A CHARACTERIZATION OF THE NIL RADICAL OF A RING

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Let $R$ be a ring and $S$ a subring of $R$. Let $\varphi$ be a ring homomorphism mapping $S$ onto a division ring $\Gamma$. Choose an ideal $P \subseteq R$ maximal with respect to the property $(P \cap S)^\varphi = (0)$. $P$ is a prime ideal of $R$. If $P$ is any prime ideal of $R$ which can be obtained in the above manner write $P = P(\Gamma, S, \varphi)$.

It is shown that all primitive ideals are of the form $P = P(\Gamma, S, \varphi)$ and that a ring $R$ is nil if and only if it has no prime ideals of the form $P = P(\Gamma, S, \varphi)$. It is shown that the nil radical of any ring is the intersection of all prime ideals $P = P(\Gamma, S, \varphi)$.

It is shown that if $P = P(\Gamma, S, \varphi)$ for all prime ideals $P \subseteq R$ then the nil and Baer radicals coincide for all homomorphic images of $R$. If the nil and Baer radicals coincide for all homomorphic images of $R$, it is shown that any prime ideal $P$ of $R$ is contained in a prime ideal $P' = P'(\Gamma, S, \varphi)$.

Finally, by consideration of prime ideals $P = P(\Gamma, S, \varphi)$, two theorems are proved giving information about rings satisfying very special conditions.

2. Certain prime ideals in rings. Let $R$ be any ring and $S$ a subring of $R$. Suppose $\varphi$ is a ring homomorphism mapping $S$ onto a division ring $\Gamma$. We may choose an ideal $P \subseteq R$ maximal with respect to the property $(P \cap S)^\varphi = (0)$. It is an easy exercise to check that $P$ will be a prime ideal of $R$. If $P$ is any prime ideal of $R$ which is a maximal ideal such that $(P \cap S)^\varphi = (0)$ for some subring $S \subseteq R$ and some ring homomorphism $\varphi: S \rightarrow \Gamma$, $\Gamma$ a division ring, we write $P = P(\Gamma, S, \varphi)$. Throughout, for any ring $R$, we let $J(R)$, $N(R)$, $\beta(R)$ denote respectively the Jacobson, nil, and Baer radicals of $R$. We start with the following simple fact.

**Theorem 1.** Let $R$ be a ring and $P$ a primitive ideal of $R$. Then $P = P(\Gamma, S, \varphi)$.

**Proof.** Let $P = (0: M)$ for some simple right $R$ module $M$. Let $\Gamma$ be the centralizer of $M$. $\Gamma$ is a division ring. As $R/P$ is primitive it is well known ([3], Th. 3, p. 33) that there exists a subring $S' \subseteq R/P$ and a homomorphism $\varphi': S' \rightarrow \Gamma$. It is easy to check $P = P(\Gamma, S, \varphi)$ with $S = (S')^{\pi^{-1}}$, $\varphi = \pi \varphi'$, $\pi$ the natural map from $R$ onto $R/P$. 

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We next consider the structure of rings which have no prime ideals of the form \( P = P(\Gamma, S, \varphi) \).

**Theorem 2.** A ring \( R \) is nil if and only if it has no prime ideals \( P \) of the form \( P = P(\Gamma, S, \varphi) \).

**Proof.** If \( R \) is nil then every subring \( S \subseteq R \) is nil and cannot be mapped onto a division ring. Thus, \( R \) has no prime ideals of the form \( P(\Gamma, S, \varphi) \).

Now assume \( R \) has no prime ideal of the form \( P(\Gamma, S, \varphi) \). This requires that every subring \( S \subseteq R \) is a Jacobson radical ring, for if \( S \) is any ring with \( J(S) \neq S \), we can find a subring \( S' \subseteq S \) which can be mapped homomorphically onto a division ring \( \Gamma' \) let \( \Gamma' \) be the centralizer of a simple \( S \) module for example.

We now show if \( R \) is a ring such that \( J(S) = S \) for all subrings \( S \) then \( R \) is nil. We wish to thank Professor S. A. Amitsur for the following simple proof of this fact. Let \( u \in R \) and \( \langle u \rangle \) denote the subring of \( R \) generated by \( u \). We know \( J(\langle u \rangle) = \langle u \rangle \). Let \( \langle u \rangle^* \) denote the ring \( \langle u \rangle \) with an identity adjoined in the usual way. Now \( \langle u \rangle^* \) is a homomorphic image of \( Z[x] \), the ring of polynomials in an indeterminate \( x \) with integral coefficients. By a result of Goldman ([2], Th. 3), we know that the Jacobson radical of any homomorphic image of \( Z[x] \) is nil. Thus \( J(\langle u \rangle^*) \) is nil, and \( \langle u \rangle = J(\langle u \rangle^*) \cap \langle u \rangle \) is nil. Thus \( u \) is nilpotent. As \( u \) was an arbitrary element of \( R \) we have \( R \) is nil. This proves the theorem.

We now obtain a result about the nil radical of an arbitrary ring.

**Theorem 3.** For any ring \( R \), \( N(R) = \bigcap_{\alpha \in T} P_\alpha \), where \( \{ P_\alpha \mid \alpha \in T \} \) is the set of all prime ideals of \( R \) of the form \( P = P(\Gamma, S, \varphi) \).

**Proof.** Let \( P = P(\Gamma, S, \varphi) \) be any prime ideal of the above type. As \( N(R) \) is nil, it is easy to check that we have \( [(N(R) + P) \cap S]^\varphi = (0) \). As \( P \) was a maximal ideal in \( R \) such that \( (P \cap S)^\varphi = (0) \), we must have \( N(R) \subseteq P \). Thus \( N(R) \subseteq \bigcap_{\alpha \in T} P_\alpha \).

We now show \( x \notin N(R) \rightarrow x \in \bigcap_{\alpha \in T} P_\alpha \). Let \( x \notin N(R) \). Then \( (x) \), the ideal generated by \( x \) in \( R \), is not nil. By Theorem 2 we have \( S \subseteq (x) \) and \( \varphi: S \rightarrow \Gamma' \), \( S \) a subring of \( (x) \), \( \Gamma' \) a division ring, \( \varphi \) a ring homomorphism onto. Let \( P = P(\Gamma, S, \varphi) \). Clearly \( P \in \{ P_\alpha \mid \alpha \in T \} \) and \( x \notin P \). This proves the theorem.

We now wish to consider rings in which all prime ideals are of the form \( P = P(\Gamma, S, \varphi) \). We obtain the following partial result.
THEOREM 4. Let $R$ be a ring such that $P$ prime in $R ightarrow P = P(\Gamma, S, \varphi)$. Then for all ideals $I \subseteq R$ we have $N[R/I] = \beta[R/I]$. If $N[R/I] = \beta[R/I]$ for all ideals $I \subseteq R$ we have $P$ prime in $R ightarrow P \subseteq P(\Gamma, S, \varphi)$.

Proof. Let $R$ be such that $P$ prime in $R ightarrow P = P(\Gamma, S, \varphi)$. Let $I$ be any ideal of $R$. We first note there is a one-to-one correspondence between all prime ideals $P/I = P/I(\Gamma, S, \varphi)$ of the ring $R/I$ and all prime ideals of the form $P(\Gamma, S, \varphi)/I$ in $R/I$ where $P(\Gamma, S, \varphi)$ is a prime ideal in $R$ containing $I$. Let $P/I = P/I(\Gamma, S, \varphi)$ where $S$ is a subring of $R/I$. Write $S$ as $S'/I$ for $S'$ a subring of $R$. Then $P/I(\Gamma, S, \varphi) = P(\Gamma, S', \pi\varphi)/I$ where $\pi$ is the natural homomorphism mapping $S'$ onto $S$. Conversely, if $P = P(\Gamma, S, \varphi)$ is a prime ideal of $R$ containing $I$ then $P/I(\Gamma, S, \varphi)/I = P/I(\Gamma', S + I/I, \lambda\varphi')$ where $\lambda$ is the natural homomorphism from $S$ onto $S + I/I$ and $\varphi': S + I/I \rightarrow \Gamma'$ is given by $(s + I)^\varphi' = s^\varphi$.

Thus we have: $N[R/I] = \bigcap_s [P/I(\Gamma, S, \varphi)]_s = \bigcap_s P(\Gamma, S, \varphi)/I = \beta[R/I]$. (Recall, by our assumption on $R$, $\{P(\Gamma, S, \varphi) \supseteq I\}$ is the set of all prime ideals of $R$ containing $I$.)

To prove the second statement of our theorem let $N[R/I] = \beta[R/I]$ for all $I$ and let $P$ be any prime ideal of $R$. We have $N[R/P] = \beta[R/P] = (0)$, thus, by Theorem 2, $R/P$ has a prime ideal $P' = P'(\Gamma, S, \varphi)$. We have $P \subseteq P'$, which finishes the proof of the theorem.

We conclude by proving two theorems about rings satisfying very special conditions. If $P = P(\Gamma, S, \varphi) \subseteq R$, we may extend $P$ to a maximal right ideal $T$ such that $(T \cap S)^\varphi = (0)$. $T$ will be a prime right ideal in the sense that if $U$ is a right ideal of $R$, $U \not\subseteq T$ and $x \in R$ with $Ux \subseteq T$, then $x \in T$. (This is weaker than the usual definition of prime right ideal which requires $x = 0$.) We have the following theorem.

THEOREM 5. If $R$ is a ring such that every prime right ideal is two sided, then every nil right ideal of $R$ is contained in $N(R)$.

Proof. Let $A$ be a nil right ideal of $R$ with $A \not\subseteq N(R)$. Then $A + RA$ is not nil and thus, by Theorem 2, contains a subring $S$ which may be mapped homomorphically onto a division ring $\Gamma$ by a map $\varphi$. As $A$ is nil, we have $(A \cap S)^\varphi = (0)$. We may extend $A$ to a maximal right ideal $T$ such that $(T \cap S)^\varphi = (0)$. By the assumption of our theorem we know $T$ is two sided. But then, $R + RA \subseteq T$, a contradiction.

THEOREM 6. Let $R$ be a ring such that if $S$ is a subring of
$R$, $I$ an ideal of $S$, then there exist $T$ an ideal of $R$ such that $T \cap S = I$. Then $J(R)$ is nil.

Proof. It is enough to show that the ring $J = J(R)$ contains no subrings $S$ which can be mapped by a ring homomorphism $\varphi$ onto a division ring $\Gamma$. Assume that $S$ is such a subring. Consider in the ring $J$ a prime ideal $P = P(\Gamma, S, \varphi)$.

Now $J/P$ contains the subring $S + P/P$ which can be mapped onto $\Gamma$ by $\pi \varphi$ where $\pi$ is the natural map from $S$ to $S + P/P$. It is easy to check that the ring $J$ inherits the condition of our theorem. Therefore, as $P$ was maximal in $J$ such that $(P \cap S)^\varphi = (0)$, we must have Kernel $\pi \varphi = (0)$. Thus $S + P/P \cong \Gamma$, a contradiction since $J/P$ is a radical ring. Thus $J$ is nil.

References


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