

# Pacific Journal of Mathematics

**A GENERALIZATION OF MARTINGALES AND TWO  
CONSEQUENT CONVERGENCE THEOREMS**

LOUIS HARVEY BLAKE

## A GENERALIZATION OF MARTINGALES AND TWO CONSEQUENT CONVERGENCE THEOREMS

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**Loeve has observed that a discrete stochastic process can be interpreted as a game and that a martingale can be interpreted as a "fair" game. In this context, the notion of a martingale is enlarged to a game which becomes "fairer with time" and then this concept is utilized to establish two convergence theorems.**

Let  $(\Omega, \mathfrak{A}, p)$  be a probability space with  $\{\mathfrak{A}_n\}_{n \geq 1}$  an increasing family of sub  $\sigma$ -algebras of  $\mathfrak{A}$  to which the process  $\{X_n\}_{n \geq 1}$  is adapted, (see [3, p. 65]). Henceforth, the process  $\{X_n\}_{n \geq 1}$  will be referred to as a game.

DEFINITION. The game  $\{X_n\}_{n \geq 1}$  will be said to become *fairer with time* if for every  $\varepsilon > 0$ .

$$p\{ |E(X_n | \mathfrak{A}_m) - X_m| > \varepsilon \} \rightarrow 0$$

as  $n, m \rightarrow \infty$  with  $n \geq m$ .

It should be noted that any martingale is a game which becomes fairer with time. An easy example of a game which is not a martingale or a sub or a super martingale but does become fairer with time is constructed by considering a game which consists of tossing a die. Here, let

$$\mathfrak{A}_n = \mathfrak{A}, \text{ all } n$$

and

$$X_n(\{i\}) \equiv i + (-1)^n/n .$$

The main results. Let  $\{\alpha_n: n \geq 1\}$  be a monotonic sequence decreasing to zero with finite sum. The game  $\{X_n\}_{n \geq 1}$  may be decomposed with respect to  $\{\alpha_n: n \geq 1\}$  as

$$(1.1) \quad X_n = Y_n - Z_n, \text{ where } \{Y_n\}_{n \geq 1} \text{ and } \{Z_n\}_{n \geq 1}$$

are defined inductively by:

$$(1.2) \quad \begin{aligned} Y_1 &= X_1 \\ &\vdots \\ Y_n &= Y_{n-1} + [X_n - E(X_n | \mathfrak{A}_{n-1})] + \alpha_{n-1} \end{aligned}$$

$$(1.3) \quad Z_n = Z_{n-1} + [X_{n-1} - E(X_n | \mathfrak{A}_{n-1})] + \alpha_{n-1}.$$

We note that  $\{Y_n\}_{n \geq 1}$  is adapted to the sequence of  $\sigma$ -algebras  $\{\mathfrak{A}_n\}_{n \geq 1}$  and forms a submartingale with respect to it.

We will call the decomposition of the game  $\{X_n\}_{n \geq 1}$  according to (1.1) – (1.3) a Doob-like decomposition. (See [3, p. 104–105].)

Also, we define the collection of sets  $\{B_{n,m}^\alpha\}$  for  $m = 1, 2, \dots$  and  $n \geq m$  by

$$B_{n,m}^\alpha \equiv \{w: |E(X_n | \mathfrak{A}^m) - X_m| > \alpha_m\}.$$

**THEOREM 1.** *Let  $\{X_n\}_{n \geq 1}$  be a uniformly integrable game and  $\{Y_n\}_{n \geq 1}$ , the submartingale associated with its Doob-like decomposition, be uniformly dominated in absolute value by an element of  $L_1(\Omega, \mathfrak{A}, p)$ . Suppose for every  $\delta > 0$  there exists an integer  $N(\delta)$ , such that*

$$(1.4) \quad P[B_{n,m}^\alpha] < \delta \text{ whenever } n \geq m \geq N(\delta),$$

and

$$(1.5) \quad \sim B_{n,m}^\alpha \subset \sim B_{k,k-1}^\alpha \text{ whenever } n \geq k \geq k-1 \geq m \geq N(\delta).$$

Then, there exists a function  $X$  in  $L_1(\Omega, \mathfrak{A}, p)$  such that

$$\lim_{n \rightarrow \infty} \int_{\Omega} |X_n - X| dp = 0.$$

*Proof.* It will be sufficient to show the game  $\{X_n\}_{n \geq 1}$  is Cauchy in the  $L_1$  norm. For every pair  $(n, m)$  of positive integers write:

$$\int_{\Omega} |X_n - X_m| dp = \int_{B_{n,m}^\alpha} |X_n - X_m| dp + \int_{\sim B_{n,m}^\alpha} |X_n - X_m| dp.$$

Since  $p[B_{n,m}^\alpha] \rightarrow 0$  as  $n, m \rightarrow \infty$  and since the game  $\{X_n\}_{n \geq 1}$  is uniformly integrable (see [1, p. 89]), it is immediate that  $\int_{B_{n,m}^\alpha} |X_n - X_m| dp$  can be made arbitrarily small for sufficiently large  $n$  and  $m$ .

By utilizing the Doob-like decomposition of  $\{X_n\}_{n \geq 1}$ , we can write

$$\int_{\sim B_{n,m}^\alpha} |X_n - X_m| dp \leq \int_{\sim B_{n,m}^\alpha} |Y_n - Y_m| dp + \int_{\sim B_{n,m}^\alpha} |Z_n - Z_m| dp.$$

Since there exists an integrable function which uniformly dominates the process  $\{Y_n\}_{n \geq 1}$  in absolute value, it is immediate that  $\{Y_n\}_{n \geq 1}$  is a convergent submartingale. Moreover, the dominated convergence theorem can be used to show that  $\int_{\sim B_{n,m}^\alpha} |Y_n - Y_m| dp$  can be made arbitrarily small for sufficiently large  $n$  and  $m$ .

Thus, it remains to show that  $\int_{\sim B_{n,m}^\alpha} |Z_n - Z_m| dp$  can be made arbitrarily small for sufficiently large  $n$  and  $m$  and the proof will be complete.

On  $\sim B_{n,m}^\alpha$  it follows that

$$X_m \geq E(X_n | \mathfrak{A}_m) - \alpha_m .$$

In particular, on  $\sim B_{n,n-1}^\alpha$

$$X_{n-1} \geq E(X_n | \mathfrak{A}_{n-1}) - \alpha_{n-1}$$

and so where

$$(1.6) \quad Z_n - Z_{n-1} = X_{n-1} - E(X_n | \mathfrak{A}_{n-1}) + \alpha_{n-1}$$

we can say

$$(1.7) \quad Z_n - Z_{n-1} \geq 0 \text{ on } \sim B_{n,n-1}^\alpha .$$

Thus, choose any  $\delta > 0$  and there exists  $N(\delta)$  such that

$$(1.8) \quad \sum_{k=m}^\infty \alpha_k < \delta/2 \text{ for } m \geq N(\delta)$$

and such that (1.5) holds. Hence, with  $n \geq m \geq N(\delta)$ , (1.5) and (1.7), write

$$(1.9) \quad Z_n - Z_{n-1} \geq 0 \text{ on } \sim B_{n,m}^\alpha .$$

By observing the fact that  $B_{n,m}^\alpha \in \mathfrak{A}_m$  for all  $n$  and  $m$ , we can write that

$$(1.10) \quad \int_{\sim B_{n,m}^\alpha} |Z_n - Z_m| dp = \int_\Omega E\{|Z_n - Z_m| I_{\sim B_{n,m}^\alpha} | \mathfrak{A}_m\} dp .$$

By (1.9),  $|Z_n - Z_m| I_{\sim B_{n,m}^\alpha} = \sum_{k=m+1}^n (Z_k - Z_{k-1}) I_{\sim B_{n,m}^\alpha}$ ; this together with (1.6) lets us continue the equality in (1.10) to

$$\begin{aligned} \int_{\sim B_{n,m}^\alpha} |Z_n - Z_m| dp &= \sum_{k=m+1}^n \left\{ \int_{\sim B_{n,m}^\alpha} E\{(X_{k-1} - E(X_k | \mathfrak{A}_{k-1}) + \alpha_{k-1} | \mathfrak{A}_m)\} dp \right\} \\ &= \int_{\sim B_{n,m}^\alpha} \{(X_m - E(X_n | \mathfrak{A}_m)) + \alpha_m + \dots + \alpha_{n-1}\} dp \\ &\leq \int_{\sim B_{n,m}^\alpha} \left\{ \alpha_m + \sum_{k=m}^{n-1} \alpha_k \right\} dp < \delta . \end{aligned}$$

By not demanding that the submartingale  $\{Y_n\}_{n \geq 1}$  associated with the Doob-like decomposition of the game  $\{X_n\}_{n \geq 1}$  be uniformly bounded above in absolute value by an element of  $L_1(\Omega, \mathfrak{A}, p)$ , we get the weaker

**THEOREM 2.** *Let  $\{X_n\}_{n \geq 1}$  be a uniformly integrable game satisfying (1.4) and (1.5). Then, there exists some constant  $c$  such that*

$$\lim_{n \rightarrow \infty} \int_{\Omega} X_n d\mathcal{P} = c .$$

*Proof.* It will be sufficient to show the sequence  $\left\{ \int_{\Omega} X_n d\mathcal{P} \right\}_{n \geq 1}$  is Cauchy. With respect to the Doob-like decomposition of  $\{X_n\}_{n \geq 1}$ , we can write

$$(1.11) \quad \left| \int_{\Omega} (X_n - X_m) d\mathcal{P} \right| \leq \left| \int_{B_{n,m}^{\alpha}} (X_n - X_m) d\mathcal{P} \right| + \left| \int_{\sim B_{n,m}^{\alpha}} (X_n - X_m) d\mathcal{P} \right| .$$

Again,  $\left| \int_{B_{n,m}^{\alpha}} (X_n - X_m) d\mathcal{P} \right|$  may be made arbitrarily small for sufficiently large  $m$  and  $n$  by using the uniform integrability of  $\{X_n\}_{n \geq 1}$ . In order to deal with the second summand in (1.11), write

$$\left| \int_{\sim B_{n,m}^{\alpha}} (X_n - X_m) d\mathcal{P} \right| \leq \left| \int_{\sim B_{n,m}^{\alpha}} (Y_n - Y_m) d\mathcal{P} \right| + \int_{\sim B_{n,m}^{\alpha}} |Z_n - Z_m| d\mathcal{P} .$$

But  $\int_{\sim B_{n,m}^{\alpha}} |Z_n - Z_m| d\mathcal{P}$  can be made arbitrarily small for sufficiently large  $m$  and  $n$  exactly as in the proof of Theorem 1. Hence, showing that  $\left| \int_{\sim B_{n,m}^{\alpha}} (Y_n - Y_m) d\mathcal{P} \right|$  can be made arbitrarily small for sufficiently large  $m$  and  $n$  will complete the proof. To this end, we use (1.2) and write

$$E((Y_n - Y_m) | \mathfrak{A}_m) = \alpha_m + \dots + \alpha_{n-1}$$

and get

$$\begin{aligned} \int_{\sim B_{n,m}^{\alpha}} (Y_n - Y_m) d\mathcal{P} &= \int_{\sim B_{n,m}^{\alpha}} E\{(Y_n - Y_m) | \mathfrak{A}_m\} d\mathcal{P} \\ &= \int_{\sim B_{n,m}^{\alpha}} (\alpha_m + \dots + \alpha_{n-1}) d\mathcal{P} \leq \sum_{k=m}^{n-1} \alpha_k . \end{aligned}$$

But since  $\sum_{k=m}^{n-1} \alpha_k$  can be made arbitrarily small for sufficiently large  $m$  and  $n$ , the result follows.

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Received March 20, 1970.

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The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.



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October, 1970

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