TRANSVERSALLY PERTURBED PLANAR DYNAMICAL SYSTEMS

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This paper investigates the behavior of limit cycles of a planar dynamical system which has been perturbed transversely. In particular, it is shown that if \( C \) is a limit cycle of the unperturbed dynamical system, then there are limit cycles of the perturbed dynamical systems arbitrarily close to \( C \). Also, if \( C \) is an exterior limit cycle of the unperturbed dynamical system, then there is an outer neighborhood of \( C \) which consists solely of cycles of the perturbed dynamical systems.

In what follows \( R \) and \( R^2 \) will denote the reals and the plane respectively.

A dynamical system is an ordered pair \((X, \pi)\) consisting of a topological space \( X \) and a mapping \( \pi \) of \( X \times R \) into \( X \) such that (where \( x \pi t = \pi(x, t) \))

1. \( x \pi t = x \) for all \( x \in X \)
2. \( (x \pi t) \pi s = x \pi (t + s) = x, (s + t) \) for all \( x \in X \) and \( s, t \in R \)
3. \( \pi \) is continuous in the product topology.

A point \( x \in X \) is called critical if and only if \( x \pi t = x \) for every \( t \in R \). A point \( x \in X \) is called periodic if and only if \( x \) is noncritical and \( x \pi t = x \) for some \( t > 0 \); if \( X \) is Hausdorff the least such \( t \) is called the fundamental period of \( x \). If \( x \) is periodic, \( x \pi R \) is called a cycle. A cycle is a simple closed curve. Hence, if \( C \) is a cycle of a planar dynamical system \((R^2, \pi)\), then \( C \) decomposes \( R^2 \) into two components; one bounded and denoted by \( \text{int} C \); the other unbounded and denoted by \( \text{ext} C \). A subset \( A \) of \( X \) is called a trajectorial arc if and only if there is an \( x \in X \) and a compact interval \([a, b] \), \( a \neq b \), such that \( A = x \pi [a, b] \).

Let \((R^2, \pi)\) be a dynamical system. A subset \( T \) of \( R^2 \) is called a transversal if and only if

1. \( T \) is homeomorphic with either \([0, 1] \) or \( S^1 \), the 1-sphere
2. there is an \( \varepsilon > 0 \) such that \( T \cap (T \pi t) = \emptyset \) for \( 0 < |t| \leq \varepsilon \).

Our investigation depends heavily upon the following three propositions which may be found in [2, VII, 4.4], [2, VII, 4.7], and [2, VII, 4.8] respectively.

**Proposition A.** Let \( C \) be a trajectory and \( T \) a transversal of a planar dynamical system. If \( C \) or \( T \) is a closed curve, they have at most one intersection point; if both are closed curves, they do not
PROPOSITION B. Let \( C \cup T \) be a simple closed curve with \( C \) a trajectorial arc and \( T \) a transversal of a planar dynamical system. Then one component of \( \mathbb{R}^2 - (C \cup T) \) is positively invariant, the second is negatively invariant, and neither is invariant. The result is also valid if \( C = \emptyset \).

PROPOSITION C. In a planar dynamical system the interior of each cycle, closed transversal, or simple closed curve consisting of a transversal and a trajectorial arc, all contain a critical point.

We are interested in studying a family of dynamical systems which is defined as follows. Let \( \pi: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2 \) be a mapping continuous in the product topology such that

(i) for each \( a \in \mathbb{R} \) the mapping \( \pi_a: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2 \) defined by \( \pi_a(x, t) = \pi(x, t, a) \) defines a dynamical system on \( \mathbb{R}^2 \).

(ii) critical points of the dynamical systems are independent of the index.

(iii) the noncritical trajectories of \( \pi_a \) are transversal to the noncritical trajectories of \( \pi_b \) if \( a \neq b \), i.e., if \( T \) is a trajectorial arc of \( \pi_a \), then \( T \) is a transversal with respect to \( \pi_b \) if \( a \neq b \).

\( C_a(x), C^a_+(x), L^a_+(x), \) and \( L^a_-(x) \) will denote the trajectory, positive semitrajectory, positive limit set, and negative limit set, respectively, of \( x \) with respect to \( \pi_a \). The family of all trajectories of \( \pi_a \), a fixed, will be called a system and the family of all trajectories will be called a complete family.

In [1] and [4] sufficient conditions are given which assure that the differential equations

\[
\dot{x} = P(x, y, a), \quad \dot{y} = Q(x, y, a),
\]

where the dots stand for differentiation with respect to the independent variable \( t \) and \( a \) is a parameter, define a complete family.

Immediate consequences of Propositions A and C are the following two propositions.

PROPOSITION 1. Cycles of distinct systems of a complete family do not intersect.

PROPOSITION 2. Let \( x \) be a noncritical point of a complete family, \( a \neq b \), and suppose that \( C_a(x) \) and \( C_b(x) \) have a point \( y, y \neq x \), in common. If the trajectorial arcs of \( C_a(x) \) and \( C_b(x) \) connecting the points \( x \) and \( y \) have only their endpoints in common, then the region
bounded by these trajectorial arcs contains a critical point.

**Proposition 3.** Let \( C \) be a cycle of \( \pi_a \). Then \( \text{int} \ C \) is positively invariant with respect to \( \pi_b \) for all \( b > a \) or \( \text{int} \ C \) is negatively invariant with respect to \( \pi_b \) for all \( b > a \), but in neither case is \( \text{int} \ C \) invariant with respect to \( \pi_b \) for any \( b > a \). A similar result holds for \( b < a \).

**Proof.** Consider the sets
\[
A = \{ b \in (a, +\infty) : \text{int} \ C \text{ is positively invariant with respect to } \pi_b \} \\
B = \{ b \in (a, +\infty) : \text{int} \ C \text{ is negatively invariant with respect to } \pi_b \}.
\]
By Proposition B, \( \text{int} \ C \) is positively invariant or negatively invariant, but not both, with respect to each \( \pi_b \), \( b > a \). Thus \( A \cup B = (a, +\infty) \) and \( A \cap B = \emptyset \). We now show that both \( A \) and \( B \) are open. If \( c \in (a, +\infty) - A = B \), then there exist \( x \in \text{int} \ C \) and \( t > 0 \) such that \( x\pi_t \in \text{ext} \ C \). Since \( \pi \) is continuous \( x\pi_t \in \text{ext} \ C \) for all \( b \) sufficiently close to \( c \). Hence \( B \) is open. Similarly \( A \) is open. The connectivity of \( (a, +\infty) \) implies either \( A \) or \( B \) must be empty. This completes the proof.

**Proposition 4.** Let \( C \) be a cycle of \( \pi_a \). If \( \text{int} \ C \) is positively invariant with respect to every \( \pi_b \), \( b > a \), then \( \text{ext} \ C \) is positively invariant with respect to every \( \pi_b \), \( b < a \). A similar result holds if \( b > a \) and \( b < a \) are interchanged.

**Proof.** Let \( x \in C \) and \( T \) be a trajectorial arc of \( C_c(x), c > a \), which contains \( x \) as a nonend point. Then \( T \) is a transversal with respect to \( \pi_b \), \( b \neq c \). Moreover, if \( \tau \) is the fundamental period of \( C \), then \( T\pi_a[-\tau, \tau] \) is a connected neighborhood of \( C \) which contains no critical points. Choose a neighborhood \( U \) of \( x \), \( 0 < \sigma < |c-a| \), and \( 0 < \varepsilon < \tau \) so small that \( U\pi_a[-\varepsilon, \varepsilon] \subset T\pi_a[\tau, \tau] \) for all \( b \in [a-\sigma, a+\sigma] \). This is possible because \( \pi \) is continuous. We can now define a mapping \( h \) of \( [a, a+\sigma] \) into \( S = \{ x\pi_b : b \in [a, a+\sigma] \} \) by \( h(b) = x\pi_b \). \( h \) is continuous since \( \pi \) is continuous. For \( b \neq d \), \( x\pi_b \) and \( x\pi_d \) cannot be equal; for if they were Proposition 2 would imply that \( T\pi_a[-\tau, \tau] \) contains a critical point. Hence \( h \) is one-to-one. Obviously, \( h \) is an onto mapping. A one-to-one continuous mapping of a compact space onto a Hausdorff space is a homeomorphism. Thus \( S \) is an arc. Since \( \text{int} \ C \) is, by assumption, positively invariant with respect to \( \pi_b \), \( b > a \), we have \( S \subset \overline{\text{int} \ C} \). Moreover, \( (x\pi_a[0, \varepsilon]) \cup S \cup (x\pi_a+\sigma[0, \varepsilon]) \) forms a simple closed curve \( J \) such that \( \text{int} \ J \subset T\pi[-\tau, \tau] \) and \( \overline{\text{int} \ J} \) is a neighborhood of \( x\pi_a \varepsilon/2 \) relative to \( \overline{\text{int} \ C} \). Let \( y \in \text{int} \ J \) and set
\[ J_t = (x_{\pi a}[0, t]) \cup (x_{\pi a+\sigma}[0, t]) \cup \{x_{\pi b}t: b \in [a, a+\sigma]\}. \]

For each \( t, 0 < t < \varepsilon \), \( J_t \) is a simple closed curve. Since \( \pi \) is continuous, \( y \in \text{ext} J_t \) for \( t \) sufficiently small. But for \( t = \varepsilon \), \( y \in \text{int} J = \text{int} J \). The continuity of \( \pi \) implies there is an \( s \in (0, \varepsilon) \) such that \( y \in J_s \). By the construction of \( J_t \) and since \( y \in \text{int} J \), \( y \) must be an element of \( \{x_{\pi b}t: b \in [a, a+\sigma]\} \). This shows that \( \overline{\text{int} J} \) consists solely of trajectorial arcs from the systems \( \pi_b \), \( b \in [a, a+\sigma] \).

Now let \( V \) be a neighborhood of \( x_{\pi a}\varepsilon/2 \) such that \( V \cap \text{int} C \subset \text{int} J \). Then there is an \( \alpha, 0 < \alpha < \sigma \), such that \( x_{\pi_b}\varepsilon/2 \in V \) for all \( b \in [\alpha, a] \).

For \( b \in [a-\alpha, 0) \), \( x_{\pi_b}\varepsilon/2 \) cannot be an element of \( \text{int} C \) for then
\[ x_{\pi_b}\varepsilon/2 \notin V \cap \text{int} C \subset \text{int} J \subset \bigcup \{x_{\pi c}[0, \varepsilon]: c \in [a, a+\sigma]\}. \]

This, by Proposition 2, implies that \( T_{\pi a}[-\tau, \tau] \) contains a critical point. Hence for \( b \in [a-\alpha, 0) \) we have \( x_{\pi_b}\varepsilon/2 \in V \) and therefore, by Proposition 2, \( C_b \) is contained in \( \text{ext} C \). Proposition 3 now implies the desired result.

Proposition 4 allows us to assume throughout the remainder of the paper that if \( C \) is a given cycle of \( \pi_a \), then \( \text{int} C \) is positively invariant with respect to every \( \pi_b \), \( b < a \), and negatively invariant with respect to every \( \pi_b \), \( b > a \). If the opposite invariance properties hold, the following propositions remain valid after the obvious modifications are made.

**Definition 5.** Let \( C \) be a cycle of \( \pi_a \). If there is an \( x \in \text{ext} C \) such that \( L^*_a(x) = C \) or \( L^-_a(x) = C \), then \( C \) is called an external limit cycle or a external negative limit cycle, respectively. Similarly, if there is an \( x \in \text{int} C \) such that \( L^\tau_a(x) = C \) or \( L^-_a(x) = C \), then \( C \) is called an internal limit cycle or a internal negative limit cycle, respectively.

**Definition 6.** Let \( U \) be a neighborhood of a simple closed curve \( C \). Then \( \text{U-int} C \) and \( \text{U-ext} C \) are called an outer neighborhood and an inner neighborhood, respectively, of \( C \).

**Proposition 7.** Let \( C \) be an external limit cycle of \( \pi_a \). Then, given any outer neighborhood \( U \) of \( C \), there exists an \( \varepsilon > 0 \) such that, for each \( b \in [a, a+\varepsilon] \), \( U \) contains both an external limit cycle and an internal limit cycle of \( \pi_b \) (the two cycles may coincide). A similar result holds for \( C \) an internal limit cycle and \( b \in [a-\varepsilon, a] \).

**Proof.** Let \( V \subset U \) be an outer neighborhood of \( C \) containing no critical points and such that \( \text{int} C \cup V \) is simply connected. Let \( x \in C \), \( y \in \text{ext} C \) be such that \( L^\tau_a(y) = C \), and \( T \subset V \) be a trajectorial arc of
$C_e(x), c < a$, containing $x$ as an endpoint. Then $T$ is a transversal with respect to $\pi_a$, $b \neq c$. Since $L^+_e(y) = C, y \in \text{ext } C,$ and $V$ is an outer neighborhood of $C$, there is a $\tau > 0$ such that $y_\pi_a[\tau, +\infty) \subset V$. Let $y_1, y_2 \in y_\pi_a[\tau, +\infty)$ be consecutive points of intersection between $C^+_e(y)$ and $T$ with $y_2 \in C^+_e(y_1)$. Then the trajectial arc of $C^+_e(y)$ and the subarc of $T$ connecting $y_1$ and $y_2$ form a simple closed curve $J \subset V$. Now $L^+_a(y_1) = L^+_e(y) = C \subset \text{int } J$ and Proposition B imply $y_\pi_e(0, +\infty) \subset \text{int } J$. Since $y_2 \in C^+_e(y_1)$ and $\pi$ is continuous there is an $\varepsilon > 0$ such that $C^+_e(y)$ intersects $\text{int } J$ for $|b-a| < \varepsilon$. If $y_\pi_e t \in \text{int } J$ for some $t > 0$, then $y_\pi_e[t, +\infty)$ must be a subset of $\text{int } J$; for it were not $y_\pi_e[t, +\infty)$ would intersect $J$ and Proposition 2 would imply $\text{int } J - \text{int } C$, and hence $V$, contains a critical point. Moreover, by the continuity of $\pi$, and the fact $L^+_e(y_1) = C$, we may assume that $\varepsilon$ was chosen so small that $C^+_e(y_1), |b-a| < \varepsilon$, intersects $T$ at least twice between $y_2$ and $x$. This is true because $C^+_e(y_1)$ intersects $T$ infinitely many times and the only limit point of the intersections is $x$, [2, VIII, 1.2] and [2, VIII, 1.5]. The trajectorial arc connecting two such consecutive points of intersection and the corresponding subarc of $T$ form a simple closed curve $J_\pi$ such that $\text{int } J_\pi \subset \text{int } J$ and $\text{int } J_\pi - \text{int } C \subset V$. Moreover, $\text{int } J_\pi$ is positively invariant with respect to $\pi_e$ by Proposition B. Thus $\text{int } J_\pi$ and $\text{ext } C$ are both positively invariant with respect to $\pi_e$. Hence $\text{int } J_\pi - \text{int } C$ is positively invariant, so that $C^+_e(x) \subset \text{int } J_\pi - \text{int } C$ which is compact and contains no critical points. By the Poincaré-Bendixson Theorem, [2, VII, 1.14], $L^+_e(x)$ is a cycle $C_e$. Since $\text{int } J_\pi$ is positively invariant, but not invariant by Proposition B, and $C_e \cap C = \emptyset$ by Proposition 1, we have $C_e \cap \partial(\text{int } J_\pi - \text{int } C) = \emptyset$. Thus $C_e$ is an internal limit cycle of $\pi_e$ contained in $\text{int } J_\pi \subset U$. For $c$ sufficiently large $y_\pi_e[c, +\infty) \subset \text{int } J_\pi$ and therefore $y_\pi_e[c, +\infty) \subset \text{int } J_\pi - \text{int } C_e$. The Poincaré-Bendixson Theorem now implies the existence of an external limit cycle. This completes the proof.

In a similar manner it can be shown that

**Proposition 8.** Let $C$ be an external negative limit cycle of $\pi_a$. Then, given any outer neighborhood $U$ of $C$, there exists an $\varepsilon > 0$ such that, for each $b \in [a-\varepsilon, a]$, $U$ contains both an external negative limit cycle and an internal negative limit cycle of $\pi_a$ (the two cycles may coincide). A similar result holds for $C$ an internal negative limit cycle and $b \in [a, a+\varepsilon]$.

**Lemma 9.** Let $D_1$ and $D_2$ be cycles of a complete family such that $D_1 \subset \text{int } D_2$ and that $\text{int } D_2 - \text{int } D_1$ contains no critical points.
If $C_1$ and $C_2$ are distinct cycles in $\text{int } D_2 - \text{int } D_1$, then $C_1 \subset \text{int } C_2$ or $C_2 \subset \text{int } C_1$.

**Proof.** Since $\text{int } D_2 - \text{int } D_1$ contains no critical points, we must have $D_i \subset \text{int } C_i$, $i = 1, 2$. Thus $\text{int } C_1 \cap \text{int } C_2 \neq \emptyset$. Then $\text{int } C_1 \subset \text{int } C_2$ or $\text{int } C_1 \cap \text{ext } C_2 \neq \emptyset$. In the first case $\text{int } C_1 \subset \text{int } C_2$. Therefore $C_1 \subset \text{int } C_2$ or $C_1 \cap C_2 \neq \emptyset$. The latter is impossible by Proposition 1. In the second case, $\partial (\text{int } C_2) \cap \text{int } C_1 \neq \emptyset$. Therefore $C_2 \cap \text{int } C_i \neq \emptyset$ and $C_2 \subset \text{int } C_1$ since $\text{int } C_1$ is either positively invariant or negatively invariant for the system containing $C_2$ (Proposition 3).

Let $D_1$ and $D_2$ be as in the statement of Lemma 9. Then

**Lemma 10.** If $C_1$ and $C_2$ are distinct cycles in $\text{int } D_2 - \text{int } D_1$ such that $C_1 \subset \text{ext } C_2$, then $C_2 \subset \text{int } C_1$.

**Proof.** By Lemma 9, $C_2 \subset \text{int } C_1$ or $C_1 \subset \text{int } C_2$. $C_i$ cannot be contained in both $\text{int } C_2$ and $\text{ext } C_2$. Therefore $C_2 \subset \text{int } C_1$.

In a topological space $X$, it is possible to define limits of nets of subsets $X_i \subset X$ as follows. Let $\lim \inf X_i$ consist of all limits of nets of points $x_i \in X_i$; let $\lim \sup X_i$ consist of all limits of subnets of points $x_i \in X_i$. Obviously $\lim \inf X_i \subset \lim \sup X_i$. If equality holds, the net $X_i$ is said to converge to its limit and we write

$$\lim X_i = \lim \inf X_i = \lim \sup X_i.$$

**Definition 11.** A net $(R^2, \pi_i)$, $i$ contained in a directed set containing 0, of dynamical system is called regular if

(i) $\pi_i \to \pi_0$ in the sense that if $x_i \to x$ and $t_i \to t$ then $x_i \pi_i t_i \to x \pi_0 t$.

(ii) critical points are independent of the index $i$.

(iii) to each noncritical point $x$ there corresponds a subset $T$ of $R^2$ which is a transversal with respect to each $\pi_i$ and contains $x$ as a nonend point.

In [3] the following theorem is proved.

**Theorem D.** Let $(R^2, \pi_i)$ be a regular net of dynamical systems. Let $C_i(x_i)$ be a cycle of $(R^2, \pi_i)$ with fundamental period $\tau_i(x_i)$. If $\lim \inf C_i(x_i) \neq \emptyset$, then

1. If $\tau_i(x_i) \to 0$, then $\lim C_i(x_i)$ exists and is a single critical point.

2. If $\lim \inf C_i(x_i)$ intersects a cycle $C_0(x)$, then $\tau_i(x_i) \to \tau_0(x)$ and $\lim C_i(x_i) = C_0(x)$. 
(3) If \( \lim \inf C_i(x_i) \) intersects a noncyclic trajectory, then \( \tau_i(x_i) \rightarrow +\infty \).

**DEFINITION 12.** Let \( C_a(x) \) be a cycle of \( \pi_a \). Then \( \tau_a(x) \) will denote the fundamental period of \( x \) with respect to \( \pi_a \).

**PROPOSITION 13.** Let \( C \) be an external limit cycle of \( \pi_a \). There exists an outer neighborhood \( U \) of \( C \) and an \( \varepsilon > 0 \) such that \( U \) consists entirely of periodic points of the systems \( \pi_b \), \( b \in [a, a + \varepsilon] \). A similar result holds for \( C \) an internal limit cycle and \( b \in [a - \varepsilon, a] \).

**Proof.** Let \( x \in C \) and \( V \) be an outer neighborhood of \( C \) which contains no other cycles of \( \pi_a \) or critical points and such that \( V \cup \text{int } C \) is simply connected. Moreover, by Theorem D, \( V \) may be chosen along with a \( \sigma > 0 \) such that if \( C_b(y) \) is a cycle of \( \pi_b \) in \( V \) with \( |b-a| < \sigma \), then \( |\tau_b(x) - \tau_a(y)| < 1/2 \tau_a(x) \). By Proposition 7 there is an \( \varepsilon, 0 < \varepsilon < \sigma \) such that, for each \( b \in [a, a + \varepsilon] \), \( V \) contains a cycle of \( \pi_a \). Thus the fundamental periods cycles of \( \pi_a + \varepsilon \) which lie in \( V \) are contained in \( [1/2 \tau_a(x), 3/2 \tau_a(x)] \). This, Theorem D with each \( i = a + \varepsilon \), and the fact that cycles of distinct systems do not intersect imply that there is a cycle \( D \) of \( \pi_{a+\varepsilon} \) which lie in \( V \) and that \( |D| - \text{int } C \) contains no cycle of \( \pi_{a+\varepsilon} \).

Set \( U = \text{int } D - \text{int } C \). \( U \) is an outer neighborhood of \( C \) by Lemma 10. Let \( A \) denote the set of periodic points of \( \pi_a \), \( b \in [a, a + \varepsilon] \), which are contained in \( U \). We will show that \( A = U \). Assume the contrary that there exists a \( w \in U - A \) and consider the sets

\[
F = \{ \text{int } C_b(y) : y \in A, C_b(y) \text{ a cycle, } w \in \text{ext } C_b(y) \}
\]

\[
G = U \cup F.
\]

Since \( w \in U \), we have \( w \in \text{ext } C = \text{ext } C_a(x) \), so that \( F \neq \emptyset \). If \( C_a(y) \subset G \subset U \), then \( \tau_b(y) \in [1/2 \tau_a(x), 3/2 \tau_a(x)] \). Proposition 7 and Theorem D now imply, respectively, that \( \partial G \cap \text{ext } C \neq \emptyset \) and \( \partial G \) consists entirely of periodic points. Lemma 9 implies that \( \partial G \cap \text{ext } C \) is a cycle \( C_d(z) \) where \( z \in U \) and \( d \in [a, a + \varepsilon] \). Moreover, since \( w \in \text{ext } C_a(y) \) for each \( \text{int } C_b(y) \) in \( F \) and \( C_b(w) \) is not a cycle for any \( b \in [a, a + \varepsilon] \), we have \( w \in \text{ext } C_d(z) \). \( d \neq a \) since \( C_d(z) = \partial G \cap \text{ext } C \subset V \) and the only cycle of \( \pi_a \) in \( V \) is \( C \). Since \( U \neq A \), \( C_d(z) \neq D \). Hence \( d \neq a + \varepsilon \). Also, by the construction of \( C_d(z) \), there is no cycle \( B \) of \( \pi_{a+\varepsilon} \), \( b \in [a, a + \varepsilon] \), in \( U \) such that \( C_d(z) \subset \text{int } B \) and \( w \in \text{ext } B \). Thus \( C_d \) is either an external limit cycle or an external negative limit cycle, [2, VIII, 3. 3]. Proposition 7 or 8, respectively, now implies the existence of a \( c \in [a, a + \varepsilon] \) such that a cycle \( C_d \) of \( \pi_a \) has the property that \( C_d(z) \subset \text{int } C \) and \( w \in \text{ext } C \). This contradiction implies \( A = U \). This completes the proof.
In a similar manner it can be shown that

**PROPOSITION 14.** Let C be an external negative limit cycle of $\pi_a$. There exists an outer neighborhood $U$ of $C$ and an $\varepsilon > 0$ such that $U$ consists entirely of periodic points of the systems $\pi_b$, $b \in [a-\varepsilon, a]$. A similar result holds for $C$ an internal negative limit cycle and $b \in [a, a+\varepsilon]$.

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