

# Pacific Journal of Mathematics

**A CHARACTERIZATION OF THE NIL RADICAL OF A RING**

WILLIAM JENNINGS WICKLESS

## A CHARACTERIZATION OF THE NIL RADICAL OF A RING

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Let  $R$  be a ring and  $S$  a subring of  $R$ . Let  $\varphi$  be a ring homomorphism mapping  $S$  onto a division ring  $\Gamma$ . Choose an ideal  $P \subseteq R$  maximal with respect to the property  $(P \cap S)^\varphi = (0)$ .  $P$  is a prime ideal of  $R$ . If  $P$  is any prime ideal of  $R$  which can be obtained in the above manner write  $P = P(\Gamma, S, \varphi)$ .

It is shown that all primitive ideals are of the form  $P = P(\Gamma, S, \varphi)$  and that a ring  $R$  is nil if and only if it has no prime ideals of the form  $P = P(\Gamma, S, \varphi)$ . It is shown that the nil radical of any ring is the intersection of all prime ideals  $P = P(\Gamma, S, \varphi)$ .

It is shown that if  $P = P(\Gamma, S, \varphi)$  for all prime ideals  $P \subseteq R$  then the nil and Baer radicals coincide for all homomorphic images of  $R$ . If the nil and Baer radicals coincide for all homomorphic images of  $R$ , it is shown that any prime ideal  $P$  of  $R$  is contained in a prime ideal  $P' = P'(\Gamma, S, \varphi)$ .

Finally, by consideration of prime ideals  $P = P(\Gamma, S, \varphi)$ , two theorems are proved giving information about rings satisfying very special conditions.

2. Certain prime ideals in rings. Let  $R$  be any ring and  $S$  a subring of  $R$ . Suppose  $\varphi$  is a ring homomorphism mapping  $S$  onto a division ring  $\Gamma$ . We may choose an ideal  $P \subseteq R$  maximal with respect to the property  $(P \cap S)^\varphi = (0)$ . It is an easy exercise to check that  $P$  will be a prime ideal of  $R$ . If  $P$  is any prime ideal of  $R$  which is a maximal ideal such that  $(P \cap S)^\varphi = (0)$  for some subring  $S \subseteq R$  and some ring homomorphism  $\varphi: S \rightarrow \Gamma$ ,  $\Gamma$  a division ring, we write  $P = P(\Gamma, S, \varphi)$ . Throughout, for any ring  $R$ , we let  $J(R)$ ,  $N(R)$ ,  $\beta(R)$  denote respectively the Jacobson, nil, and Baer radicals of  $R$ . We start with the following simple fact.

**THEOREM 1.** *Let  $R$  be a ring and  $P$  a primitive ideal of  $R$ . Then  $P = P(\Gamma, S, \varphi)$ .*

*Proof.* Let  $P = (0: M)$  for some simple right  $R$  module  $M$ . Let  $\Gamma$  be the centralizer of  $M$ .  $\Gamma$  is a division ring. As  $R/P$  is primitive it is well known ([3], Th. 3, p. 33) that there exists a subring  $S' \subseteq R/P$  and a homomorphism  $\varphi': S' \rightarrow \Gamma$ . It is easy to check  $P = P(\Gamma, S, \varphi)$  with  $S = (S')\pi^{-1}$ ,  $\varphi = \pi\varphi'$ ,  $\pi$  the natural map from  $R$  onto  $R/P$ .

We next consider the structure of rings which have no prime ideals of the form  $P = P(\Gamma, S, \varphi)$ .

**THEOREM 2.** *A ring  $R$  is nil if and only if it has no prime ideals  $P$  of the form  $P = P(\Gamma, S, \varphi)$ .*

*Proof.* If  $R$  is nil then every subring  $S \subseteq R$  is nil and cannot be mapped onto a division ring. Thus,  $R$  has no prime ideals of the form  $P(\Gamma, S, \varphi)$ .

Now assume  $R$  has no prime ideal of the form  $P(\Gamma, S, \varphi)$ . This requires that every subring  $S \subseteq R$  is a Jacobson radical ring, for if  $S$  is any ring with  $J(S) \neq S$ , we can find a subring  $S' \subseteq S$  which can be mapped homomorphically onto a division ring  $\Gamma$ -let  $\Gamma$  be the centralizer of a simple  $S$  module for example.

We now show if  $R$  is a ring such that  $J(S) = S$  for all subrings  $S$  then  $R$  is nil. We wish to thank Professor S. A. Amitsur for the following simple proof of this fact. Let  $u \in R$  and  $\langle u \rangle$  denote the subring of  $R$  generated by  $u$ . We know  $J(\langle u \rangle) = \langle u \rangle$ . Let  $\langle u \rangle^*$  denote the ring  $\langle u \rangle$  with an identity adjoined in the usual way. Now  $\langle u \rangle^*$  is a homomorphic image of  $Z[x]$ , the ring of polynomials in an indeterminate  $x$  with integral coefficients. By a result of Goldman ([2], Th. 3), we know that the Jacobson radical of any homomorphic image of  $Z[x]$  is nil. Thus  $J(\langle u \rangle^*)$  is nil, and  $\langle u \rangle = J(\langle u \rangle) = J(\langle u \rangle^*) \cap \langle u \rangle$  is nil. Thus  $u$  is nilpotent. As  $u$  was an arbitrary element of  $R$  we have  $R$  is nil. This proves the theorem.

We now obtain a result about the nil radical of an arbitrary ring.

**THEOREM 3.** *For any ring  $R$ ,  $N(R) = \bigcap_{\alpha \in T} P_\alpha$ , where  $\{P_\alpha \mid \alpha \in T\}$  is the set of all prime ideals of  $R$  of the form  $P = P(\Gamma, S, \varphi)$ .*

*Proof.* Let  $P = P(\Gamma, S, \varphi)$  be any prime ideal of the above type. As  $N(R)$  is nil, it is easy to check that we have  $[(N(R) + P) \cap S]^\varphi = (0)$ . As  $P$  was a maximal ideal in  $R$  such that  $(P \cap S)^\varphi = (0)$ , we must have  $N(R) \subseteq P$ . Thus  $N(R) \subseteq \bigcap_{\alpha \in T} P_\alpha$ .

We now show  $x \notin N(R) \rightarrow x \notin \bigcap_{\alpha \in T} P_\alpha$ . Let  $x \notin N(R)$ . Then  $(x)$ , the ideal generated by  $x$  in  $R$ , is not nil. By Theorem 2 we have  $S \subseteq (x)$  and  $\varphi: S \rightarrow \Gamma, S$  a subring of  $(x)$ ,  $\Gamma$  a division ring,  $\varphi$  a ring homomorphism onto. Let  $P = P(\Gamma, S, \varphi)$ . Clearly  $P \in \{P_\alpha \mid \alpha \in T\}$  and  $x \notin P$ . This proves the theorem.

We now wish to consider rings in which all prime ideals are of the form  $P = P(\Gamma, S, \varphi)$ . We obtain the following partial result.

**THEOREM 4.** *Let  $R$  be a ring such that  $P$  prime in  $R \rightarrow P = P(\Gamma, S, \varphi)$ . Then for all ideals  $I \subseteq R$  we have  $N[R/I] = \beta[R/I]$ . If  $N[R/I] = \beta[R/I]$  for all ideals  $I \subseteq R$  we have  $P$  prime in  $R \rightarrow P \subseteq P'(\Gamma, S, \varphi)$ .*

*Proof.* Let  $R$  be such that  $P$  prime in  $R \rightarrow P = P(\Gamma, S, \varphi)$ . Let  $I$  be any ideal of  $R$ . We first note there is a one-to-one correspondence between all prime ideals  $P/I = P/I(\Gamma, S, \varphi)$  of the ring  $R/I$  and all prime ideals of the form  $P(\Gamma, S, \varphi)/I$  in  $R/I$  where  $P(\Gamma, S, \varphi)$  is a prime ideal in  $R$  containing  $I$ . Let  $P/I = P/I(\Gamma, S, \varphi)$  where  $S$  is a subring of  $R/I$ . Write  $S$  as  $S'/I$  for  $S'$  a subring of  $R$ . Then  $P/I(\Gamma, S, \varphi) = P(\Gamma, S', \pi\varphi)/I$  where  $\pi$  is the natural homomorphism mapping  $S'$  onto  $S$ . Conversely, if  $P = P(\Gamma, S, \varphi)$  is a prime ideal of  $R$  containing  $I$  then  $P(\Gamma, S, \varphi)/I = P/I(\Gamma, S + I/I, \lambda\varphi')$  where  $\lambda$  is the natural homomorphism from  $S$  onto  $S + I/I$  and  $\varphi': S + I/I \rightarrow \Gamma$  is given by  $(s + I)^\varphi' = s^\varphi$ .

Thus we have:  $N[R/I] = \bigcap_\alpha [P/I(\Gamma, S, \varphi)]_\alpha = \bigcap_\alpha P(\Gamma, S, \varphi)_\alpha / I = \beta[R/I]$ . (Recall, by our assumption on  $R$ ,  $\{P(\Gamma, S, \varphi)_\alpha \supseteq I\}$  is the set of all prime ideals of  $R$  containing  $I$ .)

To prove the second statement of our theorem let  $N[R/I] = \beta[R/I]$  for all  $I$  and let  $P$  be any prime ideal of  $R$ . We have  $N[R/P] = \beta[R/P] = (0)$ , thus, by Theorem 2,  $R/P$  has a prime ideal  $P' = P'(\Gamma, S, \varphi)$ . We have  $P \subseteq P'$ , which finishes the proof of the theorem.

We conclude by proving two theorems about rings satisfying very special conditions. If  $P = P(\Gamma, S, \varphi) \subseteq R$ , we may extend  $P$  to a maximal right ideal  $T$  such that  $(T \cap S)^\varphi = (0)$ .  $T$  will be a prime right ideal in the sense that if  $U$  is a right ideal of  $R$ ,  $U \not\subseteq T$  and  $x \in R$  with  $Ux \subseteq T$ , then  $x \in T$ . (This is weaker than the usual definition of prime right ideal which requires  $x = 0$ .) We have the following theorem.

**THEOREM 5.** *If  $R$  is a ring such that every prime right ideal is two sided, then every nil right ideal of  $R$  is contained in  $N(R)$ .*

*Proof.* Let  $A$  be a nil right ideal of  $R$  with  $A \not\subseteq N(R)$ . Then  $A + RA$  is not nil and thus, by Theorem 2, contains a subring  $S$  which may be mapped homomorphically onto a division ring  $\Gamma$  by a map  $\varphi$ . As  $A$  is nil, we have  $(A \cap S)^\varphi = (0)$ . We may extend  $A$  to a maximal right ideal  $T$  such that  $(T \cap S)^\varphi = (0)$ . By the assumption of our theorem we know  $T$  is two sided. But then,  $R + RA \subseteq T$ , a contradiction.

**THEOREM 6.** *Let  $R$  be a ring such that if  $S$  is a subring of*

*R*, *I* an ideal of *S*, then there exist *T* an ideal of *R* such that  $T \cap S = I$ . Then  $J(R)$  is nil.

*Proof.* It is enough to show that the ring  $J = J(R)$  contains no subrings *S* which can be mapped by a ring homomorphism  $\varphi$  onto a division ring  $\Gamma$ . Assume that *S* is such a subring. Consider in the ring *J* a prime ideal  $P = P(\Gamma, S, \varphi)$ .

Now  $J/P$  contains the subring  $S + P/P$  which can be mapped onto  $\Gamma$  by  $\pi\varphi$  where  $\pi$  is the natural map from *S* to  $S + P/P$ . It is easy to check that the ring *J* inherits the condition of our theorem. Therefore, as *P* was maximal in *J* such that  $(P \cap S)^\circ = (0)$ , we must have Kernel  $\pi\varphi = (0)$ . Thus  $S + P/P \cong \Gamma$ , a contradiction since  $J/P$  is a radical ring. Thus *J* is nil.

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Received November 3, 1969. The preparation of this paper was supported in part by NSF Grant #GP-13164.

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# Pacific Journal of Mathematics

Vol. 35, No. 1

September, 1970

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