ON A SET OF POLYNOMIALS SUGGESTED BY LAGUERRE POLYNOMIALS

Tilak Raj Prabhakar
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TILAK RAJ PRABHAKAR

Generating functions, integrals and recurrence relations are obtained for the polynomials $Z_n^a(x; k)$ in $x^k$ which form one set of the biorthogonal pair with respect to the weight function $e^{-x}x^a$ over the interval $(0, \infty)$, the other set being that of polynomials in $x$.

A singular integral equation with $Z_n^a(x; k)$ in the kernel is solved in terms of a generalized Mittag-Leffler's function and a unified formula for fractional integration and differentiation of the polynomials is derived.

It is known [7] that the polynomials $Z_n^a(x; k)$ of degree $n$ in $x^k$ for positive integers $k$ and $\Re \alpha > -1$ are characterized up to a multiplicative constant by the above requirements. Konhauser [8] discussed the biorthogonality of the pair $\{Z_n^a(x; k)\}, \{Y_n^a(x; k)\}$ in the basic polynomials $x^k$ and $x$, over the interval $(0, \infty)$ and with the admissible weight function $e^{-x}x^a$ of the generalized Laguerre polynomial set $\{L_n^a(x)\}$. Indeed the polynomials have several properties of interest and Konhauser [8] obtained among other things some recurrence relations and a differential equation for the polynomials $Z_n^a(x; k)$ which are our primary concern in this paper. For $k = 2$, Preiser [11] obtained for these polynomials a generating function, a differential equation, integral representations and recurrence relations. Earlier Spencer and Fano [13] also used these polynomials for $k = 2$.

For $k = 1$, all the results proved in this paper reduce to those for $L_n^a(x)$; in particular the integral equation (3.1) either reduces to or contains as still more special cases the integral equations solved by Widder [14], Buschman [1], Khandekar [6], Rusia [12] and Prabhakar ([10], (7 · 1)). For $k = 2$, the results are essentially the same as those in [11] or [13].

2. Some properties of $Z_n^a(x; k)$. We now obtain a generating function, a contour integral representation and a fractional integration formula for $Z_n^a(x; k)$. In § 3, we need the Laplace transform and in § 4 derive a more general class of generating functions for the polynomials. Recurrence relations and a few other results will follow as natural consequences. We shall freely use the closed form ([8], (5))

$$Z_n^a(x; k) = \frac{\Gamma(kn + \alpha + 1)}{n!} \sum_{j=0}^{\infty} (-1)^j \frac{x^{kj}}{\Gamma(kj + \alpha + 1)}$$
for \( \Re\alpha > -1 \); naturally the results may be established from alternative characterizations of \( Z_\alpha^n(x; k) \) but such a discussion does not seem to be of sufficient interest.

(i) A generating function. We obtain the generating function indicated in

\[
e^t \phi (k, \alpha + 1; - x^k t) = \sum_{n=0}^{\infty} \frac{Z_\alpha^n(x; k) t^n}{\Gamma(kn + \alpha + 1)}
\]

where \( \phi (a, b; z) \) is the Bessel-Maitland function ([15], (1.3); [3], 18.1 (27)).

From (2.1), we have

\[
\sum_{n=0}^{\infty} \frac{Z_\alpha^n(x; k) t^n}{\Gamma(kn + \alpha + 1)} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{m! (n - m)! \Gamma(km + \alpha + 1)}
\]

\[
= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{m=0}^{\infty} \frac{(-x^k t)^m}{m! \Gamma(km + \alpha + 1)}
\]

\[
= e^t \phi (k, \alpha + 1; - x^k t)
\]

and (2.2) is established.

Denoting \( e^t \phi (k, \alpha + 1; - x^k t) \) by \( f(x, t) \) we at once find that \( f(x, t) \) satisfies the partial differential equation

\[
x \frac{\partial f}{\partial x} - \alpha t \frac{\partial f}{\partial t} + \alpha t f = 0.
\]

Substituting for \( f(x, t) \) from (2.2) and equating the coefficients of \( t^n \), we obtain the differential recurrence relation

\[
x Z_\alpha^n(x; k) = n k Z_\alpha^{n-1}(x; k) - k \frac{\Gamma(kn + \alpha + 1)}{\Gamma(kn + \alpha - k + 1)} Z_\alpha^n(x; k),
\]

also obtained by Konhauser ([8], (6)) by direct calculations.

(ii) Schl"afli's Contour integral. It is easy to show that

\[
Z_\alpha^n(x; k) = \frac{\Gamma(kn + \alpha + 1)}{n! 2\pi i} \int_{-\infty}^{\infty} \frac{(t^k - x^k)^n e^t}{t^{kn+\alpha+1}} dt.
\]

For

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(t^k - x^k)^n e^t}{t^{kn+\alpha+1}} dt = \sum_{j=0}^{n} (-1)^j \frac{1}{\Gamma(kj + \alpha + 1)} \int_{-\infty}^{\infty} t^{-(kj + \alpha + 1)} e^t dt
\]

\[
= \sum_{j=0}^{n} (-1)^j \frac{1}{\Gamma(kj + \alpha + 1)} \]

using Hankel's formula ([3], 1.6(2))
(2.4) \[ \frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^t t^{-z} dt; \]

finally (2.3) follows from (2.1).

For \( k = 1 \), (2.3) reduces to the known result ([2], p. 269)

\[ L_n^\alpha(x) = \frac{\Gamma(n + \alpha + 1)}{n! 2\pi i} \int_{-\infty}^{(0+)} (1 - \frac{x}{t})^n e^t \frac{dt}{t^{\alpha+1}}. \]

If \( \alpha \) is also a positive integer than the integrand in (2.3) is a single-valued analytic function of \( t \) with the only singularity \( t = 0 \).

Hence we can deform the contour into \( |t| = b \, |x| \) and the substitution \( t = xu \) then leads to

(2.5) \[ x^\alpha Z_n^\alpha(x; k) = \frac{\Gamma(kn + \alpha + 1)}{n! 2\pi i} \int_C (u^k-1)^n e^{\alpha u} u^{-(kn+\alpha+1)} du \]

where \( C \) denotes the circle \( |u| = b \). Indeed \( C \) may be replaced by any simple closed contour surrounding the point \( u = 0 \). For \( k = 2 \), (2.5) reduces to the integral representation by Preiser ([11], (5.22)).

Using (2.5), it follows that

\[ \frac{\partial^k}{\partial x^k} \left[ \frac{n! x^{\alpha+k} Z_n^{\alpha+k}(x; k)}{\Gamma(kn + k + \alpha + 1)} \right] = \frac{n! x^\alpha Z_n^\alpha(x; k)}{\Gamma(kn + \alpha + 1)} \]

and

\[ \left( \frac{\partial^k}{\partial x^k} - 1 \right) \left[ \frac{n! x^{\alpha+k} Z_n^{\alpha+k}(x; k)}{\Gamma(kn + k + \alpha + 1)} \right] = \frac{(n+1)! x^\alpha Z_n^{\alpha+1}(x; k)}{\Gamma(kn + k + \alpha + 1)} \]

which leads to the pure recurrence relation

(2.6) \[ x^k Z_n^{\alpha+k}(x; k) = (kn + \alpha + 1)_k Z_n^\alpha(x; k) - (n + 1) Z_n^{\alpha+1}(x; k). \]

For \( k = 2 \), (2.6) reduces to ([11], (5.39)).

(iii) Fractional integrals and derivatives. We show that

(2.7) \[ I^\mu [x^\alpha Z_n^\alpha(x; k)] = \frac{\Gamma(kn + \alpha + 1)}{\Gamma(kn + \alpha + \mu + 1)} x^{\alpha+\mu} Z_n^{\alpha+\mu}(x; k) \]

for \( \text{Re} \, \alpha > -1 \) and \( \text{Re} \, \mu > -\text{Re} \, (1 + \alpha) \) where for suitable \( f \) and complex \( \mu \), \( I^\mu f(x) \) denotes the \( \mu \)th order fractional integral (or fractional derivative) of \( f(x) \) (see [10], §2).

When \( \text{Re} \, \mu > 0 \), we write [10]

\[ I^\mu [x^\alpha Z_n^\alpha(x; k)] = \int_0^x \frac{(x - t)^{\mu-1}}{\Gamma(\mu)} t^\alpha Z_n^\alpha(t; k) dt \]

\[ = \frac{\Gamma(kn + \alpha + 1)}{n!} \sum_{j=0}^{\infty} \frac{(-n)_j}{\Gamma(kj + \alpha + 1)} \int_0^x t^{kj+\alpha} (x - t)^{\mu-1} dt \]
hence for \( \Re \mu > 0 \) and \( \Re \alpha > -1 \), we obtain

\[
(2.8) \quad I^\mu \left[ x^\alpha Z_n^\alpha(x; k) \right] = \frac{\Gamma(kn + \alpha + 1)}{\Gamma(kn + \alpha + \mu + 1)} x^{\alpha+\mu} Z_n^{\alpha+\mu}(x; k).
\]

But (2.8) may be written as

\[
(2.9) \quad x^\alpha Z_n^\alpha(x; k) = \frac{\Gamma(kn + \alpha + 1)}{\Gamma(kn + \alpha + \mu + 1)} I^{-\mu} \left[ x^{\alpha+\mu} Z_n^{\alpha+\mu}(x; k) \right],
\]

the inversion being valid for \( \Re \mu > 0 \) and the assumptions made.

Putting \( \mu' = -\mu, \alpha' = \alpha + \mu \), we obtain for \( \Re \mu' < 0 \)

\[
x^{\alpha+\mu'} Z_n^{\alpha+\mu'}(x; k) = \frac{\Gamma(kn + \alpha' + 1)}{\Gamma(kn + \alpha' + 1)} I^{-\mu'} \left[ x^{\alpha'} Z_n^\alpha(x; k) \right]
\]

which is (4.1) with the letters \( \alpha, \mu \) accented. Ignoring the accents we can write

\[
(2.10) \quad I^\mu \left[ x^\alpha Z_n^\alpha(x; k) \right] = \frac{\Gamma(kn + \alpha + 1)}{\Gamma(kn + \alpha + \mu + 1)} x^{\alpha+\mu} Z_n^{\alpha+\mu}(x; k)
\]

for \( \Re \mu < 0, \Re \alpha > -1 \) and \( \Re (\alpha + \mu) > -1 \).

When \( \Re \mu = 0 \), we write \( I^\mu = I^{\mu+1} I^{-1} \) and the result easily follows; thus (2.7) is established for all complex \( \mu \) with \( \Re \mu > -\Re (1 + \alpha) \).

**Remark 1.** When \( \mu \) is a negative integer say \(-m\), then (2.7) is written as

\[
\left( \frac{d}{dx} \right)^m \left[ x^\alpha Z_n^\alpha(x; k) \right] = \frac{\Gamma(kn + \alpha + 1)}{\Gamma(kn + \alpha - m + 1)} x^{\alpha-m} Z_n^{\alpha-m}(x; k)
\]

which can also be proved by direct differentiation provided \( \Re \alpha > m - 1 \).

**Remark 2.** For \( k = 1 \), (2.7) unifies the results ([3], 10.12(27)) and ([4], 13.1(49)) for Laguerre polynomials.

3. **A singular integral equation.** We show that the convolution equation

\[
(3.1) \quad \int_0^z (x-t)^\alpha Z_n^\alpha(\lambda(x-t); k) f(t) \, dt = g(x)
\]

for \( \Re \alpha > -1 \) admits a locally integrable solution \( f \) given by
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\begin{equation}
(3.2) \quad f(x) = \frac{n!}{\Gamma(kn + \alpha + 1)} \int_{0}^{x} (x-t)^{1-\alpha-2} E_{k,1-\alpha-1}^{\alpha}(\lambda(x-t))^k I^{-i} g(t) \, dt
\end{equation}

provided $I^{-i} g$ exists for $\Re l > \Re \alpha + 1$ and is locally integrable in $(0, \delta), 0 < x < \delta < \infty$.

The function

\begin{equation}
(3.3) \quad E_{a,b}^{\alpha}(z) = \sum_{j=0}^{\infty} \frac{(c)_j z^j}{\Gamma(a j + b) j !} \quad \Re a > 0
\end{equation}

is a very special case of the generalized hypergeometric functions considered by Wright [16] and is also expressible as a Fox's $H$-function [5]. On the other hand $E_{a,b}^{\alpha}(z)$ is a most natural generalization of the Mittag-Leffler's function $E_{\alpha}(z)$ ([3], 18.1; [9]) and also contains the confluent hypergeometric function $\, _1F_1(\alpha; \beta; z)$ ([3], ch.VI), the Wiman's function $E_{a,b}^{\alpha}(z)$ ([3], 18.1(19)) and several other functions as special cases. It is an entire function of order $(\Re a)^{-1}$ and indeed has a number of properties which may be of independent interest. A fact of immediate interest to us is that the polynomials $Z_{\alpha}^{\alpha}(x; k)$ bear to $E_{a,b}^{\alpha}(z)$ a relation which is analogous to that which the Laguerre polynomials $L_{\alpha}^{\alpha}(x)$ bear to the confluent hypergeometric function $\, _1F_1$; evidently

\begin{equation}
(3.4) \quad Z_{\alpha}^{\alpha}(x; k) = \frac{\Gamma(kn + \alpha + 1)}{n !} \frac{E_{k,a+1}^{\alpha-n}(x^k)}{E_{a+1}^{\alpha-n}(x^k)}.
\end{equation}

As usual let

\begin{equation}
(3.5) \quad L[f(t)] = \hat{f}(p) = \int_{0}^{\infty} e^{-p t} f(t) \, dt \quad \Re p > 0
\end{equation}

denote the Laplace transform of $f$. Then it is easily verified that for $\Re \alpha$, $\Re p > 0$,

\begin{align}
(3.6) \quad & L\{t^{b-1} E_{a,b}^{\alpha}(\lambda t)^a\} = p^{-b+\alpha} (p^a-\lambda^a)^{-c} \quad \Re b > 0, \\
(3.7) \quad & L\{t^{\alpha} Z_{\alpha}^{\alpha}(\lambda t; k)\} = \frac{\Gamma(kn + \alpha + 1)}{n !} p_{k+a+1}^{kn+\alpha+1} (p^k-\lambda^k)^n, \quad \Re \alpha > -1.
\end{align}

We next note a general result on the Laplace transform of the $r$-times repeated indefinite integral as well as the $r$th order derivative of a function; in fact, we observe that

\begin{equation}
(3.8) \quad p^\mu \hat{f}(p) = L[I^{-\mu} f(t)]
\end{equation}

for suitable $f$, complex $\mu$ and $p$ with $\Re p > 0$. Evidently both ([4], 4.1(8)) and ([4], 4.1(9)) are included in (3.8) as special cases.

We are now prepared to solve (3.1). From (3.1), (3.4) and using ([4], 4.1(20)), we have
\begin{equation}
\frac{\Gamma(kn + \alpha + 1)}{n!} \left( p^k - \lambda^k \right)^n p^{-kn - \alpha - 1} \hat{f}(p) = \hat{g}(p). \tag{3.9}
\end{equation}

For $\text{Re} \, l > \text{Re} \, (\alpha + 1)$, (3.9) can be written (compare with [1]) as
\begin{equation}
\hat{f}(p) = \frac{n!}{\Gamma(kn + \alpha + 1)} \left\{ (p^k - \lambda^k)^n p^{-kn - \alpha - 1} \right\} \{p \hat{g}(p) \}
\end{equation}

and we finally get
\begin{equation}
f(x) = \frac{n!}{\Gamma(kn + \alpha + 1)} \int_0^x (x-t)^{l-a-2} E_{\alpha+1}^\nu (\lambda(x-t))^k I^{-1} g(t) \, dt
\end{equation}

using ([4], 4.1(20)), (3.6) and (3.8).

4. A general class of generating functions. For arbitrary $\lambda$, we prove the generating relation
\begin{equation}
(1-t)^{-\lambda} E_{\alpha+1}^\nu \left( \frac{-x^k}{1-t} \right) = \sum_{n=0}^\infty \frac{(\lambda)_n Z_\alpha^n(x; k) t^n}{\Gamma(kn + \alpha + 1)}.
\tag{4.1}
\end{equation}

From (2.1), we have
\begin{align*}
\sum_{n=0}^\infty \frac{(\lambda)_n Z_\alpha^n(x; k) t^n}{\Gamma(kn + \alpha + 1)} &= \sum_{n=0}^\infty \sum_{m=0}^n \frac{(-1)^m (\lambda)_n x^m t^n}{\Gamma(km + \alpha + 1) (n-m)! m!} \\
&= \sum_{m=0}^\infty \frac{(-1)^m (\lambda)_m x^m t^{n+m}}{\Gamma(km + \alpha + 1) n!} \\
&= \sum_{m=0}^\infty \frac{(\lambda)_m (-x^k t)^m}{\Gamma(km + \alpha + 1) n!} \frac{t^n}{n!} \\
&= (1-t)^{-\lambda} E_{\alpha+1}^\nu \left( \frac{-x^k}{1-t} \right).
\end{align*}

For $k = 1$, $\lambda = 1 + \alpha$, (4.1) yields the well-known generating function ([3], 10.12(7)) for the Laguerre polynomials.

From (4.1) we obtain, on applying Taylor’s theorem
\begin{equation}
\frac{(\lambda)_n Z_\alpha^n(x; k)}{\Gamma(kn + \alpha + 1)} = \frac{1}{2\pi i} \int_C (1-t)^{-\lambda} E_{\alpha+1}^\nu \left( \frac{-x^k}{1-t} \right) t^{-n-1} \, dt,
\tag{4.2}
\end{equation}

$C$ being a closed contour surrounding $t = 0$ and lying within the disk $|t| < 1$. Putting $u = x^k/1-t$,
\begin{equation}
x^{k\lambda - k} Z_\alpha^n(x; k) = \frac{\Gamma(kn + \alpha + 1)}{(\lambda)_n 2\pi i} \int_{C'} \frac{u^{n-\lambda-1} E_{\alpha+1}^\nu \left( \frac{x^k}{u} \right) du}{(u - x^k)^{n+1}}
\end{equation}

where $C'$ is a circle $|u - x^k| = \rho$ of small radius $\rho$.

Choosing $\lambda = 1$, we have in terms of Wiman’s function
Also evaluating the integral (4.3) by the Cauchy's residue theorem, we obtain for arbitrary \( \lambda \) with \( \text{Re}\lambda > 0 \),

\[
Z_n^\alpha(x; k) = \frac{\Gamma(kn + \alpha + 1)}{n!} \frac{1}{2\pi i} \oint_C \frac{u^n E_{k, \alpha+1}(x^k - u)}{(u - x^k)^{n+1}} du.
\]

Since \( E_{1,b}^b(z) = (1/\Gamma(b))e^z \), for \( k = 1 \) and \( \lambda = \alpha + 1 \), (4.5) reduces to the Rodrigues for the Laguerre polynomials.

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REFERENCES


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RAMJAS COLLEGE
UNIVERSITY OF DELHI
DELHI-7. (INDIA)
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