

# Pacific Journal of Mathematics

**A  $v$ -INTEGRAL REPRESENTATION FOR LINEAR OPERATORS  
ON SPACES OF CONTINUOUS FUNCTIONS WITH VALUES IN  
TOPOLOGICAL VECTOR SPACES**

JOHN R. EDWARDS AND STANLEY G. WAYMENT

A  $v$ -INTEGRAL REPRESENTATION FOR LINEAR OPERATORS ON SPACES OF CONTINUOUS FUNCTIONS WITH VALUES IN TOPOLOGICAL VECTOR SPACES

J. R. EDWARDS AND S. G. WAYMENT

Suppose  $X$  and  $Y$  are topological vector spaces. This paper gives an analytic representation of continuous linear operators from  $C$  into  $Y$ , where  $C$  denotes the space of continuous functions from the interval  $[0, 1]$  into  $X$  with the topology of uniform convergence. In order to obtain an integral representation theorem analogous to the ones given by R. K. Goodrich for the locally convex setting in *Trans. Amer. Math. Soc.* 131 (1968), 246–258, certain strong hypotheses on  $C$  must be assumed because of the need to be able to extend the operators to a subset of the double dual of  $C$ . However, by using the notion of  $v$ -integral, it is possible to avoid this problem and give a representation theorem without additional hypothesis.

Let  $\mathcal{I}$  be the collection of intervals in  $(0, 1]$  of the form  $(a, b]$  and let  $L[X, Y]$  denote the space of linear operators from  $X$  into  $Y$ . Then the set function  $K$  from  $\mathcal{I}$  into  $L[X, Y]$  is said to be convex with respect to length if  $K(I) = \sum_{i=1}^n [l(I_i)/l(I)]K(I_i)$  whenever  $I = \bigcup_{i=1}^n I_i$ , and where  $l(I)$  denotes the length of  $I$ . If  $K$  is convex with respect to length, then  $K$  is said to be  $v$ -integrable with respect to  $f$  if  $\lim_{|\sigma| \rightarrow 0} \sum K(I_i)(\Delta_i f) = v \int K df$  exists in  $\bar{Y}$ , the completion of  $Y$  (by  $\Delta_i f$  we mean  $f(t_{i+1}) - f(t_i)$  where  $\{t_i\}$  is the partition  $\sigma$  of  $[0, 1]$ ).

If  $I \in \mathcal{I}$ , with endpoints  $a$  and  $b$ , then the function  $\Psi_I$  defined by  $\Psi_I(t) = 0$  for  $t \leq a$ ,  $\Psi_I(t) = (t - a)/(b - a)$  for  $a \leq t \leq b$ , and  $\Psi_I(t) = 1$  for  $t \geq b$ , is called the fundamental function associated with  $I$ . A set function  $K$  whose domain is  $\mathcal{I}$  and whose range is in  $L[X, Y]$  is said to be quasi-Gowurin if given a neighborhood  $V$  of  $\theta_Y$ , there is a neighborhood  $U$  of  $\theta_C$  such that  $\sum \Psi_{I_i} \cdot x_i \in U$  implies  $\sum [K(I_i)](x_i) \in V$ .

Finally, if  $f \in C$  and  $\sigma$  is a partition of  $[0, 1]$ , then  $pf_\sigma$  denotes the polygonal function determined by  $\sigma$  and  $f$ .

2. The representation theorem. Let  $C_\theta$  denote the subspace of  $C$  such that  $f(0) = \theta_X$ .

**THEOREM 2.1.** *Suppose  $K$  is a set function on  $\mathcal{I}$  with values*

in  $L[X, Y]$  which is convex with respect to length and which is quasi-Gowurin. Then  $T(f) = v \int Kdf$  is a continuous linear operator from  $C_\theta$  into  $\bar{Y}$ .

*Proof.* First we show that  $v \int Kdf$  exists in  $\bar{Y}$  for each  $f \in C_\theta$ . Suppose  $V$  is a neighborhood of  $\theta_Y$ . Since  $K$  is quasi-Gowurin, there is a neighborhood  $U$  of  $\theta_C$  such that  $\sum \Psi_{I_i} \cdot x_i \in U$  implies  $\sum [K(I_i)](x_i)$  is in  $V$ . Since  $pf_\sigma$  converges to  $f$  in the topology of uniform convergence, there is a  $\delta$  such that  $|\sigma_1|, |\sigma_2| < \delta$  implies  $pf_{\sigma_1} - pf_{\sigma_2} \in U$ . Let  $\sigma_1, \sigma_2$  denote the common refinement of  $\sigma_1$  and  $\sigma_2$ . It follows from 7.2 in [1] that

$$(*) \quad \sum_{\sigma_1} [K(I_i)](\Delta_i f) - \sum_{\sigma_2} [K(I_j)](\Delta_j f) = \sum_{\sigma_1 * \sigma_2} [K(I_k)](\Delta_k (pf_{\sigma_1} - pf_{\sigma_2})) .$$

Since  $\sum \Psi_{I_k} \cdot (\Delta_k (pf_{\sigma_1} - pf_{\sigma_2})) = pf_{\sigma_1} - pf_{\sigma_2} \in U$ , then it follows that  $(*)$  is in  $V$  from which we conclude that  $\{\sum_\sigma [K(I_i)](\Delta_i f)\}_\sigma$  is Cauchy. Hence,  $v \int Kdf$  exists in  $\bar{Y}$ . Suppose  $f_\alpha \rightarrow f$  in  $C_\theta$ . Suppose  $V$  is a neighborhood of  $\theta_Y$ . Then there is a neighborhood  $V'$  of  $\theta_Y$  such that  $V' + V' + V' \subset V$ . Since  $K$  is quasi-Gowurin, there is a neighborhood  $U$  of  $\theta_C$  such that  $\sum \Psi_{I_i} \cdot x_i \in U$  then  $\sum [K(I_i)](x_i) \in V$ . There exists a neighborhood  $U'$  of  $\theta_C$  such that  $U' + U' + U' \subset U$ . Since  $f_\alpha$  converges to  $f$ , then, there is a  $\beta$  such that  $\alpha > \beta$  implies  $f_\alpha - f \in U'$ . Suppose  $\alpha > \beta$ . Then there is a  $\delta$  such that  $|\sigma| < \delta$  implies each of  $p(f_\alpha)_\sigma - f_\alpha \in U'$ ,  $f - pf_\sigma \in U'$ ,  $v \int Kdf - \sum_\sigma [K(I_i)](\Delta_i f) \in V'$ , and  $\sum_\sigma [K(I_i)](\Delta_i f_\alpha) - v \int Kdf \in V'$ . Then,

$$\begin{aligned} v \int Kdf - v \int Kdf_\alpha &= v \int Kdf - \sum_\sigma [K(I_i)](\Delta_i f) \\ &\quad + \sum_\sigma [K(I_i)](\Delta_i f) - \sum_\sigma [K(I_i)](\Delta_i f_\alpha) \\ &\quad + \sum_\sigma [K(I_i)](\Delta_i f_\alpha) - v \int Kdf_\alpha \\ &\in \sum_\sigma [K(I_i)](\Delta_i (f - f_\alpha)) + V' + V' . \end{aligned}$$

However,

$$\sum_\sigma \Psi_{I_i} \cdot (\Delta_i (f - f_\alpha)) = pf_\sigma - p(f_\alpha)_\sigma = (pf_\sigma - f) + (f - f_\alpha) + (f_\alpha - p(f_\alpha)_\sigma)$$

which is in  $U' + U' + U' \subset U$ . Hence  $\sum_\sigma [K(I_i)](\Delta_i (f - f_\alpha)) \in V'$ , from which it follows that  $v \int Kdf - v \int Kdf_\alpha \in V' + V' + V' \subset V$ . Therefore,  $v \int Kdf_\alpha$  converges to  $v \int Kdf$ , and hence  $T$  is continuous.

**THEOREM 2.2.** *Suppose  $T$  is a continuous linear operator from  $C_\theta$  into  $Y$ . Then there is a set function  $\mathcal{F}$  with values in  $L[X, Y]$*

which is convex with respect to length and quasi-Gowurin such that  $T(f) = v \int Kdf$  for each  $f \in C_\theta$ .

*Proof.* Define  $K$  from  $\mathcal{S}$  into  $L[X, Y]$  by  $[K(I)](x) = T(\Psi_I \cdot x)$ ,  $x \in X$ . Then  $K$  is convex with respect to length because  $T$  is linear and because of the manner in which fundamental functions combine. Suppose  $V$  is a neighborhood of  $\theta_Y$ . Since  $T$  is continuous, there is a neighborhood  $U$  of  $\theta_C$  such that  $T(U) \subset V$ . Therefore  $\sum \Psi_{I_i} \cdot x_i \in U$  implies  $\sum [K(I_i)](x_i) = T(\sum \Psi_{I_i} \cdot x_i) \in V$ , which implies  $K$  is quasi-Gowurin. Suppose  $f \in C_\theta$ . Since  $pf_\sigma$  converges to  $f$  in  $C_\theta$ , then

$$\begin{aligned} T(f) &= \lim_{|\sigma|} T(pf_\sigma) = \lim_{|\sigma|} T(\sum_\sigma \Psi_{I_i}(\Delta_i f)) \\ &= \lim_{|\sigma|} \sum_\sigma [K(I_i)](\Delta_i f) = v \int Kdf. \end{aligned}$$

The last equality follows from 2.1. The theorem is established.

**COROLLARY 2.3.** *Suppose  $Y$  is complete. Then,  $T$  is a continuous linear operator from  $C$  into  $Y$  if and only if there is an element  $\mu \in L[X, Y]$  and a set function on  $\mathcal{S}$  with values in  $L[x, y]$  which is convex with respect to length and quasi-Gowurin such that  $T(f) = \mu(f(0)) + v \int Kdf$ .*

**3. The locally convex setting.** In this section, for the purpose of comparison, we consider the special case when  $H = [0, 1]$  of the setting in which Goodrich gives his representation theorem [3], that is, we assume additionally that  $X$  and  $Y$  are locally convex spaces. The condition of quasi-Gowurin becomes: given a neighborhood  $V$  of  $\theta_Y$  there is a neighborhood  $U$  of  $\theta_X$  such that if  $\{\sum_{i=1}^j x_i; j=1, \dots, n\} \subset U$ , then  $\sum_{i=1}^n [K(I_i)](x_i) \in V$ . This condition stated in terms of the semi-norms becomes, using Swongs notation [4], there is a pairing  $(p, q)$  and a constant  $W_{p-q}$  for each pair of semi-norms  $p$  and  $q$  in the pairing such that  $q(\sum_\sigma [K(I_i)](x_i)) \leq W_{p-q} \max_j p(\sum_{i=1}^j x_i)$  for each partition of  $(0, 1]$  and each corresponding collection  $\{x_i\}$  in  $X$ . This property is the analogy of Goodrich's bounded  $(p, q)$  variation. A set function which satisfies this property is said to be of bounded  $(p, q)$  convex variation. In order to be able to state an optimal result in the following theorem we shall assume that  $Y$  is quasi-complete, i.e., each closed and bounded set in  $Y$  is complete.

**THEOREM 3.1.** *Suppose  $T$  is a linear operator from  $C$  into  $Y$  (which is quasi-complete). Then  $T$  is continuous if and only if there exists a  $\mu \in L[X, Y]$  and a set function on  $\mathcal{S}$  with values in  $L[X, Y]$  which is convex with respect to length and of bounded*

$(p, q)$  convex variation such that  $T(f) = \mu(f(0)) + v \int Kdf$ . Furthermore if  $T'$  denotes the restriction of  $T$  to  $C_\theta$ , then  $|T'|_{p-q} = W_{p-q}$ .

The theorem follows from 2.3.

REMARK 3.2. It is immediate that the  $K$  function of 2.2, 2.3, and 3.1 is unique.

#### REFERENCES

1. J. R. Edwards and S. G. Wayment, *Representations for transformations continuous in the BV norm*, Trans. Amer. Math. Soc. (to appear)
2. R. K. Goodrich, *A Riesz representation theorem* (submitted).
3. ———, *A Riesz representation theorem in the setting of locally convex spaces*, Trans. Amer. Math. Soc. **131** (1968), 246-258.
4. K. Swong, *A representation theory of continuous linear maps*, Math. Annalen **155** (1964), 270-291.

Received January 5, 1970.

UTAH STATE UNIVERSITY

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The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

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