A GENERALIZATION OF MARTINGALES AND TWO CONSEQUENT CONVERGENCE THEOREMS

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Loeve has observed that a discrete stochastic process can be interpreted as a game and that a martingale can be interpreted as a "fair" game. In this context, the notion of a martingale is enlarged to a game which becomes "fairer with time" and then this concept is utilized to establish two convergence theorems.

Let \((\Omega, \mathcal{A}, p)\) be a probability space with \(\{\mathcal{A}_n\}_{n \geq 1}\) an increasing family of sub-\(\sigma\)-algebras of \(\mathcal{A}\) to which the process \(\{X_n\}_{n \geq 1}\) is adapted, (see [3, p. 65]). Henceforth, the process \(\{X_n\}_{n \geq 1}\) will be referred to as a game.

**DEFINITION.** The game \(\{X_n\}_{n \geq 1}\) will be said to become fairer with time if for every \(\varepsilon > 0\),

\[
\Pr[|E(X_n | \mathcal{A}_m) - X_m| > \varepsilon] \to 0
\]

as \(n, m \to \infty\) with \(n \geq m\).

It should be noted that any martingale is a game which becomes fairer with time. An easy example of a game which is not a martingale or a sub or a super martingale but does become fairer with time is constructed by considering a game which consists of tossing a die. Here, let

\[
\mathcal{A}_n = \mathcal{A}, \text{ all } n
\]

and

\[
X_n(i) \equiv i + (-1)^n/n.
\]

The main results. Let \(\{\alpha_n: n \geq 1\}\) be a monotonic sequence decreasing to zero with finite sum. The game \(\{X_n\}_{n \geq 1}\) may be decomposed with respect to \(\{\alpha_n: n \geq 1\}\) as

\[
X_n = Y_n - Z_n, \text{ where } \{Y_n\}_{n \geq 1} \text{ and } \{Z_n\}_{n \geq 1}
\]

are defined inductively by:

\[
Y_1 = X_1
\]

\[
Y_n = Y_{n-1} + [X_n - E(X_n | \mathcal{A}_{n-1})] + \alpha_{n-1}
\]

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\]

\[
Y_n = Y_{n-1} + [X_n - E(X_n | \mathcal{A}_{n-1})] + \alpha_{n-1}
\]
\[(1.3)\hspace{1cm} Z_n = Z_{n-1} + [X_{n-1} - E(X_n | \mathfrak{N}_{n-1})] + \alpha_{n-1}. \]

We note that \(\{Y_n\}_{n \geq 1}\) is adapted to the sequence of \(\sigma\)-algebras \(\{\mathfrak{N}_n\}_{n \geq 1}\) and forms a submartingale with respect to it.

We will call the decomposition of the game \(\{X_n\}_{n \geq 1}\) according to (1.1) – (1.3) a Doob-like decomposition. (See [3, p. 104–105].)

Also, we define the collection of sets \(\{B^a_{n,m}\}\) for \(m = 1, 2, \ldots\) and \(n \geq m\) by

\[B^a_{n,m} \equiv \{w: |E(X_n | \mathfrak{N}^n) - X_m| > \alpha_m\}.\]

**Theorem 1.** Let \(\{X_n\}_{n \geq 1}\) be a uniformly integrable game and \(\{Y_n\}_{n \geq 1}\), the submartingale associated with its Doob-like decomposition, be uniformly dominated in absolute value by an element of \(L_1(\Omega, \mathfrak{N}, p)\). Suppose for every \(\delta > 0\) there exists an integer \(N(\delta)\), such that

\[(1.4)\hspace{1cm} P[B^a_{n,m}] < \delta \text{ whenever } n \geq m \geq N(\delta),\]

and

\[(1.5)\hspace{1cm} \sim B^a_{n,m} \subset \sim B^a_{k,n-1} \text{ whenever } n \geq k \geq k - 1 \geq m \geq N(\delta).\]

Then, there exists a function \(X\) in \(L_1(\Omega, \mathfrak{N}, p)\) such that

\[\lim_{n \to \infty} \int_{\Omega} |X_n - X| dp = 0.\]

**Proof.** It will be sufficient to show the game \(\{X_n\}_{n \geq 1}\) is Cauchy in the \(L_1\) norm. For every pair \((n, m)\) of positive integers write:

\[\int_{\Omega} |X_n - X_m| dp = \int_{B^a_{n,m}} |X_n - X_m| dp + \int_{\sim B^a_{n,m}} |X_n - X_m| dp.\]

Since \(p[B^a_{n,m}] \to 0\) as \(n, m \to \infty\) and since the game \(\{X_n\}_{n \geq 1}\) is uniformly integrable (see [1, p. 89]), it is immediate that \(\int_{B^a_{n,m}} |X_n - X_m| dp\) can be made arbitrarily small for sufficiently large \(n\) and \(m\).

By utilizing the Doob-like decomposition of \(\{X_n\}_{n \geq 1}\), we can write

\[\int_{\sim B^a_{n,m}} |X_n - X_m| dp \leq \int_{\sim B^a_{n,m}} |Y_n - Y_m| dp + \int_{\sim B^a_{n,m}} |Z_n - Z_m| dp.\]

Since there exists an integrable function which uniformly dominates the process \(\{Y_n\}_{n \geq 1}\) in absolute value, it is immediate that \(\{Y_n\}_{n \geq 1}\) is a convergent submartingale. Moreover, the dominated convergence theorem can be used to show that \(\int_{\sim B^a_{n,m}} |Y_n - Y_m| dp\) can be made arbitrarily small for sufficiently large \(n\) and \(m\).
Thus, it remains to show that \( \int_{\sim B_{n,m}^*} |Z_n - Z_m| \, dp \) can be made arbitrarily small for sufficiently large \( n \) and \( m \) and the proof will be complete.

On \( \sim B_{n,m}^* \) it follows that

\[
X_n \geq E(X_n | \mathcal{U}_m) - \alpha_m
\]

In particular, on \( \sim B_{n,n-1}^* \)

\[
X_{n-1} \geq E(X_n | \mathcal{U}_{n-1}) - \alpha_{n-1}
\]

and so where

\[
Z_n - Z_{n-1} = X_{n-1} - E(X_n | \mathcal{U}_{n-1}) + \alpha_{n-1}
\]

we can say

\[
(1.7) \quad Z_n - Z_{n-1} \geq 0 \text{ on } \sim B_{n,n-1}^*
\]

Thus, choose any \( \delta > 0 \) and there exists \( N(\delta) \) such that

\[
(1.8) \quad \sum_{k=m}^{n} \alpha_k < \delta/2 \text{ for } m \geq N(\delta)
\]

and such that (1.5) holds. Hence, with \( n \geq m \geq N(\delta) \), (1.5) and (1.7), write

\[
(1.9) \quad Z_n - Z_{n-1} \geq 0 \text{ on } \sim B_{n,m}^*
\]

By observing the fact that \( B_{n,m}^* \in \mathcal{U}_m \) for all \( n \) and \( m \), we can write that

\[
(1.10) \quad \int_{\sim B_{n,m}^*} |Z_n - Z_m| \, dp = \int_{\mathcal{U}_m} E(|Z_n - Z_m| \mid I_{\sim B_{n,m}^*}) \, dp
\]

By (1.9), \( |Z_n - Z_m| \mid I_{\sim B_{n,m}^*} = \sum_{k=m+1}^{n} (Z_k - Z_{k-1})I_{\sim B_{n,m}^*} \); this together with (1.6) lets us continue the equality in (1.10) to

\[
\int_{\sim B_{n,m}^*} |Z_n - Z_m| \, dp = \sum_{k=m+1}^{n} \left\{ \int_{\sim B_{n,m}^*} E((X_{k-1} - E(X_k \mid \mathcal{U}_{k-1}) + \alpha_{k-1} \mid \mathcal{U}_m)) \, dp \right\}
\]

\[
= \int_{\sim B_{n,m}^*} (X_n - E(X_n \mid \mathcal{U}_m)) + \alpha_m + \cdots + \alpha_{n-1} \, dp
\]

\[
\leq \int_{\sim B_{n,m}^*} \left\{ \alpha_m + \sum_{k=m}^{n-1} \alpha_k \right\} \, dp < \delta.
\]

By not demanding that the submartingale \( \{Y_n\}_{n \geq 1} \) associated with the Doob-like decomposition of the game \( \{X_n\}_{n \geq 1} \) be uniformly bounded above in absolute value by an element of \( L_i(\Omega, \mathcal{U}, p) \), we get the weaker
THEOREM 2. Let \( \{X_n\}_{n \geq 1} \) be a uniformly integrable game satisfying (1.4) and (1.5). Then, there exists some constant \( c \) such that

\[
\lim_{n \to \infty} \int_\Omega X_n \, dp = c.
\]

Proof. It will be sufficient to show the sequence \( \left\{ \int_\Omega X_n \, dp \right\}_{n \geq 1} \) is Cauchy. With respect to the Doob-like decomposition of \( \{X_n\}_{n \geq 1} \), we can write

\[(1.11) \quad \left| \int_\Omega (X_n - X_m) \, dp \right| \leq \left| \int_{E_n \setminus E_m} (X_n - X_m) \, dp \right| + \left| \int_{E_m \setminus E_n} (X_n - X_m) \, dp \right|.
\]

Again, \( \left| \int_{E_n \setminus E_m} (X_n - X_m) \, dp \right| \) may be made arbitrarily small for sufficiently large \( m \) and \( n \) by using the uniform integrability of \( \{X_n\}_{n \geq 1} \). In order to deal with the second summand in (1.11), write

\[
\left| \int_{E_m \setminus E_n} (X_n - X_m) \, dp \right| \leq \left| \int_{E_m \setminus E_n} (Y_n - Y_m) \, dp \right| + \int_{E_m \setminus E_n} |Z_n - Z_m| \, dp.
\]

But \( \int_{E_m \setminus E_n} |Z_n - Z_m| \, dp \) can be made arbitrarily small for sufficiently large \( m \) and \( n \) exactly as in the proof of Theorem 1. Hence, showing that \( \left| \int_{E_m \setminus E_n} (Y_n - Y_m) \, dp \right| \) can be made arbitrarily small for sufficiently large \( m \) and \( n \) will complete the proof. To this end, we use (1.2) and write

\[
E(Y_n - Y_m) | \mathcal{F}_m) = \alpha_m + \cdots + \alpha_{n-1}
\]

and get

\[
\int_{E_m \setminus E_n} (Y_n - Y_m) \, dp = \int_{E_m \setminus E_n} E(Y_n - Y_m) | \mathcal{F}_m) \, dp
\]

\[
= \int_{E_m \setminus E_n} (\alpha_m + \cdots + \alpha_{n-1}) \, dp \leq \sum_{k=m}^{n-1} \alpha_k.
\]

But since \( \sum_{k=m}^{n-1} \alpha_k \) can be made arbitrarily small for sufficiently large \( m \) and \( n \), the result follows.

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Worcester Polytechnic Institute
Valentin Danilovich Belousov and Balaniappan L. Kannappan, Generalized Bol functional equation ............................................................. 259
Charles Morgan Biles, Gelfand and Wallman-type compactifications ............... 267
Louis Harvey Blake, A generalization of martingales and two consequent convergence theorems ................................................. 279
Dennis K. Burke, On p-spaces and wΔ-spaces ............................................... 285
John Ben Butler, Jr., Almost smooth perturbations of self-adjoint operators ...... 297
Michael James Cambern, Isomorphisms of C0(Y) onto C(X) ....................... 307
David Edwin Cook, A conditionally compact point set with noncompact closure . . . 313
Timothy Edwin Cramer, Countable Boolean algebras as subalgebras and homomorphs ................................................................. 321
John R. Edwards and Stanley G. Wayment, A v-integral representation for linear operators on spaces of continuous functions with values in topological vector spaces ................................................................. 327
Mary Rodriguez Embry, Similarities involving normal operators on Hilbert space ................................................................. 331
Lynn Harry Erbe, Oscillation theorems for second order linear differential equations .............................................................................. 337
William James Firey, Local behaviour of area functions of convex bodies .......... 345
Joe Wayne Fisher, The primary decomposition theory for modules .................. 359
Gerald Seymour Garfinkel, Generic splitting algebras for Pic ......................... 369
J. D. Hansard, Jr., Function space topologies .................................................. 381
Keith A. Hardie, Quasisfibration and adjunction ............................................ 389
G. Hochschild, Coverings of pro-affine algebraic groups ................................. 399
Gerald L. Itzkowitz, On nets of contractive maps in uniform spaces .................. 417
Melven Robert Krom and Myrene Laurence Krom, Groups with free nonabelian subgroups ................................................................. 425
James Robert Kuttler, Upper and lower bounds for eigenvalues by finite differences ................................................................................ 429
Dany Leviatan, A new approach to representation theory for convolution transforms ....................................................................... 441
Richard MacGibbon, Perfect subsets of definable sets of real numbers ............. 451
Brenda MacGibbon, A necessary and sufficient condition for the embedding of a Lindelof space in a Hausdorff 3σ space .............................................. 459
David G. Mead and B. D. McLemore, Ritt’s question on the Wronskian ........... 467
Edward Yoshio Mikami, Focal points in a control problem ............................. 473
Paul G. Miller, Characterizing the distributions of three independent n-dimensional random variables, X1, X2, X3, having analytic characteristic functions by the joint distribution of (X1 + X3, X2 + X3) ......................................................... 487
P. Rosenthal, On the Bergman integral operator for an elliptic partial differential equation with a singular coefficient .................................................. 493
Douglas B. Smith, On the number of finitely generated O-groups ........................ 499
J. W. Spellmann, Concerning the domains of generators of linear semigroups .... 503
Arne Stray, An approximation theorem for subalgebras of H∞ .......................... 511
Arnold Lewis Villone, Self-adjoint differential operators .................................. 517