EIGENVALUES IN THE BOUNDARY OF THE NUMERICAL RANGE

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We study eigenvalues $\lambda$ of a continuous linear operator $T$
on a complex Banach space $X$ that lie in the boundary of the
numerical range of $T$. We show that the kernel of $T - \lambda I$
is orthogonal, in the sense of G. Birkhoff, to the range of
$T - \lambda I$.

M. R. Fortet [5, Th. III, p. 32] proves that if $T$ is a continuous
linear operator of norm one on a strictly convex Banach space, then
the kernel of $T - I$ is orthogonal to the range of $T - I$. Proposition
1 is a generalisation of this result, since the numerical radius is less
than or equal to the norm [2, Th. 4.1]. Proposition 1 is also related
to the theorem of N. Nirschl and H. Schneider that an eigenvalue
in the boundary of the numerical range has ascent one [8, Th. 4,
p. 362] and [2, Th. 10.10].

If $T$ is a continuous linear operator on a Banach space $X$ (over
the complex field), the numerical range $V(T, \mathcal{B})$ of $T$ is the set

$$\{F(T): F \in \mathcal{B}^*, \|F\| = F(I) = 1\}$$

where $\mathcal{B}$ is the Banach algebra of all continuous linear operators
on $X$, $\mathcal{B}^*$ is the dual Banach space of $\mathcal{B}$, and $I$ is the identity
operator on $X$ [2, Chapter 3] and [1, §3]. The spatial numerical
range [2, Definition 9.1] $V(T)$ of $T$ is the set

$$\{f(Tx): f \in X^*, x \in X, \|f\| = \|x\| = f(x) = 1\}.$$

The numerical range of $T$ is equal to the closed convex hull of the
spatial numerical range, that is, $V(T, \mathcal{B}) = \overline{\co V(T)}$ [2, Th. 3.9]
and [1, Th. 6]. The spectrum, and hence the set of eigenvalues of
$T$, is contained in the numerical range of $T$ [2, Th. 2.6]. A linear
subspace $Y$ of $X$ is said to be orthogonal to a linear subspace $Z$ of
$X$ if $\|y\| \leq \|y + z\|$ for all $y$ in $Y$ and all $z$ in $Z$ [6] and [4, p. 93].

There is no loss of generality in assuming that 0 is the eigen-
value in the boundary of the numerical range, as we assume hence-
forth, because we may achieve this by adding a scalar multiple of
the identity to $T$.

**Proposition 1.** Let $T$ be a continuous linear operator on a
complex Banach space $X$. If 0 is in the boundary of the numerical
range of $T$, that is, $0 \in \partial \overline{\co V(T)}$, then the kernel of $T$ is orthogonal
to the range of $T$. In particular $T^{-1}(0) \oplus (TX)^- \text{ is closed in } X$.

Proof. Since 0 is in the boundary of $V(T, B)$, a closed convex subset of the complex plane, we may assume that $\max \{\text{Re } \lambda : \lambda \in V(T, B)\} = 0$, by multiplying $T$ by a suitable complex number of modulus 1. Assuming this, we have $\|\exp \alpha T\| \leq 1$ for all nonnegative real numbers $\alpha$ by [2, Th. 3.4]. If $T$ is one-to-one, the kernel of $T$ is null and the result follows because 0 is orthogonal to all vectors. We now assume that $T$ is not one-to-one. Let $y$ be an element of unit norm in $X$ annihilated by $T$, and let

$$D(y) = \{ f \in X^* : ||f|| = f(y) = 1 \}.$$

Then $D(y)$ is a nonempty $\sigma(X^*, X)$-compact convex subset of $X^*$, by the Hahn-Banach Theorem and Alaoglu's Theorem, and $\exp \alpha T^*$ is a $\sigma(X^*, X)$—continuous affine mapping on $D(y)$ for each nonnegative real $\alpha$, since $\|\exp \alpha T\| \leq 1$ and $Ty = 0$. Further $\{\exp \alpha T^* : \alpha \text{ is real}, \alpha \geq 0\}$ is a commutative semigroup on $D(y)$. The Markov-Kakutani fixed point theorem [4, Th. V. 10.6, p. 456] implies that there is an $f$ in $D(y)$ such that $\exp \alpha T^* f = f$ for all nonnegative real $\alpha$. The use of a fixed point theorem was suggested to me by the application of a generalization of Brouwer's fixed point theorem due to Kakutani in the proof of Theorem 1 of [3]. Taking the right hand derivative of $\exp \alpha T^*$ at $\alpha = 0$, and applying the equation $\exp \alpha T^* f = f$, we obtain $T^* f = 0$. Therefore $\|y + z\| \geq |f(y + z)| = f(y) = \|y\|$ for all $z$ in $TX$, and so the kernel of $T$ is orthogonal to the range of $T$. That $T^{-1}(0) \oplus (TX)^-$ is closed in $X$, follows in a routine way from the result that $T^{-1}(0)$ is orthogonal to $TX$, and hence to $(TX)^-$. This completes the proof.

Remarks 2. In general the space $T^{-1}(0) \oplus (TX)^-$ of Proposition 1 is not equal to $X$. For example let $X$ be $C[0, 1]$, the space of continuous complex valued functions on $[0, 1]$ with the supremum norm, let $g$ be a continuous real valued function on $[0, 1]$ that is zero at 0 and positive on $(0, 1]$, and let $T$ be the operation of multiplication by $g$ in $X$. Then $T$ is a hermitian operator on $X$ [2, Chapter 2], since $\|\exp itg\| = 1$ for all real $t$, so that the numerical range of $T$ is contained in the real line [2, Lemma 5.2]. Further $T^{-1}(0) \oplus (TX)^- = (TX)^-$ is the set of functions in $X$ that vanish at 0.

Proposition 1 gives another proof of the result that an eigenvalue in the boundary of $\overline{\co} V(T)$ has ascent one [8] and [2, Th. 10.10].
**Proposition 3.** Let $T$ be a nonzero continuous linear operator on a complex Banach space $X$, and let $0$ be in the spectrum of $T$ and in the boundary of the numerical range of $T$, that is,

$$0 \in \sigma(T) \cap \partial \overline{V(T)}.$$  

If $TX$ is closed in $X$, then $0$ is an eigenvalue of $T$, $X = T^{-1}\{0\} \oplus TX$, and $0$ is an isolated point of the spectrum of $T$.

**Proof.** By Proposition 1, $T^{-1}\{0\} \oplus TX$ is closed in $X$ so that if it is not equal to $X$ there is a nonzero continuous linear functional $f$ on $X$ that is zero on $T^{-1}\{0\} \oplus TX$. Let $Y^\circ$ denote the annihilator in $X^*$ of a subset $Y$ of $X$. Then $(TX)^\circ = T^{-1}\{0\}$ where $T^*$ is the adjoint of $T$ [9, Th. 4.6-C, p. 226]. Since $TX$ is closed in $X$ which is complete, $T^*X^* = (T^*X^*)^- = T^{-1}\{0\}^\circ$ [9, Problem 7, p. 227]. By construction $f$ is thus in $(T^*X^*)^-$ and in $T^*^{-1}\{0\}$. Now $T^*$ is a continuous linear operator on $X^*$ with $0$ in the boundary of the numerical range of $T^*$. That $0$ is in the boundary of the numerical range of $T^*$ follows from the equality $V(T^*, \mathcal{B}(X^*)) = V(T, \mathcal{B})$, which is an immediate consequence of Theorem 9.4(i) and Corollary 9.6(ii) of [2]. On the space $X^*$ the operator $T^*$ satisfies the assumptions of Proposition 1 so that the intersection of $(T^*X^*)^-$ and $T^*^{-1}\{0\}$ is $\{0\}$ by Proposition 1. This gives a contradiction as we have previously shown that $f$, which is not zero, is in this intersection. Hence $X = T^{-1}\{0\} \oplus TX$. Since the spectrum of $T$ is contained in the numerical range of $T$ [2, Th. 2.6], $0$ is in the boundary of the spectrum of $T$. Therefore $TX$ is not equal to $X$ by [7, Lemma 2.2], and so the kernel of $T$ is nonnull and $0$ is an eigenvalue of $T$.

Regarded as an operator on the Banach space $TX$, $T$ is invertible and so $(\lambda I - T)$ restricted to $TX$ is invertible for all $\lambda$ in a neighborhood of $0$ in the complex plane. On the space $T^{-1}\{0\}$, the operator $T$ has spectrum $\{0\}$. Since $X = T^{-1}\{0\} \oplus TX$, $\lambda I - T$ is invertible on $X$ for all $\lambda$ in a neighbourhood of $0$ but not at $0$. This shows that $0$ is an isolated point in the spectrum of $T$ and completes the proof.

**Remarks 4.** If $T$ satisfies the hypotheses of Proposition 1, and if $(T^*X^*)^- = T^{-1}\{0\}^\circ$, then part of the proof of Proposition 3 shows that $X = (TX)^- \oplus T^{-1}\{0\}$.

From the assumptions of Proposition 3 it does not follow that the range of $T$ is orthogonal to the kernel of $T$. Let $Y$ and $Z$ be closed linear subspaces of a complex Banach space $X$ such that $X = Y \oplus Z$, $Y$ is orthogonal to $Z$, and $Z$ is not orthogonal to $Y$ (spaces
with these properties exists; see [6]). Let $E$ be the projection from $X$ onto $Y$ annihilating $Z$. Then the norm of $E$ is one, so that the eigenvalue 1 of $E$ is in the boundary of the numerical range of $E$. Further $(1 - E)X = Z$ is not orthogonal to $(I - E)^{-1}\{0\} = Y$.

**Remark 5.** If we add the hypothesis that the Banach space $X$ is reflexive, then $(T^*X^*)^* = T^{-1}\{0\}^\circ$ for all continuous linear operators $T$ on $X$ [9, § 4.6, p. 226] so that if 0 is in the boundary of the numerical range of $T$, we have $X = (TX)^* \oplus T^{-1}\{0\}$ by Remark 4. As a corollary to this we have the following result.

Let $X$ be a reflexive complex Banach space, and let $T$ be a continuous linear operator on $X$ such that 0 is in the boundary of the numerical range of $T$. Then 0 is an eigenvalue of $T$ if, and only if, $TX$ is not dense in $X$, that is, if and only if 0 is an eigenvalue of $T^*$.

This follows immediately from the equation $X = (TX)^* \oplus T^{-1}\{0\}$ which holds for $T$ since $X$ is reflexive.

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**References**


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