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**ON A SET OF POLYNOMIALS SUGGESTED BY LAGUERRE
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TILAK RAJ PRABHAKAR

ON A SET OF POLYNOMIALS SUGGESTED BY LAGUERRE POLYNOMIALS

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Generating functions, integrals and recurrence relations are obtained for the polynomials $Z_n^\alpha(x; k)$ in x^k which form one set of the biorthogonal pair with respect to the weight function $e^{-x}x^\alpha$ over the interval $(0, \infty)$, the other set being that of polynomials in x .

A singular integral equation with $Z_n^\alpha(x; k)$ in the kernel is solved in terms of a generalized Mittag-Leffler's function and a unified formula for fractional integration and differentiation of the polynomials is derived.

It is known [7] that the polynomials $Z_n^\alpha(x; k)$ of degree n in x^k for positive integers k and $\text{Re } \alpha > -1$ are characterized up to a multiplicative constant by the above requirements. Konhauser [8] discussed the biorthogonality of the pair $\{Z_n^\alpha(x; k)\}$, $\{Y_n^\alpha(x; k)\}$ in the basic polynomials x^k and x , over the interval $(0, \infty)$ and with the admissible weight function $e^{-x}x^\alpha$ of the generalized Laguerre polynomial set $\{L_n^\alpha(x)\}$. Indeed the polynomials have several properties of interest and Konhauser [8] obtained among other things some recurrence relations and a differential equation for the polynomials $Z_n^\alpha(x; k)$ which are our primary concern in this paper. For $k = 2$, Preiser [11] obtained for these polynomials a generating function, a differential equation, integral representations and recurrence relations. Earlier Spencer and Fano [13] also used these polynomials for $k = 2$.

For $k = 1$, all the results proved in this paper reduce to those for $L_n^\alpha(x)$; in particular the integral equation (3.1) either reduces to or contains as still more special cases the integral equations solved by Widder [14], Buschman [1], Khandekar [6], Rusia [12] and Prabhakar ([10], (7 · 1)). For $k = 2$, the results are essentially the same as those in [11] or [13].

2. Some properties of $Z_n^\alpha(x; k)$. We now obtain a generating function, a contour integral representation and a fractional integration formula for $Z_n^\alpha(x; k)$. In § 3, we need the Laplace transform and in § 4 derive a more general class of generating functions for the polynomials. Recurrence relations and a few other results will follow as natural consequences. We shall freely use the closed form ([8], (5))

$$(2.1) \quad Z_n^\alpha(x; k) = \frac{\Gamma(kn + \alpha + 1)}{n!} \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj + \alpha + 1)}$$

for $\operatorname{Re} \alpha > -1$; naturally the results may be established from alternative characterizations of $Z_n^\alpha(x; k)$ but such a discussion does not seem to be of sufficient interest.

(i) *A generating function.* We obtain the generating function indicated in

$$(2.2) \quad e^t \phi(k, \alpha + 1; -x^k t) = \sum_{n=0}^{\infty} \frac{Z_n^\alpha(x; k) t^n}{\Gamma(kn + \alpha + 1)}$$

where $\phi(a, b; z)$ is the Bessel-Maitland function ([15], (1.3); [3], 18.1 (27)).

From (2.1), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{Z_n^\alpha(x; k) t^n}{\Gamma(kn + \alpha + 1)} &= \sum_{n=0}^{\infty} \sum_{m=0}^n \frac{(-1)^m x^{km} t^n}{m!(n-m)! \Gamma(km + \alpha + 1)} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m x^{km} t^{n+m}}{m! n! \Gamma(km + \alpha + 1)} \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{m=0}^{\infty} \frac{(-x^k t)^m}{m! \Gamma(km + \alpha + 1)} \\ &= e^t \phi(k, \alpha + 1; -x^k t) \end{aligned}$$

and (2.2) is established.

Denoting $e^t \phi(k, \alpha + 1; -x^k t)$ by $f(x, t)$ we at once find that $f(x, t)$ satisfies the partial differential equation

$$x \frac{\partial f}{\partial x} - \alpha t \frac{\partial f}{\partial t} + \alpha t f = 0.$$

Substituting for $f(x, t)$ from (2.2) and equating the coefficients of t^n , we obtain the differential recurrence relation

$$x Z_n^\alpha(x; k) = n k Z_n^\alpha(x; k) - k \frac{\Gamma(kn + \alpha + 1)}{\Gamma(kn + \alpha - k + 1)} Z_{n-1}^\alpha(x; k),$$

also obtained by Konhauser ([8], (6)) by direct calculations.

(ii) *Schl\"{a}fli's Contour integral.* It is easy to show that

$$(2.3) \quad Z_n^\alpha(x; k) = \frac{\Gamma(kn + \alpha + 1)}{n! 2\pi i} \int_{-\infty}^{(0+)} \frac{(t^k - x^k)^n e^t dt}{t^{kn + \alpha + 1}}.$$

$$\begin{aligned} \text{For } \frac{1}{2\pi i} \int_{-\infty}^{(0+)} \frac{(t^k - x^k)^n e^t dt}{t^{kn + \alpha + 1}} &= \sum_{j=0}^n (-1)^j \binom{n}{j} x^{kj} \frac{1}{2\pi i} \int_{-\infty}^{(0+)} t^{-(kj + \alpha + 1)} e^t dt \\ &= \sum_{j=0}^n \frac{(-1)^j \binom{n}{j} x^{kj}}{\Gamma(kj + \alpha + 1)} \end{aligned}$$

using Hankel's formula ([3], 1.6(2))

$$(2.4) \quad \frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^t t^{-z} dt ;$$

finally (2.3) follows from (2.1)

For $k = 1$, (2.3) reduces to the known result ([2], p. 269)

$$L_n^\alpha(x) = \frac{\Gamma(n + \alpha + 1)}{n! 2\pi i} \int_{-\infty}^{(0+)} \left(1 - \frac{x}{t}\right)^n e^t \frac{dt}{t^{\alpha+1}} .$$

If α is also a positive integer than the integrand in (2.3) is a single-valued analytic function of t with the only singularity $t = 0$. Hence we can deform the contour into $|t| = b|x|$ and the substitution $t = xu$ then leads to

$$(2.5) \quad x^\alpha Z_n^\alpha(x; k) = \frac{\Gamma(kn + \alpha + 1)}{n! 2\pi i} \int_C (u^k - 1)^n e^{xu} u^{-(kn + \alpha + 1)} du$$

where C denotes the circle $|u| = b$. Indeed C may be replaced by any simple closed contour surrounding the point $u = 0$. For $k = 2$, (2.5) reduces to the integral representation by Preiser ([11], (5.22)).

Using (2.5), it follows that

$$\frac{\partial^k}{\partial x^k} \left[\frac{n! x^{\alpha+k} Z_n^{\alpha+k}(x; k)}{\Gamma(kn + k + \alpha + 1)} \right] = \frac{n! x^\alpha Z_n^\alpha(x; k)}{\Gamma(kn + \alpha + 1)}$$

and $\left(\frac{\partial^k}{\partial x^k} - 1 \right) \left[\frac{n! x^{\alpha+k} Z_n^{\alpha+k}(x; k)}{\Gamma(kn + k + \alpha + 1)} \right] = \frac{(n+1)! x^\alpha Z_{n+1}^\alpha(x; k)}{\Gamma(kn + k + \alpha + 1)}$

which leads to the pure recurrence relation

$$(2.6) \quad x^k Z_n^{\alpha+k}(x; k) = (kn + \alpha + 1)_k Z_n^\alpha(x; k) - (n + 1) Z_{n+1}^\alpha(x; k) .$$

For $k = 2$, (2.6) reduces to ([11], (5.39)).

(iii) *Fractional integrals and derivatives.* We show that

$$(2.7) \quad I^\mu [x^\alpha Z_n^\alpha(x; k)] = \frac{\Gamma(kn + \alpha + 1)}{\Gamma(kn + \alpha + \mu + 1)} x^{\alpha+\mu} Z_n^{\alpha+\mu}(x; k)$$

for $\text{Re } \alpha > -1$ and $\text{Re } \mu > -\text{Re}(1 + \alpha)$ where for suitable f and complex μ , $I^\mu f(x)$ denotes the μ th order fractional integral (or fractional derivative) of $f(x)$ (see [10], § 2).

When $\text{Re } \mu > 0$, we write [10]

$$\begin{aligned} I^\mu [x^\alpha Z_n^\alpha(x; k)] &= \int_0^x \frac{(x-t)^{\mu-1}}{\Gamma(\mu)} t^\alpha Z_n^\alpha(t; k) dt \\ &= \frac{\Gamma(kn + \alpha + 1)}{n! \Gamma(\mu)} \sum_{j=0}^n \frac{(-n)_j}{\Gamma(kj + \alpha + 1)} \int_0^x t^{kj+\alpha} (x-t)^{\mu-1} dt \end{aligned}$$

$$= \frac{\Gamma(kn + \alpha + 1)}{n!} \sum_{j=0}^n (-n)_j \frac{x^{kj+\alpha+\mu}}{\Gamma(kj + \alpha + \mu + 1)} ;$$

hence for $\operatorname{Re} \mu > 0$ and $\operatorname{Re} \alpha > -1$, we obtain

$$(2.8) \quad I^\mu [x^\alpha Z_n^\alpha(x; k)] = \frac{\Gamma(kn + \alpha + 1)}{\Gamma(kn + \alpha + \mu + 1)} x^{\alpha+\mu} Z_n^{\alpha+\mu}(x; k) .$$

But (2.8) may be written as

$$(2.9) \quad x^\alpha Z_n^\alpha(x; k) = \frac{\Gamma(kn + \alpha + 1)}{\Gamma(kn + \alpha + \mu + 1)} I^{-\mu} [x^{\alpha+\mu} Z_n^{\alpha+\mu}(x; k)] ,$$

the inversion being valid for $\operatorname{Re} \mu > 0$ and the assumptions made.

Putting $\mu' = -\mu$, $\alpha' = \alpha + \mu$, we obtain for $\operatorname{Re} \mu' < 0$

$$x^{\alpha'+\mu'} Z_n^{\alpha'+\mu'}(x; k) = \frac{\Gamma(kn + \alpha' + \mu' + 1)}{\Gamma(kn + \alpha' + 1)} I^{\mu'} [x^{\alpha'} Z_n^{\alpha'}(x; k)]$$

which is (4.1) with the letters α , μ accented. Ignoring the accents we can write

$$(2.10) \quad I^\mu [x^\alpha Z_n^\alpha(x; k)] = \frac{\Gamma(kn + \alpha + 1)}{\Gamma(kn + \alpha + \mu + 1)} x^{\alpha+\mu} Z_n^{\alpha+\mu}(x; k)$$

for $\operatorname{Re} \mu < 0$, $\operatorname{Re} \alpha > -1$ and $\operatorname{Re}(\alpha + \mu) > -1$.

When $\operatorname{Re} \mu = 0$, we write $I^\mu = I^{\mu+1} I^{-1}$ and the result easily follows; thus (2.7) is established for all complex μ with $\operatorname{Re} \mu > -\operatorname{Re}(1 + \alpha)$.

REMARK 1. When μ is a negative integer say $-m$, then (2.7) is written as

$$\left(\frac{d}{dx}\right)^m [x^\alpha Z_n^\alpha(x; k)] = \frac{\Gamma(kn + \alpha + 1)}{\Gamma(kn + \alpha - m + 1)} x^{\alpha-m} Z_n^{\alpha-m}(x; k)$$

which can also be proved by direct differentiation provided $\operatorname{Re} \alpha > m - 1$.

REMARK 2. For $k = 1$, (2.7) unifies the results ([3], 10.12(27)) and ([4], 13.1(49)) for Laguerre polynomials.

3. A singular integral equation. We show that the convolution equation

$$(3.1) \quad \int_0^x (x-t)^\alpha Z_n^\alpha(\lambda(x-t); k) f(t) dt = g(x)$$

for $\operatorname{Re} \alpha > -1$ admits a locally integrable solution f given by

$$(3.2) \quad f(x) = \frac{n!}{\Gamma(kn + \alpha + 1)} \int_0^x (x-t)^{l-\alpha-2} E_{k,l-\alpha-1}^n(\lambda(x-t))^k I^{-l} g(t) dt$$

provided $I^{-l} g$ exists for $\text{Re } l > \text{Re } \alpha + 1$ and is locally integrable in $(0, \delta)$, $0 < x < \delta < \infty$.

The function

$$(3.3) \quad E_{a,b}^c(z) = \sum_{j=0}^{\infty} \frac{(c)_j z^j}{\Gamma(\alpha j + b) j!} \quad \text{Re } a > 0$$

is a very special case of the generalized hypergeometric functions considered by Wright [16] and is also expressible as a Fox's H -function [5]. On the other hand $E_{a,b}^c(z)$ is a most natural generalization of the Mittag-Leffler's function $E_a(z)$ ([3], 18.1; [9]) and also contains the confluent hypergeometric function ${}_1F_1(c; d; z)$ ([3], ch.VI), the Wiman's function $E_{a,b}(z)$ ([3], 18.1(19)) and several other functions as special cases. It is an entire function of order $(\text{Re } a)^{-1}$ and indeed has a number of properties which may be of independent interest. A fact of immediate interest to us is that the polynomials $Z_n^\alpha(x; k)$ bear to $E_{a,b}^c(x)$ a relation which is analogous to that which the Laguerre polynomials $L_n^\alpha(x)$ bear to the confluent hypergeometric function ${}_1F_1$; evidently

$$(3.4) \quad Z_n^\alpha(x; k) = \frac{\Gamma(kn + \alpha + 1)}{n!} E_{k,\alpha+1}^{-n}(x^k).$$

As usual let

$$(3.5) \quad L[f(t)] = \hat{f}(p) = \int_0^\infty e^{-pt} f(t) dt \quad \text{Re } p > 0$$

denote the Laplace transform of f . Then it is easily verified that for $\text{Re } \lambda, \text{Re } p > 0$,

$$(3.6) \quad L[t^{b-1} E_{a,b}^c(\lambda t)^a] = p^{-b+ac} (p^a - \lambda^a)^{-c} \quad \text{Re } b > 0,$$

$$(3.7) \quad L[t^\alpha Z_n^\alpha(\lambda t; k)] = \frac{\Gamma(kn + \alpha + 1)}{n! p^{kn+\alpha+1}} (p^k - \lambda^k)^n, \quad \text{Re } \alpha > -1.$$

We next note a general result on the Laplace transform of the r -times repeated indefinite integral as well as the r th order derivative of a function; in fact, we observe that

$$(3.8) \quad p^\mu \hat{f}(p) = L[I^{-\mu} f(t)]$$

for suitable f , complex μ and p with $\text{Re } p > 0$. Evidently both ([4], 4.1(8)) and ([4], 4.1(9)) are included in (3.8) as special cases.

We are now prepared to solve (3.1). From (3.1), (3.4) and using ([4], 4.1(20)), we have

$$(3.9) \quad \frac{\Gamma(kn + \alpha + 1)}{n!} (p^k - \lambda^k)^n p^{-kn-\alpha-1} \hat{f}(p) = \hat{g}(p).$$

For $\text{Re } l > \text{Re } (\alpha + 1)$, (3.9) can be written (compare with [1]) as

$$(3.10) \quad \hat{f}(p) = \frac{n!}{\Gamma(kn + \alpha + 1)} \{(p^k - \lambda^k)^{-n} p^{-l+kn+\alpha+1}\} \{p^l \hat{g}(p)\}$$

and we finally get

$$f(x) = \frac{n!}{\Gamma(kn + \alpha + 1)} \int_0^x (x-t)^{l-\alpha-2} E_{k,l-\alpha-1}^n(\lambda(x-t))^k I^{-l} g(t) dt$$

using ([4], 4.1(20)), (3.6) and (3.8).

4. A general class of generating functions. For arbitrary λ , we prove the generating relation

$$(4.1) \quad (1-t)^{-\lambda} E_{k,\alpha+1}^\lambda \left(\frac{-x^k t}{1-t} \right) = \sum_{n=0}^\infty \frac{(\lambda)_n Z_n^\alpha(x; k) t^n}{\Gamma(kn + \alpha + 1)}.$$

From (2.1), we have

$$\begin{aligned} \sum_{n=0}^\infty \frac{(\lambda)_n Z_n^\alpha(x; k) t^n}{\Gamma(kn + \alpha + 1)} &= \sum_{n=0}^\infty \sum_{m=0}^n \frac{(-1)^m (\lambda)_n x^k t^n}{\Gamma(km + \alpha + 1) (n-m)! m!} \\ &= \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{(-1)^m (\lambda)_{n+m} x^k t^{n+m}}{\Gamma(km + \alpha + 1) n! m!} \\ &= \sum_{m=0}^\infty \frac{(\lambda)_m (-x^k t)^m}{\Gamma(km + \alpha + 1) m!} \sum_{n=0}^\infty \frac{(\lambda+m)_n t^n}{n!} \\ &= (1-t)^{-\lambda} E_{k,\alpha+1}^\lambda \left(\frac{-x^k t}{1-t} \right). \end{aligned}$$

For $k = 1, \lambda = 1 + \alpha$, (4.1) yields the well-known generating function ([3], 10.12(7)) for the Laguerre polynomials.

From (4.1) we obtain, on applying Taylor’s theorem

$$(4.2) \quad \frac{(\lambda)_n Z_n^\alpha(x; k)}{\Gamma(kn + \alpha + 1)} = \frac{1}{2\pi i} \int_C (1-t)^{-\lambda} E_{k,\alpha+1}^\lambda \left(\frac{-x^k t}{1-t} \right) t^{-n-1} dt,$$

C being a closed contour surrounding $t = 0$ and lying within the disk $|t| < 1$. Putting $u = x^k/1-t$,

$$(4.3) \quad x^{k\lambda-k} Z_n^\alpha(x; k) = \frac{\Gamma(kn + \alpha + 1)}{(\lambda)_n 2\pi i} \int_{C'} \frac{u^{n+\lambda-1} E_{k,\alpha+1}^\lambda(x^k-u) du}{(u-x^k)^{n+1}}$$

where C' is a circle $|u - x^k| = \rho$ of small radius ρ .

Choosing $\lambda = 1$, we have in terms of Wiman’s function

$$(4.4) \quad Z_n^\alpha(x; k) = \frac{\Gamma(kn + \alpha + 1)}{n! 2\pi i} \int_{c'} \frac{u^n E_{k, \alpha+1}(x^k - u) du}{(u - x^k)^{n+1}}.$$

Also evaluating the integral (4.3) by the Cauchy's residue theorem, we obtain for arbitrary λ with $\operatorname{Re} \lambda > 0$,

$$(4.5) \quad Z_n^\alpha(x; k) = \frac{\Gamma(kn + \alpha + 1)}{(\lambda)_n n!} x^{k-k\lambda} \frac{\partial^n}{\partial u^n} [u^{\lambda+n-1} E_{k, \alpha+1}^\lambda(x^k - u)]_{u=x^k}.$$

Since $E_{1,b}^\lambda(z) = (1/\Gamma(b))e^z$, for $k=1$ and $\lambda = \alpha + 1$, (4.5) reduces to the Rodrigues for the Laguerre polynomials.

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