

# Pacific Journal of Mathematics

## **EIGENVALUES IN THE BOUNDARY OF THE NUMERICAL RANGE**

ALLAN M. SINCLAIR

## EIGENVALUES IN THE BOUNDARY OF THE NUMERICAL RANGE

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**We study eigenvalues  $\lambda$  of a continuous linear operator  $T$  on a complex Banach space  $X$  that lie in the boundary of the numerical range of  $T$ . We show that the kernel of  $T - \lambda I$  is orthogonal, in the sense of G. Birkhoff, to the range of  $T - \lambda I$ .**

M. R. Fortet [5, Th. III, p. 32] proves that if  $T$  is a continuous linear operator of norm one on a strictly convex Banach space, then the kernel of  $T - I$  is orthogonal to the range of  $T - I$ . Proposition 1 is a generalisation of this result, since the numerical radius is less than or equal to the norm [2, Th. 4.1]. Proposition 1 is also related to the theorem of N. Nirschl and H. Schneider that an eigenvalue in the boundary of the numerical range has ascent one [8, Th. 4, p. 362] and [2, Th. 10.10].

If  $T$  is a continuous linear operator on a Banach space  $X$  (over the complex field), the *numerical range*  $V(T, \mathcal{B})$  of  $T$  is the set

$$\{F(T): F \in \mathcal{B}^*, \|F\| = F(I) = 1\}$$

where  $\mathcal{B}$  is the Banach algebra of all continuous linear operators on  $X$ ,  $\mathcal{B}^*$  is the dual Banach space of  $\mathcal{B}$ , and  $I$  is the identity operator on  $X$  [2, Chapter 3] and [1, §3]. The *spatial numerical range* [2, Definition 9.1]  $V(T)$  of  $T$  is the set

$$\{f(Tx): f \in X^*, x \in X, \|f\| = \|x\| = f(x) = 1\}.$$

The numerical range of  $T$  is equal to the closed convex hull of the spatial numerical range, that is,  $V(T, \mathcal{B}) = \overline{\text{co}} V(T)$  [2, Th. 3.9] and [1, Th. 6]. The spectrum, and hence the set of eigenvalues of  $T$ , is contained in the numerical range of  $T$  [2, Th. 2.6]. A linear subspace  $Y$  of  $X$  is said to be *orthogonal* to a linear subspace  $Z$  of  $X$  if  $\|y\| \leq \|y + z\|$  for all  $y$  in  $Y$  and all  $z$  in  $Z$  [6] and [4, p. 93].

There is no loss of generality in assuming that 0 is the eigenvalue in the boundary of the numerical range, as we assume henceforth, because we may achieve this by adding a scalar multiple of the identity to  $T$ .

**PROPOSITION 1.** *Let  $T$  be a continuous linear operator on a complex Banach space  $X$ . If 0 is in the boundary of the numerical range of  $T$ , that is,  $0 \in \partial \overline{\text{co}} V(T)$ , then the kernel of  $T$  is orthogonal*

to the range of  $T$ . In particular  $T^{-1}\{0\} \oplus (TX)^-$  is closed in  $X$ .

*Proof.* Since 0 is in the boundary of  $V(T, \mathcal{B})$ , a closed convex subset of the complex plane, we may assume that  $\max \{\operatorname{Re} \lambda : \lambda \in V(T, B)\} = 0$ , by multiplying  $T$  by a suitable complex number of modulus 1. Assuming this, we have  $\|\exp \alpha T\| \leq 1$  for all nonnegative real numbers  $\alpha$  by [2, Th. 3.4]. If  $T$  is one-to-one, the kernel of  $T$  is null and the result follows because 0 is orthogonal to all vectors. We now assume that  $T$  is not one-to-one. Let  $y$  be an element of unit norm in  $X$  annihilated by  $T$ , and let

$$D(y) = \{f \in X^* : \|f\| = f(y) = 1\}.$$

Then  $D(y)$  is a nonempty  $\sigma(X^*, X)$ -compact convex subset of  $X^*$ , by the Hahn-Banach Theorem and Alaoglu's Theorem, and  $\exp \alpha T^*$  is a  $\sigma(X^*, X)$ -continuous affine mapping on  $D(y)$  for each nonnegative real  $\alpha$ , since  $\|\exp \alpha T\| \leq 1$  and  $Ty = 0$ . Further  $\{\exp \alpha T^* : \alpha \text{ is real, } \alpha \geq 0\}$  is a commutative semigroup on  $D(y)$ . The Markov-Kakutani fixed point theorem [4, Th. V. 10.6, p. 456] implies that there is an  $f$  in  $D(y)$  such that  $\exp \alpha T^* f = f$  for all nonnegative real  $\alpha$ . The use of a fixed point theorem was suggested to me by the application of a generalization of Brouwer's fixed point theorem due to Kakutani in the proof of Theorem 1 of [3]. Taking the right hand derivative of  $\exp \alpha T^*$  at  $\alpha = 0$ , and applying the equation  $\exp \alpha T^* f = f$ , we obtain  $T^* f = 0$ . Therefore  $\|y + z\| \geq |f(y + z)| = f(y) = \|y\|$  for all  $z$  in  $TX$ , and so the kernel of  $T$  is orthogonal to the range of  $T$ . That  $T^{-1}\{0\} \oplus (TX)^-$  is closed in  $X$ , follows in a routine way from the result that  $T^{-1}\{0\}$  is orthogonal to  $TX$ , and hence to  $(TX)^-$ . This completes the proof.

**REMARKS 2.** In general the space  $T^{-1}\{0\} \oplus (TX)^-$  of Proposition 1 is not equal to  $X$ . For example let  $X$  be  $\mathcal{C}[0, 1]$ , the space of continuous complex valued functions on  $[0, 1]$  with the supremum norm, let  $g$  be a continuous real valued function on  $[0, 1]$  that is zero at 0 and positive on  $(0, 1]$ , and let  $T$  be the operation of multiplication by  $g$  in  $X$ . Then  $T$  is a hermitian operator on  $X$  [2, Chapter 2], since  $\|\exp itg\| = 1$  for all real  $t$ , so that the numerical range of  $T$  is contained in the real line [2, Lemma 5.2]. Further  $T^{-1}\{0\} \oplus (TX)^- = (TX)^-$  is the set of functions in  $X$  that vanish at 0.

Proposition 1 gives another proof of the result that an eigenvalue in the boundary of  $\overline{\operatorname{co}} V(T)$  has ascent one [8] and [2, Th. 10.10].

**PROPOSITION 3.** *Let  $T$  be a nonzero continuous linear operator on a complex Banach space  $X$ , and let  $0$  be in the spectrum of  $T$  and in the boundary of the numerical range of  $T$ , that is,*

$$0 \in \sigma(T) \cap \partial \overline{\text{co}} V(T) .$$

*If  $TX$  is closed in  $X$ , then  $0$  is an eigenvalue of  $T$ ,  $X = T^{-1}\{0\} \oplus TX$ , and  $0$  is an isolated point of the spectrum of  $T$ .*

*Proof.* By Proposition 1,  $T^{-1}\{0\} \oplus TX$  is closed in  $X$  so that if it is not equal to  $X$  there is a nonzero continuous linear functional  $f$  on  $X$  that is zero on  $T^{-1}\{0\} \oplus TX$ . Let  $Y^0$  denote the annihilator in  $X^*$  of a subset  $Y$  of  $X$ . Then  $(TX)^0 = T^{*-1}\{0\}$  where  $T^*$  is the adjoint of  $T$  [9, Th. 4.6-C, p. 226]. Since  $TX$  is closed in  $X$  which is complete,  $T^*X^* = (T^*X^*)^- = T^{-1}\{0\}^0$  [9, Problem 7, p. 227]. By construction  $f$  is thus in  $(T^*X^*)^-$  and in  $T^{*-1}\{0\}$ . Now  $T^*$  is a continuous linear operator on  $X^*$  with  $0$  in the boundary of the numerical range of  $T^*$ . That  $0$  is in the boundary of the numerical range of  $T^*$  follows from the equality  $V(T^*, \mathcal{B}(X^*)) = V(T, \mathcal{B})$ , which is an immediate consequence of Theorem 9.4(i) and Corollary 9.6(ii) of [2]. On the space  $X^*$  the operator  $T^*$  satisfies the assumptions of Proposition 1 so that the intersection of  $(T^*X^*)^-$  and  $T^{*-1}\{0\}$  is  $\{0\}$  by Proposition 1. This gives a contradiction as we have previously shown that  $f$ , which is not zero, is in this intersection. Hence  $X = T^{-1}\{0\} \oplus TX$ . Since the spectrum of  $T$  is contained in the numerical range of  $T$  [2, Th. 2.6],  $0$  is in the boundary of the spectrum of  $T$ . Therefore  $TX$  is not equal to  $X$  by [7, Lemma 2.2], and so the kernel of  $T$  is nonnull and  $0$  is an eigenvalue of  $T$ .

Regarded as an operator on the Banach space  $TX$ ,  $T$  is invertible and so  $(\lambda I - T)$  restricted to  $TX$  is invertible for all  $\lambda$  in a neighborhood of  $0$  in the complex plane. On the space  $T^{-1}\{0\}$ , the operator  $T$  has spectrum  $\{0\}$ . Since  $X = T^{-1}\{0\} \oplus TX$ ,  $\lambda I - T$  is invertible on  $X$  for all  $\lambda$  in a neighbourhood of  $0$  but not at  $0$ . This shows that  $0$  is an isolated point in the spectrum of  $T$  and completes the proof.

**REMARKS 4.** If  $T$  satisfies the hypotheses of Proposition 1, and if  $(T^*X^*)^- = T^{-1}\{0\}^0$ , then part of the proof of Proposition 3 shows that  $X = (TX)^- \oplus T^{-1}\{0\}$ .

From the assumptions of Proposition 3 it does not follow that the range of  $T$  is orthogonal to the kernel of  $T$ . Let  $Y$  and  $Z$  be closed linear subspaces of a complex Banach space  $X$  such that  $X = Y \oplus Z$ ,  $Y$  is orthogonal to  $Z$ , and  $Z$  is not orthogonal to  $Y$  (spaces

with these properties exists; see [6]). Let  $E$  be the projection from  $X$  onto  $Y$  annihilating  $Z$ . Then the norm of  $E$  is one, so that the eigenvalue 1 of  $E$  is in the boundary of the numerical range of  $E$ . Further  $(1 - E)X = Z$  is not orthogonal to  $(I - E)^{-1}\{0\} = Y$ .

REMARK 5. If we add the hypothesis that the Banach space  $X$  is reflexive, then  $(T^*X^*)^- = T^{-1}\{0\}^0$  for all continuous linear operators  $T$  on  $X$  [9, § 4.6, p. 226] so that if 0 is in the boundary of the numerical range of  $T$ , we have  $X = (TX)^- \oplus T^{-1}\{0\}$  by Remark 4. As a corollary to this we have the following result.

Let  $X$  be a reflexive complex Banach space, and let  $T$  be a continuous linear operator on  $X$  such that 0 is in the boundary of the numerical range of  $T$ . Then 0 is an eigenvalue of  $T$  if, and only if,  $TX$  is not dense in  $X$ , that is, if and only if 0 is an eigenvalue of  $T^*$ .

This follows immediately from the equation  $X = (TX)^- \oplus T^{-1}\{0\}$  which holds for  $T$  since  $X$  is reflexive.

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B. D. Arendt and C. J. Stuth, <i>On the structure of commutative periodic semigroups</i> .....	1
B. D. Arendt and C. J. Stuth, <i>On partial homomorphisms of semigroups</i> ....	7
Leonard Asimow, <i>Extensions of continuous affine functions</i> .....	11
Claude Elias Billigheimer, <i>Regular boundary problems for a five-term recurrence relation</i> .....	23
Edwin Ogilvie Buchman and F. A. Valentine, <i>A characterization of the parallelepiped in <math>E^n</math></i> .....	53
Victor P. Camillo, <i>A note on commutative injective rings</i> .....	59
Larry Jean Cummings, <i>Decomposable symmetric tensors</i> .....	65
J. E. H. Elliott, <i>On matrices with a restricted number of diagonal values</i> ...	79
Garth Ian Gaudry, <i>Bad behavior and inclusion results for multipliers of type <math>(p, q)</math></i> .....	83
Frances F. Gulick, <i>Derivations and actions</i> .....	95
Langdon Frank Harris, <i>On subgroups of prime power index</i> .....	117
Jutta Hausen, <i>The hypo residuum of the automorphism group of an abelian <math>p</math>-group</i> .....	127
R. Hrycay, <i>Noncontinuous multifunctions</i> .....	141
A. Jeanne LaDuke, <i>On a certain generalization of <math>p</math> spaces</i> .....	155
Marion-Josephine Lim, <i>Rank preservers of skew-symmetric matrices</i> .....	169
John Hathway Lindsey, II, <i>On a six dimensional projective representation of the Hall-Janko group</i> .....	175
Roger McCann, <i>Transversally perturbed planar dynamical systems</i> .....	187
Theodore Windle Palmer, <i>Real <math>C^*</math>-algebras</i> .....	195
Don David Porter, <i>Symplectic bordism, Stiefel-Whitney numbers, and a Novikov resolution</i> .....	205
Tilak Raj Prabhakar, <i>On a set of polynomials suggested by Laguerre polynomials</i> .....	213
B. L. S. Prakasa Rao, <i>Infinitely divisible characteristic functionals on locally convex topological vector spaces</i> .....	221
John Robert Reay, <i>Caratheodory theorems in convex product structures</i> ....	227
Allan M. Sinclair, <i>Eigenvalues in the boundary of the numerical range</i> .....	231
David R. Stone, <i>Torsion-free and divisible modules over matrix rings</i> .....	235
William Jennings Wickless, <i>A characterization of the nil radical of a ring</i> .....	255