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**RANK PRESERVERS OF SKEW-SYMMETRIC MATRICES**

MARION-JOSEPHINE LIM

# RANK PRESERVERS OF SKEW-SYMMETRIC MATRICES

M. J. S. LIM

It is possible to study the structure of rank preservers on  $n$ -square skew-symmetric matrices over an algebraically closed field  $F$  by considering instead the linear transformations on the second Grassmann Product Space  $\wedge^2 \mathcal{U}$  ( $\mathcal{U}$  an  $n$ -dimensional vector space) over  $F$  into itself, which preserve the irreducible lengths of the products. In this paper, it is shown that preservers of irreducible length 2 are also preservers of all irreducible lengths of the products. Correspondingly, rank 4 preservers are rank  $2k$  preservers for all positive integer values of  $k$ . The structure of the preservers in each case is deduced from the fact that these preservers are in particular irreducible length 1 and rank 2 preservers respectively, whose structures are known.

A nonzero vector in  $\wedge^2 \mathcal{U}$  is said to have *irreducible length*  $k$  if it can be written as a sum of  $k$  and *not less than*  $k$  pure (decomposable) nonzero products in  $\wedge^2 \mathcal{U}$ . The set of such products is denoted by  $\mathcal{L}_k$  and  $z \in \mathcal{L}_k$  if and only if  $\mathcal{L}(z) = k$ . A linear transformation  $\mathcal{T}$  of  $\wedge^2 \mathcal{U}$  into itself is an  $\mathcal{L}$ - $k$  preserver if and only if  $\mathcal{T}(\mathcal{L}_k) \subseteq \mathcal{L}_k$ .

A linear transformation  $\mathcal{S}$  which takes the set of rank  $2k$   $n$ -square skew-symmetric matrices into itself is a  $\rho$ - $2k$  preserver.

In [7], it is shown that  $\mathcal{L}_k$  is isomorphic to the set of all rank  $2k$   $n$ -square skew-symmetric matrices. If this isomorphism is denoted by  $\varphi$ , then  $\mathcal{S} = \varphi \mathcal{T} \varphi^{-1}$  is a  $\rho$ - $2k$  preserver if and only if  $\mathcal{T}$  is a  $\mathcal{L}$ - $k$  preserver.

To obtain the results of this paper, much use is made of  $\mathcal{L}$ -2 subspaces of  $\wedge^2 \mathcal{U}$ . An  $\mathcal{L}$ - $k$  subspace of  $\wedge^2 \mathcal{U}$  is a vector subspace whose nonzero members are in  $\mathcal{L}_k$ . An  $\mathcal{L}$ -2 subspace  $H$  is called a (1, 1)-type subspace if there exist fixed nonzero vectors  $x \neq y$  such that each nonzero  $f \in H$  can be written

$$f = x \wedge x_f + y \wedge y_f .$$

## 1. Intersection of (1, 1)-type subspaces.

LEMMA 1. *If  $V_1, V_2$  are distinct (1, 1)-type subspaces of dimension  $\geq 2$  and  $\dim V_1 \cap V_2 \geq 2$ , then the 2-dimensional subspaces of  $\mathcal{U}$  determined by  $V_1, V_2$  are equal.*

*Proof.* Let  $f_1, f_2$  be independent in  $V_1 \cap V_2$ . Then  $f_1 = x \wedge x_1 + y \wedge y_1$ ,

$f_2 = x \wedge x_2 + y \wedge y_2$  in  $V_1$ ; and  $f_1 = u \wedge u_1 + v \wedge v_1$ ,  $f_2 = u \wedge u_2 + v \wedge v_2$  in  $V_2$ . Now  $\langle x, y \rangle \subset \langle u, u_1, v, v_1 \rangle \cap \langle u, u_2, v, v_2 \rangle$  which has dimension 2 or 3 (Theorem 5 of [2], and Lemma 5 of [3]), and hence  $\dim \langle x, y \rangle \cap \langle u, v \rangle \leq 1$ . Without loss of generality, let  $x$  be in this intersection; in fact, we can take  $x = u$ ; and  $\langle u_1, v, v_1 \rangle = \langle x_1, y, y_1 \rangle$  and  $\langle u_2, v, v_2 \rangle = \langle x_2, y, y_2 \rangle$  (Lemma 9 of [2]). Since  $x \wedge y \wedge f_i = 0$ ,  $i = 1, 2$ , then  $y \in \langle v, v_1 \rangle$  and  $y \in \langle v, v_2 \rangle$  (proof of Lemma 7 in [3]). If  $\langle v, v_2 \rangle = \langle v, v_1 \rangle$ , then some linear combination of  $f_1$  and  $f_2$  has irreducible length at most one, which is impossible since  $f_1, f_2$  are independent in  $\mathcal{L}$ -2 subspaces. Hence  $\langle y \rangle = \langle v, v_1 \rangle \cap \langle v, v_2 \rangle$ , and  $\langle y \rangle = \langle v \rangle$ , which implies  $\langle x, y \rangle = \langle u, v \rangle$ .

2. The  $\mathcal{L}$ -2 preservers. The structure of  $\mathcal{L}$ -1 preservers is known. In fact, in [8], it is shown that if  $\mathcal{T}$  is an  $\mathcal{L}$ -1 preserver, then  $\mathcal{T}$  is a compound (i. e., if  $x \wedge y \in \mathcal{L}_1$ , then there exists a nonsingular matrix  $A$  such that  $\mathcal{T}(x \wedge y) = Ax \wedge Ay$ ), *except* when  $\dim \mathcal{U} = 4$ , in which case it may possibly be the composite of a compound and a linear transformation induced by a correlation of the 2-dimensional subspaces of  $\mathcal{U}$ . Thus if  $\mathcal{T}$  is an  $\mathcal{L}$ -1 preserver, it is also an  $\mathcal{L}$ - $k$  preserver for all  $k$ .

We shall show that if  $\mathcal{T}$  is an  $\mathcal{L}$ -2 preserver, then it is also an  $\mathcal{L}$ -1 preserver. Since we shall make use of  $\mathcal{L}$ -2 subspaces and these are varied (see [3]), it will be necessary to consider several cases.

2a.  $\dim \mathcal{U} \geq 7$ . In [3], it is shown that if  $\dim \mathcal{U} = n \geq 7$ , then the *maximal*  $\mathcal{L}$ -2 subspaces have dimension  $(n-3)$  and are all  $(1, 1)$ -type subspaces.

LEMMA 2. *Let  $\mathcal{T}$  be an  $\mathcal{L}$ -2 preserver,  $\dim \mathcal{U} \geq 7$ . Then  $\mathcal{T}(\mathcal{L}_1) \subset \mathcal{L}_1 \cup \mathcal{L}_2 \cup \{0\}$ .*

*Proof.* Let  $u \wedge v \in \mathcal{L}_1$ . Then  $u \wedge v$  is expressible as  $u \wedge (\alpha x_1 - x_2)$  where  $\{u, x_1, x_2\}$  is independent in  $\mathcal{U}$  and  $0 \neq \alpha \in F$ ,  $\alpha \neq 1$ . Now  $\{u, x_1, x_2\}$  can be extended to a set  $\{u, x_1, \dots, x_6\}$  of seven independent vectors in  $\mathcal{U}$ . Then the following 2 subspaces:

$$V_1 = \langle u \wedge x_1 + v \wedge x_4, u \wedge x_5 + v \wedge x_6, u \wedge x_3 + v \wedge x_4 \rangle,$$

$$V_2 = \langle u \wedge x_2 + v \wedge \alpha x_4, u \wedge x_5 + v \wedge x_6, u \wedge x_3 + v \wedge x_4 \rangle$$

are both  $\mathcal{L}$ -2 subspaces and  $\dim V_1 \cap V_2 = 2$ . Moreover

$$\begin{aligned} \mathcal{T}(u \wedge v) &= \mathcal{T}(u \wedge \alpha x_1 - x_2) \\ &= \mathcal{T}(u \wedge \alpha x_1 + \alpha v \wedge x_4 - u \wedge x_2 - \alpha v \wedge x_4) \\ &= \mathcal{T}(u \wedge \alpha x_1 + \alpha v \wedge x_4) - \mathcal{T}(u \wedge x_2 + \alpha v \wedge x_4). \end{aligned}$$

The first vector is in  $\mathcal{F}(V_1)$ , the second in  $\mathcal{F}(V_2)$ . Now  $V_1, V_2$  can be extended to  $(n-3)$ -dimensional  $\mathcal{L}$ -2 subspaces (necessarily of  $(1, 1)$ -type). Hence  $\mathcal{F}(V_1), \mathcal{F}(V_2)$  are  $(1, 1)$ -type subspaces of dimension  $(n-3)$  since  $\mathcal{F}$  is an  $\mathcal{L}$ -2 preserver, and their intersection has dimension at least two. Hence the 2-dimensional subspaces (of  $\mathcal{U}$ ) determined by  $\mathcal{F}(V_1)$  and  $\mathcal{F}(V_2)$  are equal, implying that  $\mathcal{F}(u \wedge v)$  has irreducible length  $\leq 2$ .

**THEOREM 1.** *Let  $\dim \mathcal{U} = n \geq 7$ . Then  $\mathcal{F}$  is an  $\mathcal{L}$ -2 preserver if and only if  $\mathcal{F}$  is an  $\mathcal{L}$ -1 preserver, and  $\mathcal{F}$  is a compound. Moreover,  $\mathcal{F}(\mathcal{L}_k) \subseteq \mathcal{L}_k$  for all  $k$ .*

*Proof.* Suppose  $\mathcal{F}$  is an  $\mathcal{L}$ -2 preserver. If  $f \in \mathcal{L}_1$  and  $\mathcal{F}(f) = 0$ , then there exists  $g \in \mathcal{L}_1$  such that  $\mathcal{L}(f + g) = 2$  (use Theorem 7 of [2]). Then  $\mathcal{F}(f + g) = \mathcal{F}(g) \in \mathcal{L}_2$ . Hence it is sufficient to show  $\mathcal{F}(\mathcal{L}_1)$  does not intersect  $\mathcal{L}_2$ .

Suppose  $x_1 \wedge x_n \in \mathcal{L}_1$  and  $\mathcal{F}(x_1 \wedge x_n) \in \mathcal{L}_2$ . Consider the subspace  $V$  generated by  $\{z_1 = x_1 \wedge x_n, z_i = x_1 \wedge x_{i+1} + x_2 \wedge x_{i+2}\}, 2 \leq i \leq n-2$ , where  $\mathcal{U} = \langle x_1, \dots, x_n \rangle$ . Any linear combination  $z = \sum_{i=1}^{n-2} \alpha_i z_i$  has irreducible length 2 except when  $\alpha_2 = \dots = \alpha_{n-2} = 0$ , in which case  $z = \alpha_1 z_1$  and  $\mathcal{F}(\alpha_1 z_1)$  has irreducible length 2. Hence  $\mathcal{F}(V)$  is an  $\mathcal{L}$ -2 subspace of dimension  $(n-2)$ , which contradicts the fact that the maximal  $\mathcal{L}$ -2 subspaces have dimension  $(n-3)$ . Hence  $\mathcal{F}(\mathcal{L}_1) \subseteq \mathcal{L}_1$ . The converse is easy to see (cf. beginning of § 2).

2b.  $\dim \mathcal{U} = 4, 5$ . By Theorem 7 of [2], it is clear that  $\mathcal{L}_k, k \geq 3$ , is trivial when  $\dim \mathcal{U} \leq 5$ . The following lemma is immediate.

**LEMMA 3.** *Let  $\dim \mathcal{U} \leq 5$ ,  $\mathcal{F}$  an  $\mathcal{L}$ -2 preserver. Then  $\mathcal{F}(\mathcal{L}_1) \subseteq \mathcal{L}_1 \cup \mathcal{L}_2 \cup \{0\}$ .*

**THEOREM 2.** *Let  $\dim \mathcal{U} = 4$ . Then  $\mathcal{F}$  is an  $\mathcal{L}$ -2 preserver if and only if  $\mathcal{F}$  is an  $\mathcal{L}$ -1 preserver.*

*Proof.* Suppose  $\mathcal{F}$  is an  $\mathcal{L}$ -2 preserver. Suppose  $x_1 \wedge x_2 \in \mathcal{L}_1$  and  $\mathcal{F}(x_1 \wedge x_2) = 0$ . Extend  $\{x_1, x_2\}$  to a basis  $\{x_1, \dots, x_4\}$  of  $\mathcal{U}$ . Then  $x_1 \wedge x_2 + x_3 \wedge x_4$  has irreducible length 2 and hence

$$\mathcal{F}(x_1 \wedge x_2 + x_3 \wedge x_4) = \mathcal{F}(x_3 \wedge x_4).$$

has irreducible length 2. Hence the above and Lemma 3 imply it is sufficient to show only that  $\mathcal{F}(\mathcal{L}_1) \cap \mathcal{L}_2$ .

Suppose  $\mathcal{F}(x_1 \wedge x_3)$  has irreducible length 2 for  $x_1 \wedge x_3 \in \mathcal{L}_1$ . Consider the subspace  $V$  generated by the products  $z_1 = x_1 \wedge x_3$ ;

$$z_2 = x_1 \wedge x_2 + x_3 \wedge x_4 \text{ where } \mathcal{U} = \langle x_1, \dots, x_4 \rangle.$$

Then any linear combination  $z = \alpha z_1 + \beta z_2$  has irreducible length 2 unless  $\beta = 0$ , in which case  $\mathcal{I}(z) = \mathcal{I}(\alpha z_1)$  which has irreducible length 2 by assumption. Hence  $\mathcal{I}(V)$  is an  $\mathcal{L}$ -2 subspace of dimension 2. But this contradicts the fact that the  $\mathcal{L}$ -2 subspaces have dimension one and no more (Theorem 10 of [2]). The result follows. The converse is easy to see.

**THEOREM 3.** *Let  $\dim \mathcal{U} = 5$ . Then  $\mathcal{I}$  is an  $\mathcal{L}$ -2 preserver if and only if  $\mathcal{I}$  is an  $\mathcal{L}$ -1 preserver.*

*Proof.* As in the proof of Theorem 2, it is sufficient to show  $\mathcal{I}(\mathcal{L}_1) \cap \mathcal{L}_2$ . Let  $\mathcal{U} = \langle u_1, \dots, u_5 \rangle$ . Suppose  $u_1 \wedge u_5 \in \mathcal{L}_1$  and  $\mathcal{I}(u_1 \wedge u_5) \in \mathcal{L}_2$ . Then consider the subspace  $V$  generated by the products

$$\begin{aligned} z_1 &= u_1 \wedge u_5, \\ z_2 &= u_1 \wedge u_4 + u_2 \wedge u_3, \\ z_3 &= u_1 \wedge u_3 + u_2 \wedge u_5, \\ z_4 &= u_2 \wedge u_4 + u_3 \wedge u_5. \end{aligned}$$

Then  $z = \sum_{i=1}^4 \alpha_i z_i$  has irreducible length 2 except when  $\alpha_2 = 0 = \alpha_3 = \alpha_4$ , in which case  $z = \alpha_1 z_1$  and  $\mathcal{I}(\alpha_1 z_1) \in \mathcal{L}_2$ . Hence  $\mathcal{I}(V)$  is an  $\mathcal{L}$ -2 subspace of dimension 4. But this contradicts the fact that the maximal  $\mathcal{L}$ -2 subspaces have dimension 3 (see Theorem 1 of [3]).

2c.  $\dim \mathcal{U} = 6$ . The following lemma is clear from Theorem 7 of [2].

**LEMMA 4.** *Let  $\dim \mathcal{U} = 6$ ,  $\mathcal{I}$  an  $\mathcal{L}$ -2 preserver. Then*

$$\mathcal{I}(\mathcal{L}_1) \subset \left\{ \bigcup_{i=1}^3 \mathcal{L}_1 \right\} \cup \{0\}.$$

It is thus necessary to consider also the  $\mathcal{L}$ -3 subspaces.

If  $z \in \mathcal{L}_k$ , then we can associate a unique  $2k$ -dimensional subspace  $[z]$  of  $\mathcal{U}$  with  $z$  (Theorem 5 of [2]).

**LEMMA 5.** *Let  $z \in \mathcal{L}_k$  and  $x_1 \in [z]$ . Then there is a representation  $z = x_1 \wedge u_2 + u_3 \wedge u_4 + \dots + u_{2k-1} \wedge u_{2k}$  where  $\langle u_2, \dots, u_{2k} \rangle = [z] - \langle x_1 \rangle$ .*

*Proof.* Let  $x_1$  be extended to a basis  $\{x_1, \dots, x_{2k}\}$  of  $[z]$ . Then

$$\begin{aligned} z &= \sum \alpha_{i,j} x_i \wedge x_j \quad (1 \leq i < j \leq 2k) \\ &= x_1 \wedge \left( \sum_{j=2}^{2k} \alpha_{1,j} x_j \right) + \sum \alpha_{i,j} x_i \wedge x_j \quad (2 \leq i < j \leq 2k). \end{aligned}$$

By Corollary 8 of [2] and the fact that  $\mathcal{L}(z) = k$ , the second term

in the expression of  $z$  has irreducible length  $(k-1)$ . The result follows.

**THEOREM 4.** *Let  $\dim \mathcal{U} = 6$ .  $H$  an  $\mathcal{L}$ -3 subspace. Then  $\dim H = 1$ .*

*Proof.* If  $u_1 \in \mathcal{U}$  and  $f$  is any nonzero member of  $H$ , then  $u_1 \in [f]$ . Hence  $f$  can be represented  $f = u_1 \wedge u + y$ , where  $u \in \mathcal{U}$  and  $y \in \mathcal{L}_2$ ,  $[y] \subset \mathcal{U} - \langle u_1 \rangle$ ; (Lemma 5). This latter subspace has dimension 5. Thus, if  $f_1, f_2$  are any 2 nonzero members of  $H$ , then  $f_1 = u_1 \wedge u_2 + u_3 \wedge u_4 + u_5 \wedge u_6$ , where  $\mathcal{U} = \langle u_1, \dots, u_6 \rangle$ , and  $f_2$  can be expressed as  $f_2 = u_1 \wedge y_1 + u_3 \wedge y_2 + u_5 \wedge y_3$  where  $y_i = \sum_{j=2}^6 a_{ij} u_j$ , using the fact that  $\langle f_1, f_2 \rangle$  is an  $\mathcal{L}$ -3 subspace, Corollary 8 of [2] and Corollary 1 of [3].

Consider  $f = \gamma f_1 + f_2$ ,  $\gamma \in F$ . Now  $f = u_1 \wedge [(\gamma + a_{12})u_2 + a_{13}u_3 + a_{14}u_4 + a_{15}u_5 + a_{16}u_6] + u_3 \wedge [a_{22}u_2 + (\gamma + a_{24})u_4 + a_{25}u_5 + a_{26}u_6] + u_5 \wedge [a_{32}u_2 + a_{33}u_3 + a_{34}u_4 + (\gamma + a_{36})u_6] = w_1 \wedge w_2 + w_3 \wedge w_4 + w_5 \wedge w_6$ , putting  $w_1 = u_1$ ,  $w_2 = [(\gamma + a_{12})u_2 + a_{13}u_3 + a_{14}u_4 + a_{15}u_5 + a_{16}u_6]$ , and so on. Then  $\mathcal{L}(f) = 3$  if and only if the vectors  $w_1, \dots, w_6$  are independent (Theorem 7 of [2]); i. e., if and only if the determinant of the matrix  $(a_{ij})$ , where  $a_{ij}$  is the coefficient of  $u_i$  in  $w_j$ ;  $i, j = 1, \dots, 6$ ; is *non-zero*. However this determinant is a monic polynomial in  $\gamma$  of degree 3; viz.,  $(\gamma + a_{12})((\gamma + a_{24})(\gamma + a_{36}) - a_{34}a_{26}) - a_{22}(a_{14}(\gamma + a_{36}) - a_{34}a_{16}) + a_{32}(a_{14}a_{26} - a_{16}(\gamma + a_{24}))$ , whose constant term *must* be nonzero since the vectors  $u_1, u_2, u_3, y_1, y_2, y_3$  are independent. Hence there is a *nonzero* value of  $\gamma$  in  $F$  for which the determinant is zero (since  $F$  is algebraically closed). For this value of  $\gamma$ ,  $\mathcal{L}(f) < 3$ . Hence there is at most one basis member in  $H$ .

**THEOREM 5.** *Let  $\dim \mathcal{U} = 6$ . Then  $\mathcal{T}$  is an  $\mathcal{L}$ -2 preserver if and only if  $\mathcal{T}$  is an  $\mathcal{L}$ -1 preserver.*

*Proof.* It is sufficient to prove that  $\mathcal{T}(\mathcal{L}_1)$  does not intersect  $\mathcal{L}_2 \cup \mathcal{L}_3$  (cf. proof of Theorem 2 and use Lemma 4).

Suppose  $\mathcal{U} = \langle u_1, \dots, u_6 \rangle$  and  $\mathcal{T}(u_1 \wedge u_6) \in \mathcal{L}_2$ . Consider  $V = \langle z_1, \dots, z_4 \rangle$  where

$$z_1 = u_1 \wedge u_6; z_2 = u_1 \wedge u_3 + u_2 \wedge u_4; z_3 = u_1 \wedge u_4 + u_2 \wedge u_5;$$

$$z_4 = u_1 \wedge u_5 + u_2 \wedge u_6.$$

Then  $\mathcal{T}(V)$  is an  $\mathcal{L}$ -2 subspace of dimension 4, contradicting the fact that the maximal  $\mathcal{L}$ -2 subspaces have dimension 3 (Theorem 11 of [3]).

Suppose  $\mathcal{T}(u_1 \wedge u_3) \in \mathcal{L}_3$ . Let  $V = \langle z_1, z_2 \rangle$  where  $z_1 = u_1 \wedge u_3$ ;  $z_2 = u_1 \wedge u_4 + u_2 \wedge u_3 + u_6 \wedge u_5$ . Then  $\mathcal{T}(V)$  is an  $\mathcal{L}$ -3 subspace of dimension 2, contradicting Theorem 4.

### 3. The main results. We can now assert :

**THEOREM 6.**  $\mathcal{T}$  is an  $\mathcal{L}$ -2 preserver if and only if  $\mathcal{T}$  is an  $\mathcal{L}$ -1 preserver. If  $\mathcal{T}$  is an  $\mathcal{L}$ -2 preserver, then  $\mathcal{T}$  is an  $\mathcal{L}$ - $k$  preserver,  $k = 1, 2, \dots, [n/2]$ ,  $\dim \mathcal{U} = n$ , and  $\mathcal{T}$  is a compound except when  $n = 4$ , in which case  $\mathcal{T}$  may possibly be a composite of a compound and a linear transformation induced by a correlation of the 2-dimensional subspaces of  $\mathcal{U}$ .

Using the results in [7], we can also assert the following.

**THEOREM 7.**  $\mathcal{S}$  is a  $\rho$ -4 preserver if and only if  $\mathcal{S}$  is a  $\rho$ -2 preserver. If  $\mathcal{S}$  is a  $\rho$ -4 preserver, then  $\mathcal{S}$  is a  $\rho$ - $2k$  preserver,  $k = 1, 2, \dots, [n/2]$ . Moreover, if  $A$  is any  $n$ -square skew-symmetric matrix, then  $\mathcal{S}(A) = \alpha PAP'$  or  $\mathcal{S}(A) = \beta PA' P'$  for  $\alpha, \beta$  nonzero in  $F$  and some nonsingular  $n$ -square matrix  $P$  except when  $n = 4$ , in which case  $\mathcal{S}$  may possibly be of the form

$$\mathcal{S}(A) = \alpha P \begin{vmatrix} 0 & a_{34} & a_{24} & a_{23} \\ -a_{34} & 0 & a_{14} & a_{13} \\ -a_{24} & -a_{14} & 0 & a_{12} \\ -a_{23} & -a_{13} & -a_{12} & 0 \end{vmatrix} P'$$

where  $A = (a_{ij})$ ,  $a_{ij} = -a_{ji}$ .

**REMARK.** These results are not necessarily true when the underlying field  $F$  is nonalgebraically closed (cf. § 2b. and end of [2]).

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