A \( v \)-INTEGRAL REPRESENTATION FOR LINEAR OPERATORS ON SPACES OF CONTINUOUS FUNCTIONS WITH VALUES IN TOPOLOGICAL VECTOR SPACES

JOHN R. EDWARDS AND STANLEY G. WAYMENT
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Suppose \(X\) and \(Y\) are topological vector spaces. This paper gives an analytic representation of continuous linear operators from \(C\) into \(Y\), where \(C\) denotes the space of continuous functions from the interval \([0, 1]\) into \(X\) with the topology of uniform convergence. In order to obtain an integral representation theorem analogous to the ones given by R. K. Goodrich for the locally convex setting in Trans. Amer. Math. Soc. 131 (1968), 246-258, certain strong hypotheses on \(C\) must be assumed because of the need to be able to extend the operators to a subset of the double dual of \(C\). However, by using the notion of \(v\)-integral, it is possible to avoid this problem and give a representation theorem without additional hypothesis.

Let \(\mathcal{S}\) be the collection of intervals in \((0, 1]\) of the form \((a, b]\) and let \(L[X, Y]\) denote the space of linear operators from \(X\) into \(Y\). Then the set function \(K\) from \(\mathcal{S}\) into \(L[X, Y]\) is said to be convex with respect to length if \(K(I) = \sum_{i=1}^{n} [l(I_i)/l(I)] K(I_i)\) whenever \(I = \bigcup_{i=1}^{n} I_i\), and where \(l(I)\) denotes the length of \(I\). If \(K\) is convex with respect to length, then \(K\) is said to be \(v\)-integrable with respect to \(f\) if \(\lim_{\sigma \to 0} \sum K(I_i)(f)(x_i) = v\int K df\) exists in \(\bar{Y}\), the completion of \(Y\) (by \(\Delta J\) we mean \(f(t_{i+1}) - f(t_i)\) where \(\sigma\) is the partition \(\sigma\) of \([0, 1]\)).

If \(I \in \mathcal{S}\), with endpoints \(a\) and \(b\), then the function \(\Psi_I\) defined by \(\Psi_I(t) = 0\) for \(t \leq a\), \(\Psi_I(t) = (t - a)/(b - a)\) for \(a \leq t \leq b\), and \(\Psi_I(t) = 1\) for \(t \geq b\), is called the fundamental function associated with \(I\). A set function \(K\) whose domain is \(\mathcal{S}\) and whose range is in \(L[X, Y]\) is said to be quasi-Gowurin if given a neighborhood \(V\) of \(\theta_X\), there is a neighborhood \(U\) of \(\theta_Y\) such that \(\sum \Psi_I x_i \in U\) implies \(\sum [K(I_i)](x_i) \in V\).

Finally, if \(f \in C\) and \(\sigma\) is a partition of \([0, 1]\), then \(pf_\sigma\) denotes the polygonal function determined by \(\sigma\) and \(f\).

2. The representation theorem. Let \(C_0\) denote the subspace of \(C\) such that \(f(0) = \theta_X\).

**THEOREM 2.1.** Suppose \(K\) is a set function on \(\mathcal{S}\) with values
in \( L[X, Y] \) which is convex with respect to length and which is quasi-Gowurin. Then \( T(f) = v\int Kdf \) is a continuous linear operator from \( C_0 \) into \( Y \).

**Proof.** First we show that \( v\int Kdf \) exists in \( Y \) for each \( f \in C_0 \). Suppose \( V \) is a neighborhood of \( \theta_y \). Since \( K \) is quasi-Gowurin, there is a neighborhood \( U \) of \( \theta_c \) such that \( \sum \mathcal{F}_{f_i} x_i \in U \) implies \( \sum [K(I_i)](x_i) \) is in \( V \). Since \( p\sigma_{f} \) converges to \( f \) in the topology of uniform convergence, there is a \( \delta \) such that \( |\sigma_1|, |\sigma_2| < \delta \) implies \( p\sigma_{f_1} - p\sigma_{f_2} \in U \). Let \( \sigma_1, \sigma_2 \) denote the common refinement of \( \sigma_1 \) and \( \sigma_2 \). It follows from 7.2 in [1] that

\[
(\ast) \sum_{i} [K(I_i)](\alpha_i f) - \sum_{i} [K(I_i)](\alpha_{f,i} f) = \sum_{i} p[K(I_i)](\alpha_{f_1} - \alpha_{f_2})
\]

Since \( \sum \mathcal{F}_{f_k} (\alpha_{k}(p\sigma_{f_1} - p\sigma_{f_2})) = p\sigma_{f_1} - p\sigma_{f_2} \in U \), then it follows that \((\ast)\) is in \( V \) from which we conclude that \( \{\sum [K(I_i)](\alpha_{f})\} \) is Cauchy.

Hence, \( v\int Kdf \) exists in \( Y \). Suppose \( f \to f \) in \( C_0 \). Suppose \( V \) is a neighborhood of \( \theta_y \). Then there is a neighborhood \( V' \) of \( \theta_y \) such that \( V' + V' + V' \subset V \). Since \( K \) is quasi-Gowurin, there is a neighborhood \( U \) of \( \theta_c \) such that \( \sum \mathcal{F}_{f_i} x_i \in U \) then \( \sum [K(I_i)](x_i) \in V \). There exists a neighborhood \( U' \) of \( \theta_c \) such that \( U' + U' + U' \subset U \). Since \( f \to f \) then, there is a \( \beta \) such that \( \alpha > \beta \) implies \( f - f \in U' \). Suppose \( \alpha > \beta \). Then there is a \( \delta \) such that \( |\sigma| < \delta \) implies each of \( p(f_{\alpha}) - p(f_{\beta}) \in U' \), \( f - p\sigma_{f} \in U' \), \( v\int Kdf - \sum_o[K(I_i)](\alpha_{f}) \in V' \), and \( \sum_o[K(I_i)](\alpha_{f}) - v\int Kdf \in V' \). Then,

\[
v\int Kdf - v\int Kdf_{\alpha} = v\int Kdf - \sum_o[K(I_i)](\alpha_{f}) + \sum_o[K(I_i)](\alpha_{f}) - v\int Kdf_{\alpha} + V' + V'.
\]

However,

\[
\sum_o \mathcal{F}_{f_k} (\alpha_{f} - \alpha_{f}) = p\sigma_{f} - p(f_{\alpha}) = (p\sigma_{f} - f) + (f - f_{\alpha}) + (f_{\alpha} - p(f_{\alpha}))
\]

which is in \( U' + U' + U' \subset U \). Hence \( \sum_o[K(I_i)](\alpha_{f}) \in V' \), from which it follows that \( v\int Kdf - v\int Kdf_{\alpha} \in V' + V' + V' \subset V \). Therefore, \( v\int Kdf_{\alpha} \) converges to \( v\int Kdf \), and hence \( T \) is continuous.

**Theorem 2.2.** Suppose \( T \) is a continuous linear operator from \( C_0 \) into \( Y \). Then there is a set function \( \mathcal{F} \) with values in \( L[X, Y] \).
which is convex with respect to length and quasi-Gowurin such that 
\( T(f) = v \int Kdf \) for each \( f \in C_\circ \).

**Proof.** Define \( K \) from \( \mathcal{F} \) into \( L[X, Y] \) by \( (K(I))(x) = T(\psi \cdot x), \) \( x \in X \). Then \( K \) is convex with respect to length because \( T \) is linear and because of the manner in which fundamental functions combine. Suppose \( V \) is a neighborhood of \( \theta_\circ \). Since \( T \) is continuous, there is a neighborhood \( U \) of \( \theta_\circ \) such that \( T(U) \subset V \). Therefore \( \sum \psi_i \cdot x_i \in U \) implies \( \sum[K(I_i)](x_i) = T(\sum \psi_i \cdot x_i) \in V \), which implies \( K \) is quasi-Gowurin. Suppose \( f \in C_\circ \). Since \( \psi \) converges to \( f \) in \( C_\circ \), then

\[
T(f) = \lim_{|\sigma|} T(\psi \sigma) = \lim_{|\sigma|} T(\sum \psi_i \cdot \sigma_i(f)) = \lim_{|\sigma|} \sum[K(I_i)](\sigma_i(f)) = v \int Kdf.
\]

The last equality follows from 2.1. The theorem is established.

**Corollary 2.3.** Suppose \( Y \) is complete. Then, \( T \) is a continuous linear operator from \( C \) into \( Y \) if and only if there is an element \( \mu \in L[X, Y] \) and a set function on \( \mathcal{F} \) with values in \( L[X, Y] \) which is convex with respect to length and quasi-Gowurin such that 
\( T(f) = \mu(f(0)) + v \int Kdf \).

3. The locally convex setting. In this section, for the purpose of comparison, we consider the special case when \( H = [0,1] \) of the setting in which Goodrich gives his representation theorem [3], that is, we assume additionally that \( X \) and \( Y \) are locally convex spaces. The condition of quasi-Gowurin becomes: given a neighborhood \( V \) of \( \theta_\circ \), there is a neighborhood \( U \) of \( \theta_\circ \) such that if \( \{\sum_{i=1}^n x_i; j = 1, \ldots, n\} \subset U \), then \( \sum_{i=1}^n[K(I_i)](x_i) \in V \). This condition stated in terms of the semi-norms becomes, using Swongs notation [4], there is a pairing \( (p, q) \) and a constant \( W_{p-q} \) for each pair of semi-norms \( p \) and \( q \) in the pairing such that \( q(\sum_{i=1}^n[K(I_i)](x_i)) \leq W_{p-q} \max_j p(\sum_{i=1}^n x_i) \) for each partition of \( (0,1] \) and each corresponding collection \( \{x_i\} \) in \( X \). This property is the analogy of Goodrich’s bounded \( (p, q) \) variation. A set function which satisfies this property is said to be of bounded \( (p, q) \) convex variation. In order to be able to state an optimal result in the following theorem we shall assume that \( Y \) is quasi-complete, i.e., each closed and bounded set in \( Y \) is complete.

**Theorem 3.1.** Suppose \( T \) is a linear operator from \( C \) into \( Y \) (which is quasi-complete). Then \( T \) is continuous if and only if there exists a \( \mu \in L[X, Y] \) and a set function on \( \mathcal{F} \) with values in \( L[X, Y] \) which is convex with respect to length and of bounded
(p, q) convex variation such that $T(f) = \mu(f(0)) + \sqrt{Kdf}$. Furthermore if $T'$ denotes the restriction of $T$ to $C_0$, then $\|T'|_{p-q} = W_{p-q}$.

The theorem follows from 2.3.

**Remark 3.2.** It is immediate that the $K$ function of 2.2, 2.3, and 3.1 is unique.

**References**


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Valentin Danilovich Belousov and Palaniappan L. Kannappan, Generalized Bol functional equation .................................................. 259
Charles Morgan Biles, Gelfand and Wallman-type compactifications ................ 267
Louis Harvey Blake, A generalization of martingales and two consequent convergence theorems .................................................. 279
Dennis K. Burke, On $p$-spaces and $\omega\Delta$-spaces .................................. 285
John Ben Butler, Jr., Almost smooth perturbations of self-adjoint operators ...... 297
Michael James Cambern, Isomorphisms of $C_0(Y)$ onto $C(X)$ ..................... 307
David Edwin Cook, A conditionally compact point set with noncompact closure ... 313
Timothy Edwin Cramer, Countable Boolean algebras as subalgebras and homomorphs ................................................................. 321
John R. Edwards and Stanley G. Wayment, A $v$-integral representation for linear operators on spaces of continuous functions with values in topological vector spaces ................................................................. 327
Mary Rodriguez Embry, Similarities involving normal operators on Hilbert space ................................................................. 331
Lynn Harry Erbe, Oscillation theorems for second order linear differential equations ................................................................. 337
William James Firey, Local behaviour of area functions of convex bodies ......... 345
Joe Wayne Fisher, The primary decomposition theory for modules .................. 359
Gerald Seymour Garfinkel, Generic splitting algebras for Pic .......................... 369
J. D. Hansard, Jr., Function space topologies .............................................. 381
Keith A. Hardie, Quasifibration and adjunction ............................................ 389
G. Hochschild, Coverings of pro-affine algebraic groups ................................ 399
Gerald L. Itzkowitz, On nets of contractive maps in uniform spaces ............. 417
Melven Robert Krom and Myren Laurance Krom, Groups with free nonabelian subgroups ................................................................. 425
James Robert Kuttler, Upper and lower bounds for eigenvalues by finite differences ................................................................. 429
Dany Leviatan, A new approach to representation theory for convolution transforms ................................................................. 441
Richard Beech Mansfield, Perfect subsets of definable sets of real numbers ...... 451
Brenda MacGibbon, A necessary and sufficient condition for the embedding of a Lindelof space in a Hausdorff $\mathfrak{H}\sigma$-space ................................. 459
David G. Mead and B. D. McLemore, Ritt’s question on the Wronskian ......... 467
Edward Yoshio Mikami, Focal points in a control problem ............................ 473
Paul G. Miller, Characterizing the distributions of three independent $n$-dimensional random variables, $X_1, X_2, X_3$, having analytic characteristic functions by the joint distribution of $(X_1 + X_3, X_2 + X_3)$ .................................................. 487
P. Rosenthal, On the Bergman integral operator for an elliptic partial differential equation with a singular coefficient ........................................ 493
Douglas B. Smith, On the number of finitely generated $O$-groups .................. 499
J. W. Spellmann, Concerning the domains of generators of linear semigroups .... 503
Arne Stray, An approximation theorem for subalgebras of $H_\infty$ .................. 511
Arnold Lewis Villone, Self-adjoint differential operators .............................. 517