Define three linear mappings of $V$ by

$$f_i(x_i) = 0 \quad i = 1, 2, 3$$

$$f(b) = b \in B - \{x_1, x_2, x_3\}$$

Extending each mapping to $V_kV$ we obtain for each $\lambda \in F$:

$$\langle Vf_i \rangle y = \langle Vf_i \rangle z(\lambda) \quad i = 1, 2, 3.$$  

If (3) is zero for some $i$ we infer from Proposition 3 that $f_i(y_j) = 0$ for some $j = 1, \cdots, k$. This means that $\langle x_1 \rangle = \langle y_j \rangle$ is a common factor of $x$ and $y$. For each $\lambda$, the vectors $z_1(\lambda), \cdots, z_k(\lambda)$ may be chosen so that (3) and Proposition 4 imply

$$f_i(y_j) = f_i(z_j(\lambda)) \quad j = 1, \cdots, k.$$  

Let $z_1(\lambda)$ and $y_j$ have coordinates $(\alpha_{ib}(\lambda): b \in B)$ and $(\beta_{ib}: b \in B)$ respectively. For each $\lambda \in F$ (4) implies

$$\alpha_{jb}(\lambda) = \beta_{jb} \quad b \neq x_1.$$  

If $i = 2$ then (3) and Proposition 4 implies for each $\lambda \in F$

$$f_2(z_j(\lambda)) = c_j(\lambda) f_2(y_{\pi j}) \quad j = 1, \cdots, k.$$  

where $\pi \in S_k$ and the $c_j(\lambda)$ are scalars such that $\prod_{j=1}^k c_j(\lambda) = 1$.

Therefore,

$$\alpha_{jb}(\lambda) = c_j(\lambda) \beta_{z_{(j) j}} b \neq x_2 \quad j = 1, \cdots, k.$$  

If for some $j$, $\alpha_{jb}(\lambda) = 0$ for every $b \neq x_2$ then $\langle z_{(j) j} \rangle = \langle \omega \rangle$ is a common factor of $x$ and $z(\lambda)$; hence a common factor of $x$ and $y$. Accordingly, we may assume for each $j$ there is a basis element $\beta_{z_{(j) j}} b \neq x_2$ such that $\beta_{z_{(j) j}} b \neq x_2$. If for some $j$ $b(j) \neq x_1$ as well, then (5) and (6) imply

$$c_j(\lambda) = \beta_{j b(j)}^{-1} \beta_{z_{(j) j}}.$$  

On the other hand, suppose $b(j) = x_1$ for some $j$ and $\beta_{z_{(j) j}} b = 0$ for all $b$ distinct from $x_1$ and $x_2$. From (3) with $i = 3$ we obtain

$$\alpha_{jb}(\lambda) = d_j(\lambda) \beta_{\omega j} \quad j = 1, \cdots, k.$$  

where $\omega \in S_2$ and the $d_j(\lambda)$ are scalars such that $\prod_{j=1}^k d_j(\lambda) = 1$. 
Were $\beta_{\omega(j)x_2} = 0$ then $\langle z_j(\lambda) \rangle = \langle x_1 \rangle$ would be a common factor of $x$ and $z(\lambda)$, hence a factor of $y$ as well. If $\beta_{\omega(j)x_2} \neq 0$ then (5) together with $b = x_2$ in (8) imply

$$d_j(\lambda) = \beta_{jx_2} \beta_{\omega(1)x_2}^{-1} \cdot (9)$$

From (5) we know that for any $\lambda \in F$ all coordinates of $z(\lambda)$ except $b = x_1$ are in the finite set $C_1 = \{\beta_{jx_1} : j = 1, \ldots, k; \ b \in B\}$. For each $i = 1, \ldots, k$ we have from (6)

$$\alpha_{jx_1}(\lambda) = c_j(\lambda) \beta_{x(j)x_1}$$

and from (8) we obtain

$$\alpha_{jx_2}(\lambda) = c_j(\lambda) \beta_{z(j)x_1} \cdot (10)$$

Now if $b(j) \neq \omega_1$ then (7) and (10) imply

$$\alpha_{jx_2}(\lambda) = \beta_{jx_2} \beta_{\omega(1)x_2}^{-1} \beta_{x(j)x_1} \beta_{\omega(j)x_2} \cdot$$

and if $b(j) = x_1$ then (8) and (9) imply

$$\alpha_{jx_2}(\lambda) = \beta_{jx_2} \beta_{\omega(1)x_2}^{-1} \beta_{x(j)x_1}.$$

We conclude that for any $\lambda \in F$ the coordinates of each $z_j(\lambda)$ are contained in the finite set

$$C_1 \cup \{\beta_{jx_1} \beta_{\omega(1)x_2}^{-1} \beta_{x(j)x_1}, \beta_{jx_2} \beta_{\omega(j)x_2} \beta_{x(j)x_1} : j = 1, \ldots, k\}.$$

Accordingly, the number of vectors $z_j(\lambda)$ is finite and there are only a finite number of distinct products $z(\lambda) = z_1(\lambda) \lor \cdots \lor z_k(\lambda)$. But $F$ is infinite. Hence there are distinct scalars $\lambda, \lambda'$ such that $x + \lambda y = x + \lambda' y$ which implies $y = 0$. This contradicts our standing assumption that $x$ and $y$ are nonzero products and completes the proof.

We need the following lemma in order to prove Theorem 13.

**Lemma 12.** Let $V$ be a finite-dimensional vector space over a field $F$ and $\mathcal{C}$ any collection of proper subspaces of $V$. If $V = \bigcup \mathcal{C}$ then $\text{Card } F \leq \text{Card } \mathcal{C}$.

**Proof.** When $\dim V = 1$, $V$ has no proper subspaces and the conclusion is vacuously true.

If $b_1, \ldots, b_n$ is any basis of $V$ denote the $(n-1)$-dimensional subspace $\langle b_1, \ldots, b_{n-2}, b_{n-1} + \lambda b_n \rangle$ by $S$, where $\lambda$ is a scalar. Then $\text{Card } \{S_1 : \lambda \in F\} = \text{Card } F$. For, if $S_1 = S_2$, then in particular

$$b_{n-1} + \lambda b_n = \alpha_1 b_1 + \cdots + \alpha_{n-2} b_{n-2} + \alpha_{n-1} (b_{n-1} + \lambda' b_n)$$

DECOMPOSABLE SYMMETRIC TENSORS

73
for some scalars \( \alpha_1, \ldots, \alpha_{n-1} \). Thus \( \alpha_i = 0 \) for \( i = 1, \ldots, n-2 \) and \( \alpha_{n-1} = 1 \) which implies \( \lambda = \lambda' \).

Consider \( C_2 = \{ S_i \cap T : T \in \mathcal{C} \} \). Because \( V = \bigcup \mathcal{C} \) we have \( S_i = \bigcup \mathcal{C}_2 \). The set mapping from \( \mathcal{C} \) to \( \mathcal{C}_2 \) defined by \( T \rightarrow S_i \cap T \) is onto. Consequently, \( \text{Card } C_i \leq \text{Card } \mathcal{C} \). Since \( \dim S_i = n-1 \) induction yields \( \text{Card } F \leq \text{Card } \mathcal{C}_2 \), completing the proof.

If \( D \) is a decomposable subspace of \( V_k V \) and \( v \in V \) then \( D(v) \) denotes \( \{ t \in D : \langle v \rangle \text{ is a factor of } t \} \). Any \( D(v) \) is a subspace of \( D \) and is the zero subspace when \( v \) is a factor of no product in \( D \). A nontrivial decomposable subspace can have at most \( k-1 \) factors. We have already remarked that any decomposable subspace with exactly \( k-1 \) factors (counting repetitions) is contained in a type 1 subspace. At the other extreme we have:

**Lemma 13.** If \( V \) is finite dimensional over an infinite \( k \)-field either without characteristic or with characteristic \( p > k \) then the only maximal nontrivial decomposable subspaces of \( V_k V \) without factors are those of the form \( \langle a, b \rangle \).

**Proof.** Let \( D \) be a maximal decomposable subspace without factors. If \( \text{Char } F = p \) then Proposition 8 and \( p > k \) imply \( L_k \) is not a subspace. Thus, we can assume \( D \neq L_k \); i.e., \( D \) contains at least one product \( x \) with \( |x| > 1 \). We proceed by showing first that \( D \) cannot contain a product \( x \) with \( |x| > 2 \):

Assume, on the contrary, that \( x = x_1 \lor \cdots \lor x_k \) is such a product of \( D \).

For every product \( y \in D \) we have \( \langle x, y \rangle \leq D \leq \sigma (V^k) \). Lemma 11 implies each nonzero \( y \in D \) must have a factor in common with \( x \). Hence \( D = \bigcup_{i=1}^{k-1} D(x_i) \), where each \( D(x_i) \) must be a proper subspace since \( D \) is without factors. Since \( V \) is finite-dimensional Lemma 12 implies \( \text{Card } F < k \), contrary to hypothesis. Accordingly \( |x| \leq 2 \) for every \( x \in D \). Since \( D \) is not \( L_k \), \( D \) contains a product \( x \) with \( |x| = 2 \). In what follows we suppose \( x_1, x_2 \) are independent.

Were \( y \in D \) and \( |y| = 1 \) then \( y = \alpha y_1 \lor \cdots \lor y_i \). If \( y_i \in [x] \) Proposition 7 implies \( x \) and \( y \) have a common factor and so \( y_i \in \langle x \rangle \), a contradiction. Therefore \( [y] \subseteq [x] \) for every \( y \in D \) with \( |y| = 1 \).

Suppose \( y \in D \), \( |y| = 2 \) but \( [y] \nsubseteq [x] \). The rest of the proof is in two parts and we consider first such \( y \) with no factors in common with \( x \):

Complete \( x_1, x_2 \) to a basis \( B \) and define \( f \in L(V, V) \) by

\[
\begin{align*}
  f(x_i) &= x_i & i &= 1, 2 \\
  f(b) &= b & b & \in B - \{x_1, x_2\}.
\end{align*}
\]
Were \((V_x)y = 0\) then some \(y_i \in [x]\), contrary to Proposition 6. If \(|(V_x)y| = 1\) then

\begin{equation}
ax_i \lor \cdots \lor x_i + \beta f(y_i) \lor \cdots \lor f(y_k) = (V_x) z \neq 0
\end{equation}

would imply (as in §3) that the underlying field has characteristic \(p\) and \(k = p^r\) for some prime \(p\) and positive integer \(r\), contrary to hypothesis. (If \((V_x)z = 0\) then some \(z_i \in [x]\), again contradicting Proposition 6.) The remaining alternative is \(|(V_x)y| = 2\). Since we are assuming \(x\) and \(y\) have no common factors, (12) and Proposition 7 imply for some \(i = 1, \ldots, k\)

\begin{equation}
\langle x_i \rangle = \langle f(y_i) \rangle.
\end{equation}

But (11) and (13) imply \(y_i \in [x]\), a contradiction of Proposition 6 again.

It remains to consider those \(y \in D\) with \(|y| = 2\) which have factors in common with \(x\). If for such \(y\), \([y] \neq [x]\) then \(x \cap y\) is 1-dimensional. Let \(x + y = \langle u \rangle\) and assume \(\langle u \rangle\) occurs at least \(r\) times as a factor of both \(x\) and \(y\). Consider the products

\begin{align*}
\bar{x} &= x_1 \lor \cdots \lor x_{k-r} \\
\bar{y} &= y_1 \lor \cdots \lor y_{k-r}
\end{align*}

in \(\sigma(V_{k-r})\). We may suppose that \(\bar{x}\) and \(\bar{y}\) have no common factors. Since \(x + y \in \sigma(V^k)\) and iterations of the mapping \(f\) in (0) are also injective we have \(\bar{x} + \bar{y} \in \sigma(V_{k-r})\). If either \(|\bar{x}| = 2\) or \(|\bar{y}| = 2\) then Lemma 10 implies

\begin{equation}
[\bar{x}] \subseteq [\bar{y}]
\end{equation}

or

\begin{equation}
[\bar{y}] \subseteq [\bar{x}].
\end{equation}

Either statement in (14) implies \([x] = [y]\).

If \(|\bar{x}| = |\bar{y}| = 1\) then either \([\bar{x}] = [\bar{y}]\) or \(\bar{x} \cap \bar{y} = 0\). We will show \(\bar{x} \cap \bar{y} = 0\) is contradictory:

\begin{align*}
\bar{x} &= \alpha x_1 \lor \cdots \lor x_i = (\alpha^{i_1} x_i) \lor \cdots \lor (\alpha^{i_r} x_i) \\
\bar{y} &= \beta y_1 \lor \cdots \lor y_i = (\beta^{i_1} y_i) \lor \cdots \lor (\beta^{i_r} y_i).
\end{align*}

This is possible since \(F\) is an \(r\)-field for every positive \(r \leq k\). Replace \(u\) and \(v\) by \(\alpha^{i_r} x_i\) and \(\beta^{i_r} y_i\) in (1). Then Char \(F\) is a prime \(p\) and \(r = p^m\) for some positive integer \(m\). But by hypothesis \(p > k > r\), a contradiction.

We conclude \([y] \subseteq [x]\) in all cases. Thus, \(D \subseteq \langle a, b \rangle_{(k)}\) where \({a, b}\) is any basis of \([x]\). Since \(D\) was assumed maximal the proof is complete.
THEOREM. If $V$ is finite-dimensional over an infinite $k$-field $F$ either without characteristic or with characteristic $p > k$ then the maximal nontrivial decomposable subspaces of $\mathbf{V}_k V$ are:

(i) type 1 subspaces

and for every independent pair of vectors $a, b$ in $V$:

(ii) $\langle a, b \rangle_{(k)}$

(iii) $x_i \vee \cdots \vee x_{k-r} \vee \langle a, b \rangle_{(r)}$ where $x_i \in \langle a, b \rangle$ for every $i = 1, \ldots, k-r$ and $1 < r < k$.

Proof. Lemma 13 states that the only decomposable subspace without factors are those of the form (ii). The image of a decomposable subspace under the mapping $f$ in (0) is a decomposable subspace with at least one factor. Iterations of $f$ in (0) yield decomposable subspaces in spaces of greater length. Thus, when $F$ is a $k$-field, $\langle a, b \rangle_{(r)}$ is a decomposable subspace of $\mathbf{V}_r V$ for every $1 < r < k$ and subspaces of the form

$$x_i \vee \cdots \vee x_{k-r} \vee \langle a, b \rangle_{(r)}$$

are decomposable. If $x_{k-r}$, say, is in $\langle a, b \rangle$ then

$$x_i \vee \cdots \vee x_{k-r} \vee \langle a, b \rangle_{(r)} \subseteq x_i \vee \cdots \vee x_{k-r-1} \vee \langle a, b \rangle_{(r+1)}.$$

Accordingly, subspaces of this type could be maximal only when $x_i \in \langle a, b \rangle$ for each $i = 1, \ldots, k-r$.

Conversely, if a decomposable subspace has exactly $k-r$ factors it is the image of a decomposable subspace of $\mathbf{V}_r V$ without factors under a composition of $k-r$ mappings $f$ in (0). Lemma 13 states that subspace must be of the form $\langle a, b \rangle_{(r)}$. Hence (ii) and (iii) are the only types of decomposable subspaces with factors.

Routine arguments show that a space of one type cannot be properly contained in another of the same type or a different type. Since every decomposable subspace is contained in a maximal decomposable subspace the proof is completed.

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<table>
<thead>
<tr>
<th>Author(s)</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>B. D. Arendt and C. J. Stuth</td>
<td>On the structure of commutative periodic semigroups</td>
</tr>
<tr>
<td>B. D. Arendt and C. J. Stuth</td>
<td>On partial homomorphisms of semigroups</td>
</tr>
<tr>
<td>Leonard Asimow</td>
<td>Extensions of continuous affine functions</td>
</tr>
<tr>
<td>Claude Elias Billigheimer</td>
<td>Regular boundary problems for a five-term recurrence relation</td>
</tr>
<tr>
<td>Edwin Ogilvie Buchman and F. A. Valentine</td>
<td>A characterization of the parallelepiped in $E^n$</td>
</tr>
<tr>
<td>Victor P. Camillo</td>
<td>A note on commutative injective rings</td>
</tr>
<tr>
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<td>Decomposable symmetric tensors</td>
</tr>
<tr>
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<td>On matrices with a restricted number of diagonal values</td>
</tr>
<tr>
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</tr>
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</tr>
<tr>
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</tr>
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<td>The hypo residuum of the automorphism group of an abelian $p$-group</td>
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<td>On a certain generalization of $p$ spaces</td>
</tr>
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</tr>
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</tr>
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</tr>
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</tr>
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</tr>
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</tr>
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</tr>
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<td>A characterization of the nil radical of a ring</td>
</tr>
</tbody>
</table>