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ON A CERTAIN GENERALIZATION OF p SPACES

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An \mathcal{E}_p space is a product of finite-dimensional c_p spaces with a weighted ℓ_p norm on the product. The first theorem of this paper yields an isometric embedding of \mathcal{E}_p into an appropriate c_p space. From this theorem, known results about c_p are used to deduce, among other things, the Clarkson inequalities for \mathcal{E}_p , $1 < p < \infty$, and hence, the uniform convexity of \mathcal{E}_p for $1 < p < \infty$.

The second theorem characterizes the conjugate space of \mathcal{E}_p for $0 < p < 1$. This result is then used to describe some spaces of multipliers. Let \mathcal{A} and \mathcal{B} be \mathcal{E}_p spaces, $1 \leq p \leq \infty$, or \mathcal{E}_0 . The spaces $\mathcal{M}(\mathcal{A}, \mathcal{B})$ of multipliers from \mathcal{A} to \mathcal{B} have previously been identified with certain subspaces of $\mathcal{E}(I)$ and determined precisely in some cases. The third theorem is a complete description of these multiplier spaces: the cases $0 < p < 1$ are included and the spaces $\mathcal{M}(\mathcal{A}, \mathcal{B})$ are determined precisely for all pairs \mathcal{A}, \mathcal{B} .

1. Definitions. First, we repeat the definition of c_p (called C_p by Dunford and Schwartz [1], S_p by Gohberg and Krein [2], and c_p by McCarthy [6]). See also [3, D. 37] for the case where H is finite-dimensional.

DEFINITION 1.1. Let H be a Hilbert space and let X be a compact operator on H . Then XX^* is positive and compact and hence has a unique positive square root which is also compact. We denote this square root by $|X|$. Now let μ_n be the, at most countably many, nonzero eigenvalues of $|X|$ enumerated with their multiplicity and arranged in a decreasing sequence as $\mu_1 \geq \mu_2 \geq \dots \geq 0$. For $0 < p < \infty$, we define

$$\|X\|_{\phi_p} = \left(\sum_{n=1}^{\infty} \mu_n^p \right)^{1/p}$$

whether finite or infinite; and we define

$$\|X\|_{\phi_{\infty}} = \sup \{ \mu_n : 1 \leq n < \infty \} = \mu_1.$$

Equivalently, [1, p. 1089], $\|X\|_{\phi_{\infty}}$ is the operator norm of X . Then c_p consists of all compact X with $\|X\|_{\phi_p}$ finite.

See [1], [2], and [6] for a detailed treatment of c_p spaces and for additional references. Also, [3, Appendix D] contains a number of results in case H is finite-dimensional.

We proceed to define \mathcal{E}_p spaces. These spaces were introduced by R. A. Kunze [5] primarily for the purpose of having analogues of \mathcal{L}_p spaces in the study of harmonic analysis on compact non-Abelian groups. They have been studied and exploited for this purpose especially by Hewitt and Ross [3].

DEFINITION 1.2. Let I be an index set. For each $\iota \in I$, let H_ι be a finite-dimensional Hilbert space and let $a_\iota \geq 1$. We let $\mathcal{E}(I)$ denote the $*$ -algebra $\prod_{\iota \in I} \mathcal{B}(H_\iota)$ with all operations defined coordinatewise. Let $E = (E_\iota)_{\iota \in I} \in \mathcal{E}(I)$. For $0 < p < \infty$, we define

$$\|E\|_p = \left(\sum_{\iota \in I} a_\iota \|E_\iota\|_{\phi_p}^p \right)^{1/p};$$

we also define

$$\|E\|_\infty = \sup \{ \|E_\iota\|_{\phi_\infty} : \iota \in I \}.$$

For $0 < p \leq \infty$, $\mathcal{E}_p(I)$ is defined to be the set of all $E \in \mathcal{E}(I)$ for which $\|E\|_p$ is finite. In addition, $\mathcal{E}_{00}(I)$ is the set of $E \in \mathcal{E}(I)$ for which $\{\iota \in I : E_\iota \neq 0\}$ is finite; and $\mathcal{E}_0(I)$ is the set of $E \in \mathcal{E}(I)$ for which $\{\iota \in I : \|E_\iota\|_{\phi_\infty} \geq \varepsilon\}$ is finite for all $\varepsilon > 0$. Frequently we write \mathcal{E}_p in place of $\mathcal{E}_p(I)$. We notice that if each H_ι is one-dimensional, then $\mathcal{E}_p(I)$ is just the $\{a_\iota\}$ -weighted \mathcal{L}_p space which we will call L_p ; namely, $\{c_\iota\}_{\iota \in I} \in L_p$ if and only if $c_\iota \in K$ for each $\iota \in I$ and $\|c\|_p = (\sum_{\iota \in I} a_\iota |c_\iota|^p)^{1/p} < \infty$. In addition, if each $a_\iota = 1$, then $\mathcal{E}_p(I)$ is just $\mathcal{L}_p(I)$. Also, it is convenient to think of \mathcal{E}_p as a product of c_p spaces with a weighted \mathcal{L}_p norm on the product.

2. An embedding theorem and some consequences. In Hewitt and Ross [3], several basic facts about \mathcal{E}_p for $1 \leq p \leq \infty$ are proved. There it is shown that Hölder's inequality, Minkowski's inequality and certain generalizations of these hold. The major result of this section is (2.2), a theorem describing a linear isometry of \mathcal{E}_p onto a subspace of an appropriate c_p space. The theorem is then used to derive a number of inequalities for \mathcal{E}_p from results known about c_p . We begin with a description of the setting.

Let I be an index set and let H_ι be a finite-dimensional Hilbert space for each $\iota \in I$. Also, let $a_\iota \geq 1$ for each $\iota \in I$. For $0 < p \leq \infty$, $\|E\|_p$ and \mathcal{E}_p will be as in (1.2). Now from the Hilbert space direct sum $\bigoplus_{\iota \in I} H_\iota$; namely

$$\bigoplus_{\iota \in I} H_\iota = \left\{ \{\xi_\iota\} \in \prod_{\iota \in I} H_\iota : \sum_{\iota \in I} \|\xi_\iota\|^2 < \infty \right\}$$

with addition and scalar multiplication defined coordinatewise and with

an inner product defined by $\langle \{\xi_i\}, \{\eta_i\} \rangle = \sum_{i \in I} \langle \xi_i, \eta_i \rangle$. It is well known that $\bigoplus_{i \in I} H_i$ is a Hilbert space under these definitions.

DEFINITION 2.1. Let $0 < p < \infty$ and let $E = (E_i)_{i \in I} \in \mathcal{E}_p$. Define $T_p(E) = T_E$ where $T_E(\{\xi_i\}) = \{a_i^{1/p} E_i(\xi_i)\}$ for all $\{\xi_i\} \in \bigoplus_{i \in I} H_i$. If $p = \infty$ and $E \in \mathcal{E}_\infty$, let $T_\infty(E) = T_E$ where $T_E(\{\xi_i\}) = \{E_i(\xi_i)\}$.

If $p = \infty$, it is known that $T_E \in \mathcal{B}(\bigoplus_{i \in I} H_i)$ and $\|T_E\| = \|E\|_\infty$. In general we have the following theorem.

THEOREM 2.2. Let $0 < p < \infty$ and let T_p be defined as above. Then T_p is a linear, $*$ -preserving isometry of $\mathcal{E}_p(I)$ onto the subspace $e_p = \{T \in c_p(\bigoplus_{i \in I} H_i) : H_i \text{ is invariant under } T \text{ for all } i \in I\}$ of $c_p(\bigoplus_{i \in I} H_i)$.

Proof. First, let $\xi = \{\xi_i\} \in \bigoplus_{i \in I} H_i$ so that $T_E(\{\xi_i\}) = \{a_i^{1/p} E_i \xi_i\}$ for $E = (E_i)_{i \in I} \in \mathcal{E}_p$. Then using [1, p. 1093, 9 (a)] to obtain the second inequality below, we have

$$\begin{aligned} \|T_E(\{\xi_i\})\|^2 &= \sum_{i \in I} \|a_i^{1/p} E_i(\xi_i)\|^2 \\ &\leq \sum_{i \in I} a_i^{2/p} \|E_i\|_{\phi_\infty}^2 \|\xi_i\|^2 \\ &\leq \sum_{i \in I} a_i^{2/p} \|E_i\|_{\phi_p}^2 \|\xi_i\|^2 \\ &= \sum_{i \in I} (a_i \|E_i\|_{\phi_p}^p)^{2/p} \|\xi_i\|^2 \\ &\leq \sum_{i \in I} \|E\|_p^2 \|\xi_i\|^2 \\ &= \|E\|_p^2 \|\xi\|^2. \end{aligned}$$

Therefore, $T_E(\{\xi_i\}) \in \bigoplus_{i \in I} H_i$ and $\|T_E(\{\xi_i\})\| \leq \|E\|_p \|\xi\|$. Also, T_E is clearly linear. Hence, $T_E \in \mathcal{B}(\bigoplus_{i \in I} H_i)$ and $\|T_E\| \leq \|E\|_p$. It is easy to check that T_p is linear and $*$ -preserving.

We must now see that T_E is compact for $E \in \mathcal{E}_p$. Since $E \rightarrow T_E$ is continuous and \mathcal{E}_{00} is dense in \mathcal{E}_p , we need only note that T_E is compact for $E \in \mathcal{E}_{00}$. This is obvious since T_E has finite-dimensional range for $E \in \mathcal{E}_{00}$.

To see that T_p is an isometry, we make the following observation. Suppose $\{\phi_\lambda^j : j = 1, 2, \dots, d_\lambda\}$ is an orthonormal basis for H_λ of dimension d_λ for each $\lambda \in I$. For each $\lambda \in I$ and $j = 1, 2, \dots, d_\lambda$, let $\phi^{\lambda,j} = (\phi_i^{\lambda,j})_{i \in I} \in \bigoplus_{i \in I} H_i$ be defined by

$$\phi_i^{\lambda,j} = \begin{cases} \phi_\lambda^j & \text{if } i = \lambda \\ 0 & \text{if } i \neq \lambda. \end{cases}$$

Then it is easy to see that $\{\phi^{\lambda,j} : \lambda \in I, j = 1, 2, \dots, d_\lambda\}$ is an orthonormal basis for $\bigoplus_{i \in I} H_i$. Now, let $E \in \mathcal{E}_p$ and let $\{\beta_\lambda^{(j)} : j = 1, 2, \dots, d_\lambda\}$ be

the eigenvalues of $|E_\lambda|$ for each $\lambda \in I$. For each $\lambda \in I$, we choose $\{\phi_\lambda^j: j = 1, 2, \dots, d_\lambda\}$ to be an orthonormal basis for H_λ consisting of eigenvectors corresponding to the eigenvalues $(\beta_\lambda^{(j)})^2$ of $E_\lambda E_\lambda^*$; that is, $E_\lambda E_\lambda^* \phi_\lambda^j = (\beta_\lambda^{(j)})^2 \phi_\lambda^j$. Letting $\phi^{\lambda,j}$ be as above, we have that $T_E T_E^* \phi^{\lambda,j} = T_E T_E^* \phi^{\lambda,j} = \{\eta_\iota\}_{\iota \in I}$ where $\eta_\iota = a_\lambda^{2/p} E_\lambda E_\lambda^* \phi_\lambda^j = a_\lambda^{2/p} (\beta_\lambda^{(j)})^2 \phi_\lambda^j$, if $\iota = \lambda$ and $\eta_\iota = 0$ for $\iota \neq \lambda$. That is, $T_E T_E^* \phi^{\lambda,j} = (a_\lambda^{1/p} \beta_\lambda^{(j)})^2 \phi^{\lambda,j}$; or $\{\phi^{\lambda,j}: \lambda \in I, j = 1, \dots, d_\lambda\}$ is an orthonormal basis for $\bigoplus_{\iota \in I} H_\iota$ consisting of eigenvectors corresponding to the eigenvalues $(a_\lambda^{1/p} \beta_\lambda^{(j)})^2$ of $T_E T_E^*$. Hence, by definition, we have

$$\begin{aligned} \|T_E\|_p^p &= \sum_{\substack{\lambda \in I \\ j=1,2,\dots,d_\lambda}} (a_\lambda^{1/p} \beta_\lambda^{(j)})^p \\ &= \sum_{\lambda \in I} a_\lambda \sum_{j=1}^{d_\lambda} (\beta_\lambda^{(j)})^p \\ &= \sum_{\lambda \in I} a_\lambda \|E_\lambda\|_{\phi_p}^p = \|E\|_p^p. \end{aligned}$$

Thus, T_p is an isometry.

Finally, we show that T_p maps \mathcal{E}_p onto $e_p(\bigoplus_{\iota \in I} H_\iota)$. Consider S in $e_p(\bigoplus_{\iota \in I} H_\iota)$. For each $\iota \in I$, we let $E_\iota = a_\iota^{-1/p} S|_{H_\iota}$. Since H_ι is invariant under S , $E_\iota \in \mathcal{B}(H_\iota)$ for each $\iota \in I$. Also, we notice that H_ι is invariant under S^* for each $\iota \in I$. Hence, for $\xi_\iota, \eta_\iota \in H_\iota$, we have

$$\begin{aligned} \langle E_\iota \xi_\iota, \eta_\iota \rangle &= \langle a_\iota^{-1/p} S|_{H_\iota} \xi_\iota, \eta_\iota \rangle \\ &= a_\iota^{-1/p} \langle \xi_\iota, S^*|_{H_\iota} \eta_\iota \rangle \\ &= \langle \xi_\iota, a_\iota^{-1/p} S^*|_{H_\iota} \eta_\iota \rangle \end{aligned}$$

and so $E_\iota^* = a_\iota^{-1/p} S^*|_{H_\iota}$ for each $\iota \in I$. Now we essentially repeat an earlier argument. Namely, let $\{\beta_\lambda^{(j)}: j = 1, 2, \dots, d_\lambda\}$ be eigenvalues of $|E_\lambda|$ for each $\lambda \in I$ and let $\{\phi_\lambda^j: j = 1, \dots, d_\lambda\}$ be an orthonormal basis for H_λ consisting of eigenvectors corresponding to the eigenvalues $\{(\beta_\lambda^{(j)})^2: j = 1, \dots, d_\lambda\}$ of $E_\lambda E_\lambda^*$. Then, as above, $SS^* \phi^{\lambda,j} = a_\lambda^{2/p} (\beta_\lambda^{(j)})^2 \phi^{\lambda,j}$ so that $\|S\|_p^p = \|E\|_p^p$ where $E = (E_\iota)_{\iota \in I}$, and hence $E \in \mathcal{E}_p$. Clearly, $S(\xi) = T_E(\xi)$ for all $\xi \in H_\iota$, $\iota \in I$; thus, by linearity, $S(\xi) = T_E(\xi)$ for all $\xi \in \bigoplus_{\iota \in I} H_\iota$ with $\xi_\iota \neq 0$ for only finitely many $\iota \in I$. By the density of the latter set in $\bigoplus_{\iota \in I} H_\iota$, $S(\xi) = T_E(\xi)$ for all $\xi \in \bigoplus_{\iota \in I} H_\iota$. Hence $T_p(E) = S$ and so T_p maps onto $e_p(\bigoplus_{\iota \in I} H_\iota)$.

We state several corollaries which follow immediately from results for c_p spaces found in [1, XI, § 9], [2, III, § 7] and [6]. Also, compare [3, § 28].

COROLLARY 2.3. *Let $0 < p \leq q \leq \infty$. Then $\mathcal{E}_p(I) \subset \mathcal{E}_q(I)$ and $\|E\|_q \leq \|E\|_p$.*

COROLLARY 2.4. *Suppose $0 < p \leq 1$; let $E, F \in \mathcal{E}_p(I)$. Then*

$$\|E + F\|_p^p \leq \|E\|_p^p + \|F\|_p^p.$$

Thus, $\mathcal{E}_p(I)$ is a metric space with metric ρ where $\rho(A, B) = \|A - B\|_p^p$.

Inequalities (i) and (ii) in the following are due to McCarthy [6, Th. 2.7] for c_p spaces.

COROLLARY 2.5. (*Clarkson's inequalities*). *Let $E, F \in \mathcal{E}(I)$. Then, for $1/p + 1/p' = 1$, we have*

- (i) $2^{p-1}(\|E\|_p^p + \|F\|_p^p) \leq \|E + F\|_p^p + \|E - F\|_p^p \leq 2(\|E\|_p^p + \|F\|_p^p)$
- $0 < p \leq 2$,
- (ii) $\|E + F\|_p^{p'} + \|E - F\|_p^{p'} \leq 2(\|E\|_p^p + \|F\|_p^p)^{p'/p}$ $1 < p \leq 2$,
- (iii) $2(\|E\|_p^p + \|F\|_p^p) \leq \|E + F\|_p^p + \|E - F\|_p^p \leq 2^{p-1}(\|E\|_p^p + \|F\|_p^p)$
- $2 \leq p < \infty$,
- (iv) $2(\|E\|_p^p + \|F\|_p^p)^{p'/p} \leq \|E + F\|_p^{p'} + \|E - F\|_p^{p'} \leq 2 \leq p < \infty$.

COROLLARY 2.6. *For $1 < p < \infty$, $\mathcal{E}_p(I)$ is uniformly convex. (Recall that a normed linear space X is said to be uniformly convex if $\delta(\varepsilon) = \inf \{1 - 1/2|x + y| : |x| = |y| = 1, |x - y| = \varepsilon\}$ is strictly positive in some range $0 < \varepsilon < \varepsilon_0$.)*

Proof. Use the Clarkson inequalities (2.5) (ii) and the right hand half of (2.5) (iii) to obtain

$$\|E + F\|_p^{p'} \leq 2^{p'} - \|E - F\|_p^{p'} \text{ for } 1 < p \leq 2$$

and

$$\|E + F\|_p^p \leq 2^p - \|E - F\|_p^p \text{ for } 2 \leq p < \infty$$

when $\|E\|_p = \|F\|_p = 1$. If, in addition, $\|E - F\|_p = \varepsilon$, we have

$$1 - \frac{1}{2} \|E + F\|_p \geq 1 - \frac{1}{2} (2^{p'} - \varepsilon^{p'})^{1/p'} \text{ for } 1 < p \leq 2,$$

and

$$1 - \frac{1}{2} \|E + F\|_p \geq 1 - \frac{1}{2} (2^p - \varepsilon^p)^{1/p} \text{ for } 2 \leq p < \infty.$$

The uniform convexity of \mathcal{E}_p for $1 < p < \infty$ is now clear.

COROLLARY 2.7. (*Radon-Riesz theorem*). *Let $1 < p < \infty$. Let $(E^{(n)})$ be a sequence in $\mathcal{E}_p(I)$ and $E \in \mathcal{E}_p(I)$ such that $E^{(n)} \rightarrow E$ weakly and $\|E^{(n)}\|_p \rightarrow \|E\|_p$. Then $\|E^{(n)} - E\|_p \rightarrow 0$.*

Proof. $\mathcal{E}_p(I)$ is locally uniformly convex; see [4, 15.17 (a)]. Hence, apply [4, 15.17 (a)].

3. The conjugate space of \mathcal{E}_p for $0 < p < 1$. Theorem (3.4) below is a characterization of the conjugate space of \mathcal{E}_p for $0 < p < 1$. The conjugate spaces of \mathcal{E}_p for $1 \leq p < \infty$ are described in [3, § 28]. We first state and prove some easy results which will be used in the proof of (3.4).

LEMMA 3.1. *Let H be a finite-dimensional Hilbert space and let $0 < p, q \leq \infty$. For each $A \in \mathcal{B}(H)$, there exists $B \in \mathcal{B}(H)$ such that $\|B\|_{\phi_p} = 1$ and $\|A\|_{\phi_\infty} = \|AB\|_{\phi_q} = \text{tr}(AB)$.*

Proof. (Compare [3, D.54].) Let a be the eigenvalue of $|A|$ such that $a = \|A\|_{\phi_\infty}$. By [3, D.30] there is an operator V in $\mathcal{U}(H)$ such that $AV = |A|$. Let $\{\zeta_1, \zeta_2, \dots, \zeta_n\}$ be a basis for H such that $|A|\zeta_1 = a\zeta_1$. Let P be the operator on H such that $P\zeta_1 = \zeta_1$ and $P\zeta_j = 0$ for $j > 1$. Finally, let $B = VP$. By [1, p. 1090, 4 (c)], we have $\|B\|_{\phi_p} = \|P\|_{\phi_p} = 1$. Since $AB = AVP = |A|P$, we have $AB = aP$, and hence

$$\|AB\|_{\phi_q} = \|aP\|_{\phi_q} = a = \|A\|_{\phi_\infty},$$

and

$$\text{tr}(AB) = \text{tr}(aP) = a = \|A\|_{\phi_\infty}.$$

LEMMA 3.2. *Let H be a finite-dimensional Hilbert space, and let $A \in \mathcal{B}(H)$. Then*

(i) *For $0 < p \leq q \leq \infty$, we have*

$$\|A\|_{\phi_\infty} = \sup \{\|AB\|_{\phi_q} : B \in \mathcal{B}(H) \text{ and } \|B\|_{\phi_p} \leq 1\},$$

and

(ii) *for $0 < p \leq 1$, we have*

$$\|A\|_{\phi_\infty} = \sup \{|\text{tr}(AB)| : B \in \mathcal{B}(H) \text{ and } \|B\|_{\phi_p} \leq 1\}.$$

Also, the supremum is attained in (i) and (ii).

Proof. Let $\alpha = \sup \{\|AB\|_{\phi_q} : \|B\|_{\phi_p} \leq 1\}$ for $0 < p \leq q \leq \infty$. Then by [1, p. 1093, 9 (d) and 9 (a)],

$$\|AB\|_{\phi_q} \leq \|A\|_{\phi_\infty} \|B\|_{\phi_q} \leq \|A\|_{\phi_\infty} \|B\|_{\phi_p} \leq \|A\|_{\phi_\infty},$$

so that $\alpha \leq \|A\|_{\phi_\infty}$.

For $0 < p \leq 1$, let $\beta = \sup \{|\text{tr}(AB)| : \|B\|_{\phi_p} \leq 1\}$. By [3, D.46], we have

$$|\text{tr}(AB)| \leq \|AB\|_{\phi_1} \leq \|A\|_{\phi_\infty} \|B\|_{\phi_1} \leq \|A\|_{\phi_\infty} \|B\|_{\phi_p} \leq \|A\|_{\phi_\infty}$$

so that $\beta \leq \|A\|_{\phi_\infty}$.

The opposite inequalities and the fact that the supremum is attained in (i) and (ii) follow from (3.1).

LEMMA 3.3. *Let $0 < p < 1$, $E \in \mathcal{E}_p(I)$ and $F \in \mathcal{E}_\infty(I)$. Then EF and FE are in $\mathcal{E}_p(I)$,*

- (i) $\|EF\|_p \leq \|E\|_p \|F\|_\infty$, and
- (ii) $\|FE\|_p \leq \|F\|_\infty \|E\|_p$.

Proof. Use [1, p. 1093, 9 (d)] to write

$$\begin{aligned} \|EF\|_p^p &= \sum_{i \in I} a_i \|E_i F_i\|_{\phi_p}^p \leq \sum_{i \in I} a_i \|E_i\|_{\phi_p}^p \|F_i\|_{\phi_\infty}^p \\ &\leq \|F\|_\infty^p \sum_{i \in I} a_i \|E_i\|_{\phi_p}^p = \|F\|_\infty^p \|E\|_p^p. \end{aligned}$$

Assertion (ii) follows similarly.

THEOREM 3.4. *Let $0 < p < 1$, and let $F \in \mathcal{E}(I)$. If there exists a real number $c > 0$ such that $\|F_i\|_{\phi_\infty} \leq c a_i^{(1/p)-1}$ for all $i \in I$, then T_F , defined on $\mathcal{E}_p(I)$ by $T_F(E) = \langle E, F \rangle = \sum_{i \in I} a_i \operatorname{tr}(E_i F_i^*)$, is a continuous linear functional on $\mathcal{E}_p(I)$. Conversely, if T is a continuous linear functional on $\mathcal{E}_p(I)$, then $T = T_F$ for some $F \in \mathcal{E}(I)$ such that $\|F_i\|_{\phi_\infty} \leq c a_i^{(1/p)-1}$ for some $c > 0$ and all $i \in I$.*

Proof. First, suppose there exists $c > 0$ such that $\|F_i\|_{\phi_\infty} \leq c a_i^{(1/p)-1}$ for all $i \in I$. Then, for $E \in \mathcal{E}_p(I)$, the number $T_F(E) = \sum_{i \in I} a_i \operatorname{tr}(E_i F_i^*)$ is well-defined (the series converges absolutely) since by (3.2) and an observation below, we have

$$\begin{aligned} |T_F(E)| &= \left| \sum_{i \in I} a_i \operatorname{tr}(E_i F_i^*) \right| \\ &\leq \sum_{i \in I} a_i |\operatorname{tr}(E_i F_i^*)| \\ &\leq \sum_{i \in I} a_i \|E_i\|_{\phi_p} \|F_i\|_{\phi_\infty} \\ (1) \quad &\leq \sum_{i \in I} c a_i^{1/p} \|E_i\|_{\phi_p} \\ &= c \sum_{i \in I} (a_i \|E_i\|_{\phi_p}^p)^{1/p} \\ &\leq c \left[\sum_{i \in I} a_i \|E_i\|_{\phi_p}^p \right]^{1/p} = c \|E\|_p. \end{aligned}$$

The last inequality follows since $1 < 1/p$ so that $\|b\|_{1/p} \leq \|b\|_1$ for $b \in \mathcal{L}_1$, and in particular for $b = \{b_i\}$ where $b_i = a_i \|E_i\|_{\phi_p}^p$.

The linearity of T_F follows immediately from the linearity of tr [3, D.16]. The inequality (1) also shows that T_F is continuous at 0, hence on $\mathcal{E}_p(I)$. (Recall that $\mathcal{E}_p(I)$ is a metric spaces with $\rho(A, B) = \|A - B\|_p$.) Thus, T_F is a continuous linear functional on $\mathcal{E}_p(I)$.

Conversely, let T be a continuous linear functional on $\mathcal{E}_p(I)$. Let $\mathcal{N}_\iota = \{E \in \mathcal{E}_p(I) : E_\lambda = 0 \text{ for } \lambda \neq \iota\}$. Then \mathcal{N}_ι is isomorphic with $\mathcal{B}(H_\iota)$. Restricting T to \mathcal{N}_ι , we use elementary algebra to see that there exists $F_\iota \in \mathcal{B}(H_\iota)$ such that $T(E) = a_\iota \operatorname{tr}(E_\iota F_\iota^*)$, for all $E \in \mathcal{N}_\iota$. The linearity of T shows that

$$T(E) = \sum_{\iota \in I} a_\iota \operatorname{tr}(E_\iota F_\iota^*)$$

for all $E \in \mathcal{E}_{00}(I)$. Let $F = (F_\iota)_{\iota \in I}$, so that $T = T_F$ on $\mathcal{E}_{00}(I)$.

Now suppose that for every real number $c > 0$, there exists $\iota \in I$ such that $\|F_\iota\|_{\phi_\infty} > ca_\iota^{(1/p)-1}$. In particular, for $n \in \{1, 2, \dots\}$, let $\iota_n \in I$ be such that $\iota_n \neq \iota_m$ for $m \neq n$ and $\|F_{\iota_n}\|_{\phi_\infty} > n^k a_{\iota_n}^{(1/p)-1}$, where k is a real number greater than zero and such that $2/(1+k) < p$.

For each $n \in \{1, 2, \dots\}$, let $B_{\iota_n} \in \mathcal{B}(H_{\iota_n})$ be such that $\|B_{\iota_n}\|_{\phi_p} = 1$ and $\|F_{\iota_n}\|_{\phi_\infty} = \operatorname{tr}(F_{\iota_n} B_{\iota_n})$ as in (3.1). Let $b_n = (a_{\iota_n} n^2)^{-1/p}$ for each n , and define $E = (E_\iota)_{\iota \in I}$, where $E_\iota = b_n B_{\iota_n}^*$ if $\iota = \iota_n$ for some n , and $E_\iota = 0$ otherwise. Then

$$\begin{aligned} \|E\|_p^p &= \sum_{\iota \in I} a_\iota \|E_\iota\|_{\phi_p}^p = \sum_{n=1}^{\infty} a_{\iota_n} \|b_n B_{\iota_n}^*\|_{\phi_p}^p \\ &= \sum_{n=1}^{\infty} a_{\iota_n} b_n^p \|B_{\iota_n}\|_{\phi_p}^p = \sum_{n=1}^{\infty} a_{\iota_n} (a_{\iota_n} n^2)^{-1} \\ &= \sum_{n=1}^{\infty} n^{-2} < \infty \end{aligned}$$

so that $E \in \mathcal{E}_p(I)$.

For each positive integer N , define $E^{(N)} = (E_\iota^{(N)})_{\iota \in I}$, where $E_\iota^{(N)} = E_\iota$ if $\iota = \iota_n$ with $n \leq N$, and $E_\iota^{(N)} = 0$ otherwise. Then $E^{(N)} \in \mathcal{E}_{00}(I)$ and $\|E^{(N)}\|_p^p \leq \|E\|_p^p$ for each N . However,

$$\begin{aligned} T(E^{(N)}) &= T_F(E^{(N)}) = \sum_{\iota \in I} a_\iota \operatorname{tr}(E_\iota^{(N)} F_\iota^*) \\ &= \sum_{n=1}^N a_{\iota_n} \operatorname{tr}(E_{\iota_n} F_{\iota_n}^*) \\ &= \sum_{n=1}^N a_{\iota_n} \operatorname{tr}(b_n B_{\iota_n}^* F_{\iota_n}^*) \\ &= \sum_{n=1}^N a_{\iota_n} b_n \operatorname{tr}((F_{\iota_n} B_{\iota_n})^*) \\ &= \sum_{n=1}^N a_{\iota_n} b_n \overline{\operatorname{tr}(F_{\iota_n} B_{\iota_n})} \\ &= \sum_{n=1}^N a_{\iota_n} (a_{\iota_n} n^2)^{-1/p} \|F_{\iota_n}\|_{\phi_\infty} \\ &> \sum_{n=1}^N a_{\iota_n} (a_{\iota_n} n^2)^{-1/p} n^k a_{\iota_n}^{(1/p)-1} \\ &= \sum_{n=1}^N n^{k-2/p} > \sum_{n=1}^N 1/n. \end{aligned}$$

A simple argument now shows that T is discontinuous, a contradiction. Therefore, there exists $c > 0$ so that $\|F_\epsilon\|_{\mathcal{E}_\infty} \leq c\alpha_\epsilon^{(1/p)-1}$ for all $\epsilon \in I$. Thus, T_F and T are continuous linear functionals on $\mathcal{E}_p(I)$ which agree on $\mathcal{E}_{00}(I)$, a dense subspace of $\mathcal{E}_p(I)$, so that $T = T_F$ on $\mathcal{E}_p(I)$.

Several easy corollaries follow and will be stated without proof. The notation is as in (3.4).

COROLLARY 3.5. *If $0 < p < 1$ and if $\sup_{\epsilon \in I} a_\epsilon < \infty$, then $\mathcal{E}_p^* = \{T_F: F \in \mathcal{E}_\infty\}$.*

COROLLARY 3.6. *Let $0 < p < 1$ and let L_p be a weighted \mathcal{L}_p space; say $\|b\|_p = (\sum_{\epsilon \in I} a_\epsilon |b_\epsilon|^p)^{1/p}$ for $\{b_\epsilon\} \in L_p$. For $b = \{b_\epsilon\} \in L_p$ and $c = \{c_\epsilon\}$, let $T_c(b) = \sum_{\epsilon \in I} a_\epsilon b_\epsilon \bar{c}_\epsilon$. Then*

$$L_p^* = \{T_c: |c_\epsilon| \leq k a_\epsilon^{(1/p)-1} \text{ for some } k > 0 \text{ and all } \epsilon \in I\}.$$

COROLLARY 3.7. *If $0 < p < 1$, then $\mathcal{L}_p^* = \{T_c: c \in \mathcal{L}_\infty\}$.*

4. Some multiplier theorems. Theorem (4.2) is a collection of results concerning $(\mathcal{E}_p, \mathcal{E}_q)$ -multipliers. We use the following definition: Let \mathcal{A} and \mathcal{B} be subsets of $\mathcal{E}(I)$. We say that E in $\mathcal{E}(I)$ is an $(\mathcal{A}, \mathcal{B})$ -multiplier if $EA \in \mathcal{B}$ for all $A \in \mathcal{A}$. The set of all $(\mathcal{A}, \mathcal{B})$ -multipliers is denoted by $\mathcal{M}(\mathcal{A}, \mathcal{B})$.

Clearly, multipliers may be discussed in a context much wider than that of \mathcal{E}_p spaces. For example, it is known that $\mathcal{L}_r = \mathcal{M}(\mathcal{L}_q, \mathcal{L}_p)$ for $0 < p < q < \infty$ with $1/r = 1/p - 1/q$. Also, it is shown in McCarthy [6, Ths. 2.3 and 5.1] that $\mathcal{M}(c_q, c_p) = c_r$ for p, q and r as above.

In Hewitt and Ross [3, 35.4] $\mathcal{M}(\mathcal{A}, \mathcal{B})$ is described for any pair $(\mathcal{A}, \mathcal{B})$ chosen from the spaces $\mathcal{E}_p, \mathcal{E}_q, \mathcal{E}_0, \mathcal{E}_\infty$ with $1 \leq p < q < \infty$ with the following exceptions: if $\sup_{\epsilon \in I} a_\epsilon = \infty$, it is shown only that $\mathcal{M}(\mathcal{A}, \mathcal{B}) \supsetneq \mathcal{E}_\infty$, where $\mathcal{A} = \mathcal{E}_p$ and $\mathcal{B} = \mathcal{E}_q$ or $\mathcal{B} = \mathcal{E}_0$ with $1 \leq p < q < \infty$. Our theorem which follows extends the results of [3, 35.4] to all p and q with $0 < p < q < \infty$. Also, it identifies $\mathcal{M}(\mathcal{A}, \mathcal{B})$ precisely in the exceptions mentioned above when $\sup_{\epsilon \in I} a_\epsilon = \infty$. The major tool used in the identification of $\mathcal{M}(\mathcal{A}, \mathcal{B})$ in the cases where $\sup_{\epsilon \in I} a_\epsilon = \infty$ is (3.4), our characterization of \mathcal{E}_p^* for $0 < p < 1$.

Before stating our theorem we note that the following lemma may easily be verified using [6, Th. 2.3] and the generalized Hölder inequality.

LEMMA 4.1. *Let $0 < p, q, r < \infty$ with $1/p + 1/q = 1/r$. If $E \in \mathcal{E}_p(I)$, $F \in \mathcal{E}_q(I)$, then $EF \in \mathcal{E}_r(I)$ and $\|EF\|_r \leq \|E\|_p \|F\|_q$.*

THEOREM 4.2. *Let $0 < p < q < \infty$ and let r be so that $1/r = 1/p - 1/q$. For each space \mathcal{A} listed to the left of the matrix below and each space \mathcal{B} listed above the matrix, the corresponding entry of the matrix is exactly $\mathcal{M}(\mathcal{A}, \mathcal{B})$.*

	\mathcal{E}_p	\mathcal{E}_q	\mathcal{E}_0	\mathcal{E}_∞
\mathcal{E}_∞	\mathcal{E}_p	\mathcal{E}_q	\mathcal{E}_0	\mathcal{E}_∞
\mathcal{E}_0	\mathcal{E}_p	\mathcal{E}_q	\mathcal{E}_∞	\mathcal{E}_∞
\mathcal{E}_q	\mathcal{E}_r	\mathcal{E}_∞	\mathcal{E}_s^* $s = \frac{q}{1+q}$	\mathcal{E}_s^* $s = \frac{q}{1+q}$
\mathcal{E}_p	\mathcal{E}_∞	\mathcal{E}_s^* $s = \frac{pq}{q-p+pq}$	\mathcal{E}_s^* $s = \frac{p}{1+p}$	\mathcal{E}_s^* $s = \frac{p}{1+p}$

The proof of the above theorem will be broken into several parts.

Part I. For $0 < p \leq \infty$, $\mathcal{M}(\mathcal{E}_p, \mathcal{E}_p) = \mathcal{E}_\infty$.

Proof. In case $1 \leq p \leq \infty$, we use the proof of [3, 35.4, Part II] with d_{σ_n} replaced by a_{σ_n} throughout.

Now let $0 < p < 1$. The fact that $\mathcal{E}_\infty \subset \mathcal{M}(\mathcal{E}_p, \mathcal{E}_p)$ follows from (3.3). The proof of the opposite inclusion is similar to the proof of [3, 35.4, Part II]. Namely, suppose $E \notin \mathcal{E}_\infty(I)$. Then there is a sequence $\{\iota_n\}_{n=1}^\infty$ of distinct elements in I such that $\|E_{\iota_n}\|_{\phi_\infty} > n$ for each n . By (3.1), there exists B_{ι_n} in $\mathcal{B}(H_{\iota_n})$ such that $\|E_{\iota_n} B_{\iota_n}\|_{\phi_p} > n$ and $\|B_{\iota_n}\|_{\phi_p} = 1$. For $n \in \{1, 2, \dots\}$, let $\alpha_n = (a_{\iota_n} n^{1+p})^{-1/p}$. Define $A \in \mathcal{E}(I)$ as follows: $A_{\iota_n} = \alpha_n B_{\iota_n}$ for $n \in \{1, 2, \dots\}$ and $A_{\iota} = 0$ for all other ι 's in I . Since

$$\begin{aligned} \|A\|_p^p &= \sum_{n=1}^{\infty} a_{\iota_n} \|\alpha_n B_{\iota_n}\|_{\phi_p}^p \\ &= \sum_{n=1}^{\infty} n^{-(1+p)} < \infty, \end{aligned}$$

we have that $A \in \mathcal{E}_p(I)$. On the other hand, EA does not belong to $\mathcal{E}_p(I)$ because

$$\begin{aligned} \|EA\|_p^p &= \sum_{n=1}^{\infty} a_{\iota_n} \|\alpha_n E_{\iota_n} B_{\iota_n}\|_{\phi_p}^p \geq \sum_{n=1}^{\infty} a_{\iota_n} (a_{\iota_n} n^{1+p})^{-1} n^p \\ &= \sum_{n=1}^{\infty} 1/n = \infty. \end{aligned}$$

Thus, $E \notin \mathcal{M}(\mathcal{E}_p, \mathcal{E}_p)$ and so $\mathcal{M}(\mathcal{E}_p, \mathcal{E}_p) \subset \mathcal{E}_\infty(I)$. Hence, entries (1, 4), (3, 2), and (4, 1) are verified.

Part II. For $0 < p < \infty$, we have that $\mathcal{E}_p = \mathcal{M}(\mathcal{E}_0, \mathcal{E}_p) = \mathcal{M}(\mathcal{E}_\infty, \mathcal{E}_p)$. This will verify entries (1, 1), (1, 2), (2, 1), and (2, 2).

Proof. Using (3.3) we see that, for $0 < p < 1$, $\mathcal{E}_p \subset \mathcal{M}(\mathcal{E}_\infty, \mathcal{E}_p) \subset \mathcal{M}(\mathcal{E}_0, \mathcal{E}_p)$. The rest of the assertion is proved in [3, 35.4, Part VII] if we replace d_σ by a_σ throughout.

Part III. Let $0 < p < q < \infty$ and let $s = pq/(q - p + pq)$. Then $\mathcal{E}_s^* = \mathcal{M}(\mathcal{E}_p, \mathcal{E}_q)$.

Proof. Consider $T_F \in \mathcal{E}_s^*$ with s as above. Then $0 < s < 1$ so that by (3.4), there exists a real number $c > 0$ such that $\|F_t\|_{\phi_\infty} \leq ca_t^{(1/s)-1}$. Let $E \in \mathcal{E}_p$. The following is seen to be true by using $\|\cdot\|_q \leq \|\cdot\|_p$ for $0 < p < q < \infty$ and the results (3.3), [3, D.52.i.], and (2.3).

$$\begin{aligned} \|FE\|_q &= \left[\sum_{t \in I} (a_t^{1/q} \|F_t E_t\|_{\phi_q})^q \right]^{1/q} \\ &\leq \left[\sum_{t \in I} (a_t^{1/q} \|F_t E_t\|_{\phi_q})^p \right]^{1/p} \\ &\leq \left[\sum_{t \in I} a_t^{p/q} \|F_t\|_{\phi_\infty}^p \|E_t\|_{\phi_q}^p \right]^{1/p} \\ &\leq \left[\sum_{t \in I} a_t^{p/q} c^p a_t^{(p/s)-p} \|E_t\|_{\phi_p}^p \right]^{1/p} \\ &= c \left[\sum_{t \in I} a_t \|E_t\|_{\phi_p}^p \right]^{1/p} \\ &= c \|E\|_p. \end{aligned}$$

Thus, $FE \in \mathcal{E}_q$ so that $F \in \mathcal{M}(\mathcal{E}_p, \mathcal{E}_q)$. Hence, $\mathcal{E}_s^* \subset \mathcal{M}(\mathcal{E}_p, \mathcal{E}_q)$.

On the other hand, suppose $T_F \notin \mathcal{E}_s^*$. Again, by (3.4), we have that for every $c > 0$, there exists $t \in I$ such that $\|F_t\|_{\phi_\infty} > ca_t^{(1/s)-1}$. Or, in particular, for each $n \in \{1, 2, \dots\}$, let t_n be such that $t_n \neq t_m$ for $n \neq m$ and

$$\|F_{t_n}\|_{\phi_\infty} > n^k a_{t_n}^{(1/s)-1}$$

where k is a real number satisfying $k \geq 2/p - 1/q$; that is, $1 \geq q(2/p - k)$. For each $n \in \{1, 2, \dots\}$, let $B_{t_n} \in \mathcal{B}(H_{t_n})$ be such that $\|B_{t_n}\|_{\phi_p} = 1$ and $\|F_{t_n} B_{t_n}\|_{\phi_q} = \|F_{t_n}\|_{\phi_\infty}$ as in (3.1). Let $b_n = (a_{t_n} n^2)^{-1/p}$ and define $E_t = b_n B_{t_n}$ if $t = t_n$ and $E_t = 0$ otherwise. Let $E = (E_t)_{t \in I}$. Then

$$\|E\|_p^p = \sum_{n=1}^{\infty} a_{t_n} \|b_n B_{t_n}\|_{\phi_p}^p$$

$$= \sum_{n=1}^{\infty} a_{\iota_n} (a_{\iota_n} n^2)^{-1} = \sum_{n=1}^{\infty} 1/n^2 < \infty ,$$

so that $E \in \mathcal{E}_p$. However,

$$\begin{aligned} \|FE\|_q^q &= \sum_{n=1}^{\infty} a_{\iota_n} \|F_{\iota_n} b_n B_{\iota_n}\|_{\phi_q}^q \\ &= \sum_{n=1}^{\infty} a_{\iota_n} (a_{\iota_n} n^2)^{-q/p} \|F_{\iota_n}\|_{\phi_{\infty}}^q \\ &\geq \sum_{n=1}^{\infty} a_{\iota_n} (a_{\iota_n} n^2)^{-q/p} n^{qk} a_{\iota_n}^{c/s-q} \\ &= \sum_{n=1}^{\infty} n^{q(k-2/p)} \geq \sum_{n=1}^{\infty} 1/n = \infty . \end{aligned}$$

Thus, $FE \notin \mathcal{E}_q$ so that $F \notin \mathcal{M}(\mathcal{E}_p, \mathcal{E}_q)$. We have, therefore, that $\mathcal{M}(\mathcal{E}_p, \mathcal{E}_q) \subset \mathcal{E}_s^*$ and (4, 2) is verified.

Part IV. We verify entries (3, 3), (3, 4), (4, 3) and (4, 4) by showing that for $0 < p < \infty$,

$$\mathcal{E}_s^* = \mathcal{M}(\mathcal{E}_p, \mathcal{E}_0) = \mathcal{M}(\mathcal{E}_p, \mathcal{E}_{\infty}) \text{ where } s = \frac{p}{1+p} .$$

Proof. Let $T_F \in \mathcal{E}_s^*$. We will first show that $F \in \mathcal{M}(\mathcal{E}_p, \mathcal{E}_0)$. By (3.4), there exists a constant $c > 0$ such that $\|F_{\iota}\|_{\phi_{\infty}} \leq c a_{\iota}^{1/s-1} = c a_{\iota}^{1/p}$ for all $\iota \in I$. Let $E \in \mathcal{E}_p$ so that $\sum_{\iota \in I} a_{\iota} \|E_{\iota}\|_{\phi_p}^p < \infty$. Then, for $\varepsilon > 0$, $a_{\iota} \|E_{\iota}\|_{\phi}^p \leq (\varepsilon/c)^p$ for all except finitely many $\iota \in I$. Thus,

$$\begin{aligned} \|F_{\iota} E_{\iota}\|_{\phi_{\infty}} &\leq \|F_{\iota}\|_{\phi_{\infty}} \|E_{\iota}\|_{\phi_{\infty}} \leq \|F_{\iota}\|_{\phi_{\infty}} \|E_{\iota}\|_{\phi_p} \\ &\leq c a_{\iota}^{1/p} \|E_{\iota}\|_{\phi_p} \leq c \cdot \frac{\varepsilon}{c} = \varepsilon \end{aligned}$$

for all except finitely many $\iota \in I$. Hence, $FE \in \mathcal{E}_0$ so that $F \in \mathcal{M}(\mathcal{E}_p, \mathcal{E}_0)$. Clearly $\mathcal{M}(\mathcal{E}_p, \mathcal{E}_0) \subset \mathcal{M}(\mathcal{E}_p, \mathcal{E}_{\infty})$ so that it remains only to show that $\mathcal{M}(\mathcal{E}_p, \mathcal{E}_{\infty}) \subset \mathcal{E}_s^*$.

Suppose $T_F \in \mathcal{E}_s^*$. Then by (3.4), for each $n \in \{1, 2, \dots\}$, we can choose distinct $\iota_n \in I$ with the property that $\|F_{\iota_n}\|_{\phi_{\infty}} > n^{2/p+1} a_{\iota_n}^{-1/p}$. As in (3.1), for each $\iota \in I$, let $B_{\iota} \in \mathcal{B}(H_{\iota})$ be such that $\|B_{\iota}\|_{\phi_p} = 1$ and $\|F_{\iota}\|_{\phi_{\infty}} = \|F_{\iota} B_{\iota}\|_{\phi_{\infty}}$. For each $n \in \{1, 2, \dots\}$ let $b_n = (a_{\iota_n} n^2)^{-1/p}$ and let $E = (E_{\iota})_{\iota \in I}$ where $E_{\iota} = b_n B_{\iota_n}$ if $\iota = \iota_n$ and $E_{\iota} = 0$ otherwise. As in Part III, it is clear that $E \in \mathcal{E}_p$. However, $\|F_{\iota_n} E_{\iota_n}\|_{\phi_{\infty}} = \|F_{\iota_n} b_n B_{\iota_n}\|_{\phi_{\infty}} = b_n \|F_{\iota_n}\|_{\phi_{\infty}} > n$ for $n \in \{1, 2, \dots\}$. Thus, $\|FE\|_{\infty}$ is not finite so that $FE \notin \mathcal{E}_{\infty}$. Hence, $F \notin \mathcal{M}(\mathcal{E}_p, \mathcal{E}_{\infty})$ and so $\mathcal{M}(\mathcal{E}_p, \mathcal{E}_{\infty}) \subset \mathcal{E}_s^*$.

Part V. If $0 < p < q < \infty$ and $1/p - 1/q = 1/r$, then $\mathcal{M}(\mathcal{E}_q, \mathcal{E}_p) = \mathcal{E}_r$.

Proof. This result is proved for $1 \leq p < q < \infty$ in [3, 35.4, Part VI]. That proof does not carry over to our wider range for p, q and r , however.

The inclusion $\mathcal{E}_r \subset \mathcal{M}(\mathcal{E}_q, \mathcal{E}_p)$ follows immediately from (4.1). To see the opposite inclusion, suppose that $E = (E_i)_{i \in I}$ is in $\mathcal{E}(I)$ but not in \mathcal{E}_r . We will show that $E \notin \mathcal{M}(\mathcal{E}_q, \mathcal{E}_p)$.

Let $\gamma_i = a_i^{1/r} \|E_i\|_{\phi_r}$. Since $E \notin \mathcal{E}_r$, $\{\gamma_i\}$ does not belong to $\mathcal{L}_r(I)$. However, since $\mathcal{L}_r = \mathcal{M}(\mathcal{L}_q, \mathcal{L}_p)$, there exists $\{\beta_i\} \in \mathcal{L}_q$ such that $\{\gamma_i \beta_i\} \notin \mathcal{L}_p$. We may, and will, choose β_i so that $\beta_i \geq 0$ for all $i \in I$. Using [6, Th. 2.3] choose F_i so that $\|E_i F_i\|_{\phi_p} = \|E_i\|_{\phi_r} \|F_i\|_{\phi_q}$ for each $i \in I$ and such that $E_i \neq 0$ if and only if $F_i \neq 0$. [For example, let $F_i = |E_i|^{r/q}$. That the above equality holds in this case may be seen directly using conditions for equality in Hölder's inequality for \mathcal{L}_p .]

For our convenience below let $\Phi = \{i \in I : \gamma_i \neq 0\}$. Note also that $\Phi = \{i \in I : E_i \neq 0\}$. For $i \in \Phi$, let $c_i = \beta_i a_i^{-1/q} \|F_i\|_{\phi_q}^{-1}$, otherwise $c_i = 0$. For all $i \in I$, let $F'_i = c_i F_i$ and let $F' = (F'_i)_{i \in I}$. Then

$$\|F'\|_q^q = \sum_{i \in I} a_i \|F'_i\|_{\phi_q}^q = \sum_{i \in \Phi} a_i \beta_i^q a_i^{-1} \|F_i\|_{\phi_q}^{-q} \|F_i\|_{\phi_q}^q = \sum_{i \in \Phi} \beta_i^q \leq \sum_{i \in I} \beta_i^q < \infty$$

since $\{\beta_i\} \in \mathcal{L}_q$. Thus, $F' \in \mathcal{E}_q$. However,

$$\begin{aligned} \|EF'\|_p^p &= \sum_{i \in I} a_i \|E_i F'_i\|_{\phi_p}^p \\ &= \sum_{i \in I} a_i c_i^p \|E_i F_i\|_{\phi_p}^p \\ &= \sum_{i \in \Phi} a_i \beta_i^p a_i^{-p/q} \|F_i\|_{\phi_q}^{-p} \|E_i\|_{\phi_r}^p \|F_i\|_{\phi_q}^p \\ &= \sum_{i \in \Phi} a_i^{1-p/q} \beta_i^p \|E_i\|_{\phi_r}^p \\ &= \sum_{i \in \Phi} (a_i^{1/r} \|E_i\|_{\phi_r})^p \beta_i^p \\ &= \sum_{i \in \Phi} (\gamma_i \beta_i)^p = \sum_{i \in I} (\gamma_i \beta_i)^p = \infty \end{aligned}$$

since $\{\gamma_i \beta_i\} \notin \mathcal{L}_p$. Hence $E \notin \mathcal{M}(\mathcal{E}_q, \mathcal{E}_p)$ and (3, 1) is verified.

Part VI. $\mathcal{M}(\mathcal{E}_0, \mathcal{E}_0) = \mathcal{M}(\mathcal{E}_0, \mathcal{E}_\infty) = \mathcal{E}_\infty$.

Proof. The proof in [3, 35.4, Part III] can be adapted to our somewhat more general setting. However, an easy direct proof will be given.

Since \mathcal{E}_0 is an ideal of \mathcal{E}_∞ , we have $\mathcal{E}_\infty \subset \mathcal{M}(\mathcal{E}_0, \mathcal{E}_0)$. Also, clearly, $\mathcal{M}(\mathcal{E}_0, \mathcal{E}_0) \subset \mathcal{M}(\mathcal{E}_0, \mathcal{E}_\infty)$. Thus we need to show only that $\mathcal{M}(\mathcal{E}_0, \mathcal{E}_\infty) \subset \mathcal{E}_\infty$. Consider any E in $\mathcal{E}(I)$ that is not in \mathcal{E}_∞ . Then for each $n \in \{1, 2, \dots\}$, let ι_n be such that $\iota_n \neq \iota_m$ for $n \neq m$ and $\|E_{\iota_n}\|_{\phi_\infty} > n^2$. Let $F = (F_i)_{i \in I}$ where $F_i = (1/n)I_{d_{\iota_n}}$ for $i = \iota_n$ and $F_i =$

0 otherwise. Then we have $F \in \mathcal{E}_0$ and $EF \notin \mathcal{E}_\infty$, so that $E \notin \mathcal{M}(\mathcal{E}_0, \mathcal{E}_\infty)$. Hence, entries (2, 3) and (2, 4) are verified.

Part VII. It remains only to verify (1, 3) by showing that $\mathcal{M}(\mathcal{E}_\infty, \mathcal{E}_0) = \mathcal{E}_0$.

Proof. The proof is easy. Namely, $\mathcal{E}_0 \subset \mathcal{M}(\mathcal{E}_\infty, \mathcal{E}_0)$ since \mathcal{E}_0 is an ideal in \mathcal{E}_∞ . Finally, suppose $E \notin \mathcal{E}_0$. If $F_i = I_{d_i}$, then $F \in \mathcal{E}_\infty$ but $EF \notin \mathcal{E}_0$ so that $E \notin \mathcal{M}(\mathcal{E}_\infty, \mathcal{E}_0)$.

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