Pacific Journal of Mathematics

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Vol. 35, No. 2 October 1970

PACIFIC JOURNAL OF MATHEMATICS
Vol. 35, No. 2, 1970

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In this paper the relationships between p-spaces and $w\Delta$-spaces are investigated. It is shown that strict p-spaces, p-spaces, and $w\Delta$-spaces are all equivalent in the class of completely regular θ-refinable spaces. There is an example of a completely regular, countably compact space (and thus a $w\Delta$-space) which is not a p-space. An example is given of a T_2 locally compact space (and thus a p-space) which is not a $w\Delta$-space. In the last section we give some conditions for p-spaces or $w\Delta$-spaces to be developable.

1. Relationships between p-spaces and $w\Delta$-spaces. Unless otherwise stated no separation axioms are assumed; however regular and completely regular spaces are always assumed to be T_1. The set of positive integers is denoted by N.

A sequence $\{\mathscr{U}_n\}_{n=1}^\infty$ of open covers of a topological space X is called a *development* for X if for any $x \in X$ and any open set O about x, there is an integer $n \in N$ such that $St(x, \mathscr{U}_n) = \cup\{U \in \mathscr{U}_n : x \in U\} \subset O$. A regular developable space is a *Moore space*.

A completely regular space X is called a *p-space* [1] if in the Stone-Cech compactification $\beta(X)$ there is a sequence $\{\gamma_n\}_{n=1}^\infty$ of open covers of X such that $\bigcap_{n=1}^\infty St(x, \gamma_n) \subset X$ for each $x \in X$. The sequence $\{\gamma_n\}_{n=1}^\infty$ is called a *pluming* for X in $\beta(X)$. A space X is called a *strict p-space* if it has a pluming $\{\gamma_n\}_{n=1}^\infty$ with the following additional property: For any $x \in X$ and $n \in N$ there is $n' \in N$ such that $\overline{St(x, \gamma_{n'})} \subset St(x, \gamma_n)$. In this case we call $\{\gamma_n\}_{n=1}^\infty$ a *strict pluming*.

Since any T_2 locally compact space X is open in its compactification $\beta(X)$ it is clear that if we let $\gamma_n = \{X\}$, then $\{\gamma_n\}_{n=1}^\infty$ will be a pluming for X. Also any metric space or completely regular Moore space is a strict p-space [2].

A sequence $\{A_n(x)\}_{n=1}^\infty$ of subsets of X, with $x \in A_n(x)$ for each $n \in N$, is called an *x-sequence* if $x_n \in A_n(x)$ implies that $\{x_n\}_{n=1}^\infty$ has a cluster point in X. A space X is called a *$w\Delta$-space* (compare [4]) if X has a sequence $\{\mathscr{U}_n : n \in N\}$ of open covers such that $\{St(x, \mathscr{U}_n) : n \in N\}$ is an x-sequence for each $x \in X$.

Clearly any countably compact space and any Moore space is a $w\Delta$-space.

The following theorem was proved in [6]:

THEOREM 1.1. *A completely regular space X is a strict p-space if and only if there is a sequence $\{\mathscr{S}_n\}_{n=1}^\infty$ of open covers of X*

satisfying:

(a) $P_x = \bigcap_{n=1}^\infty St(x, \mathscr{G}_n)$ *is a compact set for each* $x \in X$.

(b) *The family* $\{St(x, \mathscr{G}_n) : n \in N\}$ *is a neighborhood base for the set* P_x.

Notice that if $\{\mathscr{G}_n\}_{n=1}^\infty$ is a sequence of open covers satisfying (a) and (b) of Theorem 1.1 and we assume $\mathscr{G}_{n+1}$ refines $\mathscr{G}_n$, it is easily verified that $\{St(x, \mathscr{G}_n): n \in N\}$ is an x-sequence for each $x \in X$. The following corollary follows immediately.

COROLLARY 1.2. *A strict p-space is a* $w\Delta$*-space.*

Similar to Theorem 1.1 we have a theorem which characterizes p-spaces without the use of the compactification $\beta(X)$. This theorem helps to illustrate the relationship between p-spaces and $w\Delta$-spaces.

THEOREM 1.3. *A completely regular space* X *is a p-space if and only if there is a sequence* $\{\mathscr{G}_n\}_{n=1}^\infty$ *of open covers of* X *satisfying*: *If* $x \in X$ *and* $G_n \in \mathscr{G}_n$ *such that* $x \in G_n$, *then*

(a) $\bigcap_{n=1}^\infty \bar{G}_n$ *is compact.*

(b) $\{\bigcap_{n=1}^k \bar{G}_n : k \in N\}$ *is an* x*-sequence.*

Proof. Before proceeding with the proof of the theorem, notice that (a) and (b) above are equivalent to (a) and (b′) where:

(b′) If O is any open set containing $\bigcap_{n=1}^\infty \bar{G}_n$, there is $k \in N$ such that $\bigcap_{n=1}^k \bar{G}_n \subset O$.

It will be convenient to prove the theorem using the statements (a) and (b′).

Suppose $\{\gamma_n\}_{n=1}^\infty$ is a pluming for X in $\beta(X)$. For each $n \in N$, let $\mathscr{G}_n$ be a cover of X, open in X, such that $\{(G)^-_{\beta(X)} : G \in \mathscr{G}_n\}$ refines γ_n. For a given $x \in X$, let G_n be an arbitrary element of $\mathscr{G}_n$ such that $x \in G_n$. Then $\bigcap_{n=1}^\infty (G_n)^-_{\beta(X)}$ is compact and

$$\bigcap_{n=1}^\infty (G_n)^-_{\beta(X)} \subset \bigcap_{n=1}^\infty St(x, \gamma_n) \subset X .$$

Thus
$$\bigcap_{n=1}^\infty (G_n)^-_{\beta(X)} = \bigcap_{n=1}^\infty [X \cap (G_n)^-_{\beta(X)}] = \bigcap_{n=1}^\infty \bar{G}_n .$$

Hence $\bigcap_{n=1}^\infty \bar{G}_n$ is a compact set. Now let $O \subset X$ be an open set containing $\bigcap_{n=1}^\infty \bar{G}_n$, and let O' be open in $\beta(X)$ such that $O' \cap X = O$. If $\bigcap_{n=1}^k \bar{G}_n$ is not contained in O' for any $k \in N$, then $\bigcap_{n=1}^k (G_n)^-_{\beta(X)} - O' \neq \varnothing$ for each $k \in N$. Hence $\{\bigcap_{n=1}^k (G_n)^-_{\beta(X)} - O' : k \in N\}$ is a decreasing sequence of compact sets. It follows that

$$\bigcap_{n=1}^\infty (G_n)^-_{\beta(X)} - O' \neq \varnothing$$

which is impossible since

$$\bigcap_{n=1}^{\infty} (G_n)^-_{\beta(X)} = \bigcap_{n=1}^{\infty} \bar{G}_n \subset O = O' \cap X .$$

Thus there exists an integer k such that $\bigcap_{n=1}^{k} \bar{G}_n \subset O'$ and so $\bigcap_{n=1}^{k} \bar{G}_n \subset O$. Hence (a) and (b′) are true.

Now suppose $\{\mathscr{G}_n\}_{n=1}^{\infty}$ is a sequence of open covers of X such that (a) and (b′) are true. For $n \in N$ define γ_n to be the collection of all sets $G' \subset \beta(X)$ such that G' is open in $\beta(X)$ and $G' \cap X \in \mathscr{G}_n$. We show $\{\gamma_n\}_{n=1}^{\infty}$ is a pluming for X. Let $x \in X$ and $y \in \beta(X) - X$. If $y \in \bigcap_{n=1}^{\infty} St(x, \gamma_n)$ there is a set $G'_n \in \gamma_n$ such that $x, y \in G'_n$ for each $n \in N$. Then $x \in G_n = G'_n \cap X \in \mathscr{G}_n$ and $\bigcap_{n=1}^{\infty} \bar{G}_n$ is a compact set which does not contain y. Let 0 be an open set in $\beta(X)$ such that

$$\bigcap_{n=1}^{\infty} \bar{G}_n \subset O \subset (O)^-_{\beta(X)} \subset \beta(X) - \{y\} .$$

Then there is $k \in N$, such that $\bigcap_{n=1}^{k} \bar{G}_n \subset O$. Thus $\bigcap_{n=1}^{k} G'_n - (O)^-_{\beta(X)} = \emptyset$ since it is an open set contained in $\beta(X) - X$. Thus $y \notin \bigcap_{n=1}^{\infty} G'_n$ which is a contradiction. So $y \notin \bigcap_{n=1}^{\infty} St(x, \gamma_n)$ and y was an arbitrary element of $\beta(X) - X$. Hence $\bigcap_{n=1}^{\infty} St(x, \gamma_n) \subset X$ and the theorem is proved.

THEOREM 1.4. *A completely regular $w\Delta$-space X is a p-space if every closed countably compact subset of X is compact.*

Proof. Let $\{\mathscr{U}_n\}_{n=1}^{\infty}$ be a sequence of open covers of X such that $\{St(x, \mathscr{U}_n) \colon n \in N\}$ is an x-sequence for each $x \in X$. For each $n \in N$ let $\mathscr{G}_n$ be an open cover of X such that $\{\bar{G} \colon G \in \mathscr{G}_n\}$ refines $\mathscr{U}_n$. Let $x \in X$ and $G_n \in \mathscr{G}_n$ such that $x \in G_n$. Then $\{\bigcap_{n=1}^{k} \bar{G}_n \colon k \in N\}$ is an x-sequence since $\bigcap_{n=1}^{k} \bar{G}_n \subset St(x, \mathscr{U}_n)$. Also, $\bigcap_{n=1}^{\infty} \bar{G}_n$ is countably compact since $\{x_k\}_{k=1}^{\infty} \subset \bigcap_{n=1}^{\infty} \bar{G}_n$ implies $x_k \in \bigcap_{n=1}^{k} \bar{G}_n$, and so $\{x_k\}_{k=1}^{\infty}$ must have a cluster point. By hypothesis, $\bigcap_{n=1}^{\infty} \bar{G}_n$ must then be a compact set. Thus (a) and (b) of Theorem 1.3 are satisfied.

A space X is said to be *θ-refinable* [17] if, for every open covering $\mathscr{U}$ of X, there is a sequence $\{\mathscr{U}_n\}_{n=1}^{\infty}$ of open refinements of $\mathscr{U}$ such that, if $x \in X$, there is $m(x) \in N$ such that x is in at most a finite number of elements of $\mathscr{U}_{m(x)}$.

In Theorem 1.7 we show that p-spaces and $w\Delta$-spaces are equivalent in the class of completely regular θ-refinable spaces. Before proving this theorem we want to point out the relationship between the θ-refinable property and two other covering properties.

A topological space X is called *metacompact* if every open cover of X has a point-finite open refinement. It is clear that all metacompact spaces are θ-refinable.

A space X is called *subparacompact* [5] if it satisfies any one of conditions (a) through (d) stated in the following theorem:

THEOREM 1.5. *For a topological space X the following conditions are equivalent*:

(a) *For any open covering $\mathscr{U}$ of X there is a sequence $\{\mathscr{U}_n\}_{n=1}^{\infty}$ of open coverings of X such that, if $x \in X$, there is $m(x) \in N$ and some set $U \in \mathscr{U}$ with $St(x, \mathscr{U}_{m(x)}) \subset U$.*

(b) *Every open cover of X has a σ-discrete closed refinement.*

(c) *Every open cover of X has a σ-locally-finite closed refinement.*

(d) *Every open cover of X has a σ-closure-preserving closed refinement.*

Theorem 1.5 was proved in [5] and we use this theorem to prove Theorem 1.6, from which it follows that all subparacompact spaces are θ-refinable.

THEOREM 1.6. *For a space X to be subparacompact it is necessary and sufficient that every open cover of X has a sequence $\{\mathscr{U}_n\}_{n=1}^{\infty}$ of open refinements with the property that, if $x \in X$, there is $m(x) \in N$ such that x is in exactly one element of $\mathscr{U}_{m(X)}$.*

Proof. By Theorem 1.5 it is enough to prove that the condition is necessary for X to be subparacompact. So suppose X is subparacompact and $\mathscr{U}$ is an open cover of X. Let $\mathscr{P} = \bigcup_{n=1}^{\infty} \mathscr{P}_n$ be a closed refinement of $\mathscr{U}$ where $\mathscr{P}_n$ is discrete for each $n \in N$. For each $P \in \mathscr{P}$ let $U(P)$ be a fixed element of $\mathscr{U}$ such that $P \subset U(P)$. For each $x \in X$ let $U(x)$ be a fixed element of $\mathscr{U}$ such that $x \in U(x)$.

Fix $n \in N$. If $x \in X$ and $x \in X - \bigcup\{P\colon P \in \mathscr{P}_n\}$, define

$$U_n(x) = U(x) - \bigcup\{P\colon P \in \mathscr{P}_n\}\,.$$

If $x \in \bigcup\{P\colon P \in \mathscr{P}_n\}$, say $x \in P \in \mathscr{P}_n$, define

$$U_n(x) = U(P) - \bigcup\{P' \in \mathscr{P}_n : x \notin P'\}\,.$$

Then $\mathscr{U}_n = \{U_n(x) : x \in X\}$ is an open refinement of $\mathscr{U}$ for each $n \in N$. It is clear that $x \in P \in \mathscr{P}_n$ implies that $U_n(x)$ is the only element in $\mathscr{U}_n$ which contains x. Since every $x \in X$ is in some element of $\mathscr{P}$, it follows that $\{\mathscr{U}_n\}_{n=1}^{\infty}$ is a sequence of open refinements of $\mathscr{U}$ satisfying the required properties.

THEOREM 1.7. *For a completely regular θ-refinable space X, the following conditions are equivalent*:

(a) *X is a p-space.*

(b) *X is a strict p-space.*

(c) *X is a $w\Delta$-space.*

Proof. That (b) $\Rightarrow$ (c) follows from Corollary 1.2.

In a θ-refinable space closed countably compact subsets are compact [17]. Hence (c) $\Rightarrow$ (a) follows from Theorem 1.4.

To prove that (a) $\Rightarrow$ (b) let $\{\gamma_n\}_{n=1}^{\infty}$ be a pluming for X in $\beta(X)$. We will construct a strict pluming for X. Since X is θ-refinable, we can find a sequence $\{\gamma_{1,n}\}_{n=1}^{\infty}$ of covers of X, open in $\beta(X)$, such that the following is true:

(1) For each $n \in N$, the collection $\{(G)^{-}_{\beta(X)}: G \in \gamma_{1,n}\}$ refines γ_1.

(2) For each $x \in X$, there is $m \in N$ such that x is in at most a finite number of elements of $\gamma_{1,m}$.

Continue by induction and assume $\{\gamma_{k,n}\}_{n=1}^{\infty}$ is defined for $k \in N$. Then we can find a sequence $\{\gamma_{k+1,n}\}_{n=1}^{\infty}$ of covers of X, open in $\beta(X)$, such that the following is true:

(3) For each $n \in N$, the collection $\{(G)^{-}_{\beta(X)}: G \in \gamma_{k+1,n}\}$ refines γ_{k+1} and refines $\gamma_{r,s}$ where $r, s \in N$, $r + s = k + 1$.

(4) For each $x \in X$, there is $m \in N$ such that x is in at most a finite number of elements of $\gamma_{k+1,m}$.

It is clear that the sequence $\{\gamma_{n,1}\}_{n=1}^{\infty}$ is a pluming for X since $St(x, \gamma_{n,1}) \subset St(x, \gamma_n)$ for any $x \in X$. To show that $\{\gamma_{n,1}\}_{n=1}^{\infty}$ is a strict pluming, let $n \in N$ and $x \in X$. Let $m \in N$ such that x is in at most a finite number of element of $\gamma_{n+1,m}$. Then by (3)

$$\begin{aligned}\overline{St(x, \gamma_{n+1,m})} &= (\bigcup\{G \in \gamma_{n+1,m}: x \in G\})^{-} \\ &= \bigcup\{\bar{G}: x \in G \in \gamma_{n+1,m}\} \subset St(x, \gamma_{n,1}) .\end{aligned}$$

Also $\gamma_{n+m+1,1}$ refines $\gamma_{n+1,m}$. Thus

$$\overline{St(x, \gamma_{n+m+1,1})} \subset \overline{St(x, \gamma_{n+1,m})} \subset St(x, \gamma_{n,1})$$

and $\{\gamma_{n,1}\}_{n=1}^{\infty}$ is a strict pluming.

Since subparacompact spaces and metacompact spaces are θ-refinable the next corollary is obvious.

COROLLARY 1.8. *For a completely regular subparacompact (metacompact) space X, the following conditions are equivalent*:

(a) *X is a p-space.*

(b) *X is a strict p-space.*

(c) *X is a $w\Delta$-space.*

REMARK 1.9. If a regular $w\Delta$-space X is θ-refinable it is possible to construct a sequence $\{\mathscr{G}_n\}_{n=1}^{\infty}$ of open covers of X satisfying (a) and (b) of Theorem 1.1 even if X is not completely regular. We will use this fact in the next section.

We finish this section by giving two examples. The first is a completely regular countably compact space (and thus a $w\Delta$-space) which is not a p-space. The last example is a p-space which is not a $w\Delta$-space.

Recall that a space X is a *k-space* if any set $A \subset X$ is closed if and only if $A \cap C$ is closed in C for every compact set $C \subset X$. In Example 1.10 we need to know that a p-space is a k-space [1].

EXAMPLE 1.10. A completely regular countably compact space which is not a p-space.

Let $\beta(N)$ be the Stone-Cech compactification of N. J. Novak [12] constructed two countably compact subsets $X_1 \subset \beta(N)$ and $X_2 \subset \beta(N)$ such that $X_1 \cap X_2 = N$, $X_1 \cup X_2 = \beta(N)$, and $D = \{(x, x): x \in N\}$ is an infinite, discrete, closed subset of $X_1 \times X_2$. We show that X_1 is not a p-space by showing that it is not a k-space. Let C be any compact subset of X_1 and consider the set $C \cap N$. If $C \cap N$ is finite, it is certainly closed in C; assume $C \cap N = A$ is infinite. Now $(A)^-_{X_1} \times (A)^-_{X_2}$ is a countably compact subset of $X_1 \times X_2$ since $(A)^-_{X_1} \subset C$ is compact and $(A)^-_{X_2}$ is at least countably compact (see Theorem 5 in [12]). However, $\{(x, x): x \in A\} \subset D$, and hence $\{(x, x): x \in A\}$ is an infinite, discrete, closed subset of $(A)^-_{X_1} \times (A)^-_{X_2}$, which is impossible. Thus $C \cap N$ is always finite for every compact $C \subset X_1$; hence $C \cap N$ is closed in C for every compact C. But N is not closed in X_1, so X_1 is not a k-space.

EXAMPLE 1.11. A T_2 locally compact space (and thus a p-space) which is not a $w\Delta$-space.

Let ω, ω_1, ω_2 be the first ordinals of cardinalities $\aleph_0$, $\aleph_1$, $\aleph_2$ respectively. Let $\Gamma = [0, \omega_2)$. Before constructing the example we prove the following lemma:

LEMMA 1.12. *For each $\alpha \in \Gamma$, suppose Γ_α is a countable subset of Γ. There is a sequence $\{\alpha_n\}_{n=1}^\infty \subset \Gamma$, $\alpha_1 < \alpha_2 < \cdots$, such that $\alpha_i \notin \Gamma_{\alpha_j}$ if $\alpha_i \neq \alpha_j$.*

Proof. We will define the α_k inductively. For each $\alpha \in \Gamma$ let

$$D_\alpha = \{\beta \in \Gamma:\ \beta \geqq \alpha, \alpha \notin \Gamma_\beta\}\ .$$

Suppose that card $D_\alpha \leqq \aleph_1$ for all $\alpha \in \Gamma$. Let $\alpha_0 \in \Gamma$ such that $\alpha_0 \geqq \omega_1$. Then

$$[\alpha_0, \omega_2) - \bigcup_{\alpha < \alpha_0} D_\alpha \neq \emptyset$$

since card $[\alpha_0, \omega_2) = \aleph_2$ and card $(\bigcup_{\alpha<\alpha_0} D_\alpha) \leqq \aleph_1$. Let $\beta_0 \in [\alpha_0, \omega_2) - \bigcup_{\alpha<\alpha_0} D_\alpha$. Then $\alpha \in [0, \alpha_0)$ implies $\beta_0 \notin D_\alpha$; so $\alpha \in \Gamma_{\beta_0}$. Thus

$$\operatorname{card} \Gamma_{\beta_0} \geqq \operatorname{card} [0, \alpha_0) = \aleph_1 ,$$

which is a contradiction. Hence $\{\alpha \in \Gamma : \operatorname{card} D_\alpha > \aleph_1\} \neq \emptyset$. Let α_1 be the smallest element in the set. Now suppose that $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ are defined such that the following is true:

(1) $\alpha_1 < \alpha_2 < \cdots < \alpha_{n-1}$

(2) $\alpha_i \notin \Gamma_{\alpha_j}$, for $i \neq j$, $i, j = 1, 2, \cdots, n-1$

(3) Card $(D_{\alpha_1} \cap D_{\alpha_2} \cap \cdots \cap D_{\alpha_{n-1}}) > \aleph_1$.

Let $\gamma_0 = \sup\{\beta : \beta \in \Gamma_{\alpha_i},\ 1 \leqq i \leqq n-1\} + 1$ and let

$$\Gamma^{(n)} = D_{\alpha_1} \cap D_{\alpha_2} \cap \cdots \cap D_{\alpha_{n-1}} - [0, \gamma_0) .$$

Then card $\Gamma^{(n)} > \aleph_1$. For $\alpha \in \Gamma^{(n)}$ let $D_\alpha^{(n)} = D_\alpha \cap \Gamma^{(n)}$. Suppose that card $D_\alpha^{(n)} \leqq \aleph_1$, for all $\alpha \in \Gamma^{(n)}$, and let $\alpha'_0 \in \Gamma$ such that card $(\Gamma^{(n)} \cap [\gamma_0, \alpha'_0]) = \aleph_1$. Then

$$[\gamma_0, \omega_2) \cap \Gamma^{(n)} - \cup \{D_\alpha^{(n)} : \alpha \in \Gamma^{(n)} \cap [\gamma_0, \alpha'_0]\} \neq \emptyset$$

so let β'_0 be an element of this set. If $\alpha \in \Gamma^{(n)} \cap [\gamma_0, \alpha'_0]$, we have $\alpha \in \Gamma_{\beta'_0}$. This implies

$$\operatorname{card} \Gamma_{\beta'_0} \geqq \operatorname{card} (\Gamma^{(n)} \cap [\gamma_0, \alpha'_0]) = \aleph_1 ,$$

which is a contradiction. Therefore

$$\{\alpha \in \Gamma^{(n)} : \operatorname{card} D_\alpha^{(n)} > \aleph_1\} \neq \emptyset ,$$

and we let α_n be the first element of this set.

Notice that $\alpha_n > \sup\{\beta : \beta \in \Gamma_{\alpha_i},\ 1 \leqq i \leqq n-1\}$, so $\alpha_n \notin \Gamma_{\alpha_i}$, for $1 \leqq i \leqq n-1$. Also, $\alpha_n \in \Gamma^{(n)} \subset D_{\alpha_1} \cap D_{\alpha_2} \cap \cdots \cap D_{\alpha_{n-1}}$, so $\alpha_i \notin \Gamma_{\alpha_n}$, for $1 \leqq i \leqq n-1$. Thus (1) and (2) are satisfied for $\alpha_1, \alpha_2, \cdots, \alpha_n$. Now

$$D_{\alpha_n}^{(n)} = \Gamma^{(n)} \cap D_{\alpha_n} \subset D_{\alpha_1} \cap D_{\alpha_2} \cap \cdots \cap D_{\alpha_{n-1}} \cap D_{\alpha_n} ,$$

so

$$\operatorname{card} (D_{\alpha_1} \cap D_{\alpha_2} \cap \cdots \cap D_{\alpha_n}) \geqq \operatorname{card} D_{\alpha_n}^{(n)} > \aleph_1 .$$

Thus (3) is satisfied and the lemma is proved.

For each $\alpha \in \Gamma$, let $X_\alpha = \{0, 1\}$ with the discrete topology. Let $Y = \prod_\alpha X_\alpha$ have the product topology, and define $X = Y - \{g\}$, where g is the element in Y such that $g(\alpha) = 0$ for all $\alpha \in \Gamma$. Since X is open in the compact space Y, it follows that X is a T_2 locally compact space. Let $\{\mathscr{U}_n\}_{n=1}^\infty$ be any sequence of open covers of X. To show that X is not a $w\Delta$-space, we will find an element $x \in X$ such that $\{St(x, \mathscr{U}_n) : n \in N\}$ is not an x-sequence. For a given $\alpha \in \Gamma$, let

f_α be the element of X such that $f_\alpha(\alpha)=1$ and $f_\alpha(\beta)=0$ for all $\beta\neq\alpha$. Let $U_n(\alpha)\in\mathscr{U}_n$ such that $f_\alpha\in U_n(\alpha)$. Since $U_n(\alpha)$ is also open in Y, there exists a finite set $\Gamma_{n,\alpha}\subset\Gamma$ such that $\alpha\in\Gamma_{n,\alpha}$ and

$$\prod_\beta Z^\alpha_\beta(n)\subset U_n(\alpha)\ ,$$

where

$$Z^\alpha_\alpha(n)=\{1\}\ ,$$

$$Z^\alpha_\beta(n)=\{0\}\qquad(\beta\in\Gamma_{n,\alpha}-\{\alpha\})\ ,$$

and

$$Z^\alpha_\beta(n)=X_\beta\qquad(\beta\in\Gamma-\Gamma_{n,\alpha})\ .$$

Let $\Gamma_\alpha=\bigcup_{n=1}^\infty\Gamma_{n,\alpha}$. Then Γ_α is a countable subset of Γ for each $\alpha\in\Gamma$. By the lemma, there is a sequence $\{\alpha_k\}_{k=1}^\infty$, of distinct elements of Γ, such that $\alpha_i\notin\Gamma_{\alpha_j}$ for $i\neq j$. Define

$$Z_{\alpha_k}=\{1\}\qquad(k\in N)\ ,$$

$$Z_\beta=\{0\}\qquad(\beta\in\Gamma_{\alpha_k}-\{\alpha_k\},\,k\in N)\ ,$$

and

$$Z_\beta=X_\beta\qquad(\beta\notin\bigcup_{k=1}^\infty\Gamma_{\alpha_k})\ .$$

Let $g_0\in\prod_\beta Z_\beta$. It follows that $g_0\in U_n(\alpha_k)$ for each $n,k\in N$. Thus $f_{\alpha_n}\in St(g_0,\mathscr{U}_n)$ for each $n\in N$; however, the sequence $\{f_{\alpha_n}\}_{n=1}^\infty$ does not have a cluster point in X. Hence X is not a $w\Delta$-space.

2. Developable p-spaces and $w\Delta$-spaces. In this section we give some conditions for p-spaces and $w\Delta$-spaces to be developable. As is suggested by Theorem 1.7 it turns out that p-spaces and $w\Delta$-spaces can be used interchangeably in many theorems. We state Theorems 2.1 and 2.2 as an illustration of this.

A collection $\mathscr{P}$ of subsets of a space X is called a *network* for X if for any open set $O\subset X$ and $x\in O$ there is a set $P\in\mathscr{P}$ such that $x\in P\subset O$. A space with a σ-locally-finite network is called a σ-space [13]. It is proved in [16] that existence of a σ-closure preserving closed network in X implies the existence of a σ-discrete closed network; hence a regular space has a σ-discrete network if and only if it has a σ-closure-preserving network.

Let X be a topological space and d a nonnegative real valued symmetric function defined on $X\times X$ such that $d(x,y)=0$ if and only if $x=y$. The function d is called a *symmetric* [2] for the topology on X provided: $A\subset X$ is closed if and only if $\inf\{d(x,z):z\in A\}>0$ for any $x\in X-A$. The function d is called a *semi-metric* for X provided: For $A\subset X$, $x\in\bar{A}$ if and only if $\inf\{d(x,z): z\in A\}=0$. It is easily shown that a symmetric space X is a semi-metric space if and only if X is first countable.

THEOREM 2.1. *For a completely regular space X the following conditions are equivalent:*

(a) *X is developable.*
(b) *X is a p-space with a σ-discrete network.*
(c) *X is a semi-metrizable p-space.*
(d) *X is a symmetrizable p-space.*

THEOREM 2.2. *For a regular space X the following conditions are equivalent:*

(a) *X is developable.*
(b) *X is a $w\Delta$-space with a σ-discrete network.*
(c) *X is a semi-metrizable $w\Delta$-space.*
(d) *X is a symmetrizable $w\Delta$-space.*

Theorem 2.1 was proved in [6]. Siwiec has shown in [15] that parts (a) and (c) of Theorem 2.2 are equivalent. To complete Theorem 2.2 it is only necessary to show that the spaces described in (b) and (d) are first countable and therefore semi-metrizable. This is a relatively easy exercise.

A collection $\mathscr{B}$ of closed subsets of X is called a *ct-net* [16] for X if, when $x, y \in X$ such that $x \neq y$, there is an element $B \in \mathscr{B}$ such that $x \in B$ and $y \notin B$. It is obvious that a closed network in a T_1 space is a ct-net and it can be shown that a semi-metric space has a σ-discrete ct-net.

We need the following theorem from [9]:

THEOREM 2.3 (*Heath*). *A T_1 space X is semi-metrizable if and only if each point $x \in X$ has a decreasing open neighborhood base $\{U_n(x)\}_{n=1}^\infty$ such that if $x_n \in X$, with $x \in U_n(x_n)$ for each $n \in N$, then $x_n \longrightarrow x$.*

THEOREM 2.4. *A regular $w\Delta$-space (or strict p-space) X is developable if and only if it has a σ-closure-preserving ct-net.*

Proof. Since a strict p-space is a $w\Delta$-space we show only for the case when X is a regular $w\Delta$-space.

If we assume X has a σ-closure-preserving ct-net then to show that X is developable it is sufficient to show that X is semi-metrizable and apply Theorem 2.2. Let $\mathscr{P} = \bigcup_{n=1}^\infty \mathscr{P}_n$ be a ct-net for X where each $\mathscr{P}_n$ is a closure-preserving collection of closed sets. Suppose $\{\mathscr{G}_n\}_{n=1}^\infty$ is a sequence of open covers of X such that $\{St(x, \mathscr{G}_n)\colon n \in N\}$ is an x-sequence for each $x \in X$ and assume $\mathscr{G}_{n+1}$ refines $\mathscr{G}_n$. For $x \in X$, $n \in N$, define

$$V_n(x) = X - \bigcup\{P \in \mathscr{P}_k \colon x \notin P,\ 1 \leqq k \leqq n\}\ .$$

Note that $z \in V_n(x)$ implies that $z \notin P \in \mathscr{P}_k$ whenever $x \notin P \in \mathscr{P}_k$, for $1 \leqq k \leqq n$. Hence $V_n(z) \subset V_n(x)$.

Now let $U_n(X)$ be an open neighborhood of x such that

$$\overline{U_n(x)} \subset V_n(x) \cap St(x, \mathscr{G}_n)\ .$$

We may assume $U_{n+1}(x) \subset U_n(x)$ for each $n \in N$. Since $\mathscr{P}$ is a ct-net for X, it follows that $\bigcap_{n=1}^{\infty} V_n(x) = \{x\}$. Hence $\bigcap_{n=1}^{\infty} \overline{U_n(x)} = \{x\}$. Let O be any open set containing x. If $U_n(x)$ is not contained in O for any n, there is an element $y_n \in \overline{U_n(x)} - O$. Since $U_n(x) \subset St(x, \mathscr{G}_n)$ the sequence $\{y_n\}_{n=1}^{\infty}$ has a cluster point y not in O. But $y_n \in \overline{U_n(x)} - O$ implies $y \in \bigcap_{n=1}^{\infty} \overline{U_n(x)} - O$, which is a contradiction since $\bigcap_{n=1}^{\infty} \overline{U_n(x)} = \{x\}$. So there is $n \in N$ such that $U_n(x) \subset O$ and $\{U_n(x)\}_{n=1}^{\infty}$ is a decreasing open neighborhood base at x. To show that X is semi-metrizable, we show that $\{U_n(x)\}_{n=1}^{\infty}$ satisfies the conditions in Theorem 2.3. Suppose $x_n \in X$ such that $x \in U_n(x_n)$ for each $n \in N$. Then $x \in St(x_n, \mathscr{G}_n)$ which implies $x_n \in St(x, \mathscr{G}_n)$. Thus $\{x_n\}_{n=1}^{\infty}$ has a cluster point $y \in X$. Suppose $y \neq x$. Then there is an integer $m \in N$ such that there is $P \in \mathscr{P}_m$ with $x \in P$ and $y \notin P$. So $x \notin V_m(y)$. Since y is a cluster point of $\{x_n\}_{n=1}$ there is $m_1 \in N$, $m_1 \geqq m$ such that $x_{m_1} \in V_m(y)$. Hence $V_m(x_{m_1}) \subset V_m(y)$. It follows that

$$x \in V_{m_1}(x_{m_1}) \subset V_m(x_{m_1}) \subset V_m(y)$$

which is a contradiction. Thus $y = x$ and x is the only cluster point of $\{x_n\}_{n=1}^{\infty}$. If $\{x_{n_k}\}_{k=1}^{\infty}$ is any subsequence of $\{x_n\}_{n=1}^{\infty}$, then

$$x_{n_k} \in St(x, \mathscr{G}_{n_k}) \subset St(x, \mathscr{G}_k)\ .$$

Thus $\{x_{n_k}\}_{k=1}^{\infty}$ must have x as a cluster point. Since every subsequence of $\{x_n\}_{n=1}^{\infty}$ has x as a cluster point, it follows that $x_n \to x$ and X is semi-metrizable.

The converse is trivial so the theorem is proved.

It was stated in [2] that if a strict p-space X can be mapped onto a Moore space by a one-to-one continuous map then X is a Moore space. The following corollary is a generalization of this.

COROLLARY 2.5. *Suppose X is a strict p-space (regular $w\Delta$-space) and X is mapped onto a T_2 space Y by a one-to-one continuous map. Then X is developable if any one of the following conditions hold:*

(a) *Y is developable.*

(b) *Y has a σ-discrete network.*

(c) *Y is semi-metrizable.*

(d) *Y has a σ-closure-preserving ct-net.*

Proof. The corollary follows immediately from Theorem 2.4 when you show that any one of conditions (a) through (d) implies that X has a σ-closure-preserving ct-net.

QUESTION 2.6. Is Theorem 2.4 or Corollary 2.5 true if X is required to be a p-space instead of a strict p-space?

A point-countable collection of subsets of a space X which is an open base for the topology on X is called a *point-countable base*. Filippov [8] has proved the following result:

THEOREM 2.7. *A paracompact p-space with a point-countable base is metrizable.*

We generalize Theorem 2.7 in Theorems 2.8 and 2.10 by showing that a p-space X with a point-countable base is a Moore space if it is either metacompact or subparacompact. Theorem 2.8 was proved in [6] for the case when X is a p-space. The proof when X is a regular $w\Delta$-space is similar if we use Remark 1.9 and assume X has a sequence of open covers satisfying (a) and (b) of Theorem 1.1.

THEOREM 2.8. *A metacompact p-space (regular $w\Delta$-space) with a point-countable base $\mathscr{B}$ is a Moore space.*

Before proceeding with the statement and proof of Theorem 2.10 we need the following lemma which can be found in [8].

LEMMA 2.9. *If $\mathscr{B}$ is a point-countable collection of subsets of a space X and $A \subset X$, then the family of all minimal finite covers of A with elements from $\mathscr{B}$ is countable.*

THEOREM 2.10. *A subparacompact p-space (regular $w\Delta$-space) X with a point-countable base $\mathscr{B}$ is a Moore space.*

Proof. Let $\{\mathscr{G}_n\}_{n=1}^{\infty}$ be a sequence of open covers of X satisfying (a) and (b) of Theorem 1.1. For each $n \in N$, let $\mathscr{P}_n = \bigcup_{m=1}^{\infty} \mathscr{P}_{n,m}$ be a σ-discrete closed refinement of $\mathscr{G}_n$ where each $\mathscr{P}_{n,m}$ is discrete. By Lemma 2.9, each $P \in \mathscr{P}_n$ has at most a countable number of minimal finite covers with elements of $\mathscr{B}$, say $P(1, n), P(2, n), \cdots$ (if they exist).

Let $\mathscr{S}_n = \{P \cap B \colon P \in \mathscr{P}_n,\ B \in P(k, n),\ k \in N\}$. It follows that $\mathscr{S}_n$ is a σ-locally-finite collection, and so $\mathscr{S} = \bigcup_{n=1}^{\infty} \mathscr{S}_n$ is also a

σ-locally-finite collection. We show that $\mathscr{S}$ is a network for X. Let O be an open subset of X and $x \in O$. Since $\mathscr{B}$ is a base there is a set $B \in \mathscr{B}$ such that $x \in B \subset O$. Since $P_x = \bigcap_{n=1}^{\infty} St(x, \mathscr{G}_n)$ is compact, we can find a finite subcollection $\{B_1, B_2, \cdots, B_k\} \subset \mathscr{B}$ such that $P_x \subset \bigcup_{i=1}^{k} B_i$ and $B = B_1$ is the only element of $\{B_1, B_2, \cdots, B_k\}$ which contains x. Let $n \in N$ such that $St(x, \mathscr{G}_n) \subset \bigcup_{i=1}^{k} B_i$. Let $P \in \mathscr{P}_n$ such that $x \in P$. Then $P \subset \bigcup_{i=1}^{k} B_i$, so there is a minimal finite cover of P, say $P(j, n)$, such that $B_1 = B \in P(j, n)$. Hence $x \in P \cap B \subset O$ and $P \cap B \in \mathscr{S}_n$. Therefore $\mathscr{S}$ is a σ-locally-finite network for X. Hence X has a σ-discrete network and is developable by Theorems 2.1 and 2.2.

QUESTION 2.11. Is a p-space (or regular $w\Delta$-space) with a point-countable base a Moore space?

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Received January 29, 1970.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Pacific Journal of Mathematics

Vol. 35, No. 2 October, 1970