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A CHARACTERIZATION OF THE NIL RADICAL OF A RING

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Let R be a ring and S a subring of R . Let φ be a ring homomorphism mapping S onto a division ring Γ . Choose an ideal $P \subseteq R$ maximal with respect to the property $(P \cap S)^\varphi = (0)$. P is a prime ideal of R . If P is any prime ideal of R which can be obtained in the above manner write $P = P(\Gamma, S, \varphi)$.

It is shown that all primitive ideals are of the form $P = P(\Gamma, S, \varphi)$ and that a ring R is nil if and only if it has no prime ideals of the form $P = P(\Gamma, S, \varphi)$. It is shown that the nil radical of any ring is the intersection of all prime ideals $P = P(\Gamma, S, \varphi)$.

It is shown that if $P = P(\Gamma, S, \varphi)$ for all prime ideals $P \subseteq R$ then the nil and Baer radicals coincide for all homomorphic images of R . If the nil and Baer radicals coincide for all homomorphic images of R , it is shown that any prime ideal P of R is contained in a prime ideal $P' = P'(\Gamma, S, \varphi)$.

Finally, by consideration of prime ideals $P = P(\Gamma, S, \varphi)$, two theorems are proved giving information about rings satisfying very special conditions.

2. Certain prime ideals in rings. Let R be any ring and S a subring of R . Suppose φ is a ring homomorphism mapping S onto a division ring Γ . We may choose an ideal $P \subseteq R$ maximal with respect to the property $(P \cap S)^\varphi = (0)$. It is an easy exercise to check that P will be a prime ideal of R . If P is any prime ideal of R which is a maximal ideal such that $(P \cap S)^\varphi = (0)$ for some subring $S \subseteq R$ and some ring homomorphism $\varphi: S \rightarrow \Gamma, \Gamma$ a division ring, we write $P = P(\Gamma, S, \varphi)$. Throughout, for any ring R , we let $J(R), N(R), \beta(R)$ denote respectively the Jacobson, nil, and Baer radicals of R . We start with the following simple fact.

THEOREM 1. *Let R be a ring and P a primitive ideal of R . Then $P = P(\Gamma, S, \varphi)$.*

Proof. Let $P = (0: M)$ for some simple right R module M . Let Γ be the centralizer of M . Γ is a division ring. As R/P is primitive it is well known ([3], Th. 3, p. 33) that there exists a subring $S' \subseteq R/P$ and a homomorphism $\varphi': S' \rightarrow \Gamma$. It is easy to check $P = P(\Gamma, S, \varphi)$ with $S = (S')\pi^{-1}$, $\varphi = \pi\varphi'$, π the natural map from R onto R/P .

We next consider the structure of rings which have no prime ideals of the form $P = P(\Gamma, S, \varphi)$.

THEOREM 2. *A ring R is nil if and only if it has no prime ideals P of the form $P = P(\Gamma, S, \varphi)$.*

Proof. If R is nil then every subring $S \subseteq R$ is nil and cannot be mapped onto a division ring. Thus, R has no prime ideals of the form $P(\Gamma, S, \varphi)$.

Now assume R has no prime ideal of the form $P(\Gamma, S, \varphi)$. This requires that every subring $S \subseteq R$ is a Jacobson radical ring, for if S is any ring with $J(S) \neq S$, we can find a subring $S' \subseteq S$ which can be mapped homomorphically onto a division ring Γ -let Γ be the centralizer of a simple S module for example.

We now show if R is a ring such that $J(S) = S$ for all subrings S then R is nil. We wish to thank Professor S. A. Amitsur for the following simple proof of this fact. Let $u \in R$ and $\langle u \rangle$ denote the subring of R generated by u . We know $J(\langle u \rangle) = \langle u \rangle$. Let $\langle u \rangle^*$ denote the ring $\langle u \rangle$ with an identity adjoined in the usual way. Now $\langle u \rangle^*$ is a homomorphic image of $Z[x]$, the ring of polynomials in an indeterminate x with integral coefficients. By a result of Goldman ([2], Th. 3), we know that the Jacobson radical of any homomorphic image of $Z[x]$ is nil. Thus $J(\langle u \rangle^*)$ is nil, and $\langle u \rangle = J(\langle u \rangle) = J(\langle u \rangle^*) \cap \langle u \rangle$ is nil. Thus u is nilpotent. As u was an arbitrary element of R we have R is nil. This proves the theorem.

We now obtain a result about the nil radical of an arbitrary ring.

THEOREM 3. *For any ring R , $N(R) = \bigcap_{\alpha \in T} P_\alpha$, where $\{P_\alpha | \alpha \in T\}$ is the set of all prime ideals of R of the form $P = P(\Gamma, S, \varphi)$.*

Proof. Let $P = P(\Gamma, S, \varphi)$ be any prime ideal of the above type. As $N(R)$ is nil, it is easy to check that we have $[(N(R) + P) \cap S]^\varphi = (0)$. As P was a maximal ideal in R such that $(P \cap S)^\varphi = (0)$, we must have $N(R) \subseteq P$. Thus $N(R) \subseteq \bigcap_{\alpha \in T} P_\alpha$.

We now show $x \notin N(R) \rightarrow x \notin \bigcap_{\alpha \in T} P_\alpha$. Let $x \notin N(R)$. Then (x) , the ideal generated by x in R , is not nil. By Theorem 2 we have $S \subseteq (x)$ and $\varphi: S \rightarrow \Gamma, S$ a subring of (x) , Γ a division ring, φ a ring homomorphism onto. Let $P = P(\Gamma, S, \varphi)$. Clearly $P \in \{P_\alpha | \alpha \in T\}$ and $x \notin P$. This proves the theorem.

We now wish to consider rings in which all prime ideals are of the form $P = P(\Gamma, S, \varphi)$. We obtain the following partial result.

THEOREM 4. *Let R be a ring such that P prime in $R \rightarrow P = P(\Gamma, S, \varphi)$. Then for all ideals $I \subseteq R$ we have $N[R/I] = \beta[R/I]$. If $N[R/I] = \beta[R/I]$ for all ideals $I \subseteq R$ we have P prime in $R \rightarrow P \subseteq P'(\Gamma, S, \varphi)$.*

Proof. Let R be such that P prime in $R \rightarrow P = P(\Gamma, S, \varphi)$. Let I be any ideal of R . We first note there is a one-to-one correspondence between all prime ideals $P/I = P/I(\Gamma, S, \varphi)$ of the ring R/I and all prime ideals of the form $P(\Gamma, S, \varphi)/I$ in R/I where $P(\Gamma, S, \varphi)$ is a prime ideal in R containing I . Let $P/I = P/I(\Gamma, S, \varphi)$ where S is a subring of R/I . Write S as S'/I for S' a subring of R . Then $P/I(\Gamma, S, \varphi) = P(\Gamma, S', \pi\varphi)/I$ where π is the natural homomorphism mapping S' onto S . Conversely, if $P = P(\Gamma, S, \varphi)$ is a prime ideal of R containing I then $P(\Gamma, S, \varphi)/I = P/I(\Gamma, S + I/I, \lambda\varphi')$ where λ is the natural homomorphism from S onto $S + I/I$ and $\varphi': S + I/I \rightarrow \Gamma$ is given by $(s + I)^{\varphi'} = s^{\varphi}$.

Thus we have: $N[R/I] = \bigcap_{\alpha} [P/I(\Gamma, S, \varphi)]_{\alpha} = \bigcap_{\alpha} P(\Gamma, S, \varphi)_{\alpha}/I = \beta[R/I]$. (Recall, by our assumption on R , $\{P(\Gamma, S, \varphi)_{\alpha} \supseteq I\}$ is the set of all prime ideals of R containing I .)

To prove the second statement of our theorem let $N[R/I] = \beta[R/I]$ for all I and let P be any prime ideal of R . We have $N[R/P] = \beta[R/P] = (0)$, thus, by Theorem 2, R/P has a prime ideal $P' = P'(\Gamma, S, \varphi)$. We have $P \subseteq P'$, which finishes the proof of the theorem.

We conclude by proving two theorems about rings satisfying very special conditions. If $P = P(\Gamma, S, \varphi) \subseteq R$, we may extend P to a maximal right ideal T such that $(T \cap S)^{\varphi} = (0)$. T will be a prime right ideal in the sense that if U is a right ideal of R , $U \not\subseteq T$ and $x \in R$ with $Ux \subseteq T$, then $x \in T$. (This is weaker than the usual definition of prime right ideal which requires $x = 0$.) We have the following theorem.

THEOREM 5. *If R is a ring such that every prime right ideal is two sided, then every nil right ideal of R is contained in $N(R)$.*

Proof. Let A be a nil right ideal of R with $A \not\subseteq N(R)$. Then $A + RA$ is not nil and thus, by Theorem 2, contains a subring S which may be mapped homomorphically onto a division ring Γ by a map φ . As A is nil, we have $(A \cap S)^{\varphi} = (0)$. We may extend A to a maximal right ideal T such that $(T \cap S)^{\varphi} = (0)$. By the assumption of our theorem we know T is two sided. But then, $R + RA \subseteq T$, a contradiction.

THEOREM 6. *Let R be a ring such that if S is a subring of*

R, *I* an ideal of *S*, then there exist *T* an ideal of *R* such that $T \cap S = I$. Then $J(R)$ is nil.

Proof. It is enough to show that the ring $J = J(R)$ contains no subrings *S* which can be mapped by a ring homomorphism φ onto a division ring Γ . Assume that *S* is such a subring. Consider in the ring *J* a prime ideal $P = P(\Gamma, S, \varphi)$.

Now J/P contains the subring $S + P/P$ which can be mapped onto Γ by $\pi\varphi$ where π is the natural map from *S* to $S + P/P$. It is easy to check that the ring *J* inherits the condition of our theorem. Therefore, as *P* was maximal in *J* such that $(P \cap S)^\circ = (0)$, we must have Kernel $\pi\varphi = (0)$. Thus $S + P/P \cong \Gamma$, a contradiction since J/P is a radical ring. Thus *J* is nil.

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