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S. Naimpally [3] introduced the graph topology, Γ, for function spaces. H. Poppe [5] showed that if the graph topology is finer than the topology of uniform convergence, τ_u, or finer than the finest of the σ-topologies of Arens and Dugundji, τ, and if the range space is the real line, R, then the domain is countably compact.

We assume our range space is R and that our domain space X is T_1. In most of this paper we deal with topologies on $C(X)$ the set of continuous real-valued functions on X. We show that $\Gamma = \tau = \tau_u$ on $C(X)$ if and only if X is countably compact. Further, we find that when X is locally connected, $\tau_u \subset \tau$ on $C(X)$ if and only if X has finitely many components.

In order to determine conditions under which $\tau \subset \tau_u$, we introduce a map extension property between complete regularity and normality and show that for domain spaces X having this property, $\tau \subset \tau_u$ on $C(X)$ if and only if X is countably compact. We indicate further applications of this map extension property and compare it to weak normality.

We let Y^X denote the set of all functions from X to Y. For $f \in Y^X$, let $G(f) = \{(x, f(x)) : x \in X\}$, and for $U \subset X \times Y$, let $F_U = \{f \in Y^X : G(f) \subset U\}$. If $K \subset X$ and $U \subset Y$, define $(K, U) = \{f \in Y^X : f(K) \subset U\}$. For $\varepsilon > 0$, let $V_\varepsilon(f) = \{g \in R^X : |f(x) - g(x)| < \varepsilon$ for all $x \in X\}$. Also, define $N_\varepsilon(x) = \{y \in R : |y - x| < \varepsilon$, and for any set K, let cK be the complement of K.

The graph topology, defined in [3], has a basis consisting of the sets of the form F_U where U is open in $X \times Y$. The finest of the σ-topologies, defined in [1], has a subbasis consisting of the collection of all sets of the form (K, U) where $K \subset X$ is closed and $U \subset Y$ is open. The topology of uniform convergence has a basis consisting of all sets of the form $V_\varepsilon(f)$ where $f \in R^X$ and $\varepsilon > 0$.

1. Two lemmas.

The first of our lemmas is a characterization of τ which we find convenient to use throughout this paper. This result provides us with a basis for τ. Because of the nature of these basic elements, the relation between τ and Γ is immediately made clear, and we are able to think of τ, intuitively, as a special kind of graph topology rather than as a set-open topology.

H. Poppe [5] showed that τ has a subbasis consisting of sets of the form $[K \times L] = \{f : G(f) \cap K \times L = \emptyset$, $K \subset X$ closed, $L \subset Y$ closed$\}$. Thus a basic open set of functions in τ, $\bigcap_1^n [K_i \times L_i]$, is completely determined by the set of points $\bigcup_1^n K_i \times L_i$.

LEMMA 1. *The topology τ has a basis consisting of sets of the form F_U where $U = \bigcup_{i \in \mathscr{A}} A_i \times B_i$ and $A_i \subset X$ and $B_i \subset Y$ are open for each $i \in \mathscr{A}$ and $\mathscr{A}$ is finite.*

Proof. Choose $[K \times L] \in \tau$. Clearly $[K \times L] = F_V$ where $V = cK \times Y \cup X \times cL$, so that $[K \times L]$ is open in the topology generated by sets of the form F_U described above. Now consider F_U where $U = \bigcup_{i \in \mathscr{A}} A_i \times B_i$, $A_i \subset X$ and $B_i \subset Y$ are open for each $i \in \mathscr{A}$ and $\mathscr{A}$ is finite. In order to see that $cU = c(\bigcup_{i \in \mathscr{A}} A_i \times B_i) = \bigcup_{\mathscr{B} \subset \mathscr{A}} (c\bigcup_{i \notin \mathscr{B}} A_i \times c\bigcup_{i \in \mathscr{B}} B_i)$, choose $(x, y) \in cU$. Then for each i, either $x \notin A_i$ or $y \notin B_i$. If $\mathscr{B} < \mathscr{A}$ is chosen so that $i \in \mathscr{B}$ if and only if $x \in A_i$, then $(x, y) \in (c\bigcup_{i \notin \mathscr{B}} A_i \times c\bigcup_{i \in \mathscr{B}} B_i)$. If $(x, y) \in \bigcup_{\mathscr{B} \subset \mathscr{A}} (c\bigcup_{i \notin \mathscr{B}} A_i \times c\bigcup_{i \in \mathscr{B}} B_i)$, then there exists a $\mathscr{B} \subset \mathscr{A}$ such that $(x, y) \in (c\bigcup_{i \notin \mathscr{B}} A_i \times c\bigcup_{i \in \mathscr{B}} B_i)$ implying that $x \notin A_i$ for $i \notin \mathscr{B}$ while $y \notin B_i$ for $i \in \mathscr{B}$. This establishes set equality and from this we see that

$$F_U = [\bigcup_{\mathscr{B} \subset \mathscr{A}} (c\bigcup_{i \notin \mathscr{B}} A_i \times c\bigcup_{i \in \mathscr{B}} B_i)] = \bigcap_{\mathscr{B} \subset \mathscr{A}} [(c\bigcup_{i \notin \mathscr{B}} A_i \times c\bigcup_{i \in \mathscr{B}} B_i)]$$

showing that F_U is open in τ.

LEMMA 2. *If X is countably compact, then for $f \in C(X)$ and $G(f) \subset U$ where U is open in $X \times R$, there exists an $\varepsilon > 0$ such that $\{x\} \times N_\varepsilon(f(x)) \subset U$ for any $x \in X$.*

Proof. If the condition fails, then for $\varepsilon = 1, 1/2, 1/3, \cdots$ there exist points $x_1, x_2, x_3, \cdots$, respectively, such that $\{x_n\} \times N_{1/n}(f(x_n)) \not\subset U$. Suppose $\{x_n\}$ clusters to the point x. Then $(x, f(x)) \in U$ and there exists an open set V containing x and an $\varepsilon > 0$ such that $V \times N_\varepsilon(f(x)) \subset U$. We can suppose $f(V) \subset N_{\varepsilon/2}(f(x))$ using the continuity of f. Then choosing n_0 such that $1/n_0 < \varepsilon/2$ and $m > n_0$ such that $x_m \in V$, we see that $\{x_m\} \times N_{1/m}(f(x_m)) \subset V \times N_\varepsilon(f(x)) \subset U$ which is a contradiction.

We remark that the converse of this lemma is also true. This follows from the fact that if X is not countably compact, then there exists a sequence, $\{x_n\}$, with no cluster points. Let U be open and $G(f) \subset U$. Then let $V = U - [\bigcup_1^\infty \{x_n\} \times cN_{1/n}(f(x_n))]$. The set V is open, contains the gragh of f, and no $\varepsilon > 0$ exists satisfying the condition.

This lemma together with its converse states that the complement of an open set containing the graph of an element of $C(X)$ can get "close" to the graph of f if and only if X is countably compact.

2. Comparisons of Γ with τ_u and τ.

THEOREM 3. *A space X is countably compact if and only if $\Gamma = \tau_u$ on $C(X)$.*

Proof. Poppe [5] showed that when $\Gamma \subset \tau_u$ on $C(X)$, then X is countably compact. The converse of this result follows immediately from Lemma 2.

A proof analogous to Poppe's can be made with R replaced by any first countable T_1 space containing a nonisolated point. Also, the proof of Lemma 2 above remains valid with R replaced by any metric space. Hence, Theorem 3 is true for the case when R is replaced by any metric space containing a nonisolated point.

We note that $\tau_u \subset \Gamma$ on $C(X)$ always holds since $\{(x, y): |y\text{-}f(x)| < \varepsilon\}$ is an open set for any $\varepsilon > 0$ and for $f \in C(X)$. In fact, the function f is continuous if and only if the set is open for every $\varepsilon > 0$.

THEOREM 4. *A space X is countably compact if and only if $\Gamma = \tau$ on $C(X)$.*

Proof. Poppe [5] showed that when $\Gamma \subset \tau$ on $C(X)$, then X is countably compact. It is clear from Lemma 1 that $\tau \subset \Gamma$. Therefore, we have only to show that when X is countably compact, then $\Gamma \subset \tau$. Suppose X is countably compact and choose $F_U \in \Gamma$ with $f \in F_U$. By Lemma 2, there exists and $\varepsilon > 0$ such that $\{x\} \times N_\varepsilon(f(x)) \subset U$ for every $x \in X$. Let $\{W_i: 1 = 1, 2, \cdots, n\}$ be an open cover of $f(X)$ such that the diameter of each W_i *is less* than ε, and define $V = \bigcup_1^n f^{-1}(W_i) \times W_i$. Clearly, $F_V \in \tau$ and it is easy to show that $F_V \subset F_U$ by showing that $V \subset U$.

We remark that Theorem 4 can be established for the case when R is replaced by any Lindelöf metric space containing a nonisolated point. We need the metric space property to use Lemma 2, the Lindelöf property to insure that $f(X)$ is compact when X is countably compact, and the nonisolated point (together with first countable and T_1) for Poppe's proof.

When Poppe showed that $\Gamma \subset \tau$ on $C(X)$ implies that X is countably compact, he remarked that if X is completely regular and if $\Gamma \subset \tau$ on $C(X)$, then X must be compact. Theorem 4 shows that his statement is incorrect. Poppe made a similar statement when comparing Γ with τ_u and Theorem 3 indicates that it also is incorrect.

Naimpally [3] showed that when X is a compact T_2 space, the graph topology and the compact-open topology coincide. This fact together with Theorem 4 shows that when X is a compact T_2 space, the compact-open topology is the same as τ for $C(X)$ and so has basic open sets of the form F_U where U is expressible as a finite union of Cartesian products of open sets.

3. Comparisons of τ with τ_u. S. Naimpally and C. Pareek ([4], Ex. 5.10) assert that $\tau_u \not\subset \tau$ on $C(X)$ when $X = R$. The following result shows that their assertion is wrong.

LEMMA 5. *When X is connected, $\tau_u \subset \tau$ on $C(X)$.*

Proof. Assume X is connected and choose $V_\varepsilon(f) \in \tau_u$. Fix $x \in X$ and define $p_k = k(\varepsilon/2) + f(x)$ for $k \in I$ where I is the set of integers. Then define $B_1 = N_{\varepsilon/2}(p_0)$, $B_2 = \bigcup_{i \in I} N_{\varepsilon/2}(p_{3i+1})$, $B_3 = \bigcup_{i \in I} N_{\varepsilon/2}(p_{3i+2})$, $B_4 = \bigcup_{i \neq 0} N_{\varepsilon/2}(p_{3i})$, and $A_i = f^{-1}(B_i)$ for $i = 1, 2, 3, 4$. We assert that $f \in F_W \subset V_\varepsilon(f)$ where $W = \bigcup_1^4 A_i \times B_i$.

In order to establish this assertion, notice that $U = \bigcup_{i \in I} f^{-1}(N_{\varepsilon/2}(p_i)) \times N_{\varepsilon/2}(p_i) \subset W$ and certainly $G(f) \subset U$, so that $f \in F_w$. If $g \in F_w$, then $G(g)$ is connected, and $G(g) \cap f^{-1}(N_{\varepsilon/2}(p_0)) \times N_{\varepsilon/2}(p_0) \neq \emptyset$ implying that $G(g) \cap U \neq \emptyset$. We will show that $W = U \cup (W - U)$ is a separation of W, so that $G(g) \subset U$. This will establish the fact that $F_W \subset F_U$ and surely $F_U \subset V_\varepsilon(f)$, which will complete the proof.

Now we show that $W = U \cup (W - U)$ is a separation. Call a set of the form $f^{-1}(N_{\varepsilon/2}(p_i)) \times N_{\varepsilon/2}(p_j)$ a diagonal set if $i = j$. We prove that the i^{th} diagonal set intersects only the $(i-1)^{st}$ and $(i+1)^{st}$ diagonal sets, so that $W - U$ is exactly the union of nondiagonal sets, and hence open. Suppose

$$\{f^{-1}(N_{\varepsilon/2}(p_i)) \times N_{\varepsilon/2}(p_i)\} \cap \{f^{-1}(N_{\varepsilon/2}(p_j)) \times N_{\varepsilon/2}(p_k)\} \neq \emptyset \ .$$

Clearly, both j and k must assume values from $\{i-1, i, i+1\}$. Assume that $k = i + 1$. By the definition of W, $|j - k| = 3n$, so that $j = i + 1$ also. The argument for $j = i$ or $i - 1$ is the same.

THEOREM 6. *Let X be locally connected. Then X has finitely many components if and only if $\tau_u \subset \tau$ for $C(X)$.*

Proof. Suppose that X is locally connected and has finitely many components. Then clearly the process used in Lemma 5 can be repeated a finite number of times to yield the desired result.

Now suppose that $X = \bigcup_0^\infty X_i$ where X_i, for $i > 0$, is a component of X. Define $f(X_i) = i$ for every i. It is true that $V_{1/2}(f)$ is not open in τ. In order to see this, choose $F_U \in \tau$ with $f \in F_U$ and $U =$

$\bigcup_{i \in \mathscr{A}} A_i \times B_i$. Then observe that $\{A_i : j \in B_i\}$ must cover X_j and that for some $j \neq k$, $\{A_i : k \in B_i\} = \{A_i : j \in B_i\}$. If $g(X_i) = i$ for $i \neq j$ and $g(X_j) = k$, then $g \in F_U$, but $g \notin V_{1/2}(f)$.

Let $C^*(X)$ represent the collection of elements of $C(X)$ which are bounded. Our next lemma will enable us to show that when X is locally connected, then $C^*(X)$ is closed in $C(X)$ with topology τ if and only if $\tau_u \subset \tau$.

LEMMA 7. *Let X be locally connected. Then $C^*(X)$ is closed in $C(X)$ with topology τ if and only if X has finitely many components.*

Proof. Let X be locally connected and suppose that X has finitely many components, so that $\tau_u \subset \tau$. If $C^*(X) = C(X)$, then $C^*(X)$ is closed. If $C^*(X) \neq C(X)$, choose $f \in C(X) - C^*(X)$. Then $V_{1/2}(f) \in \tau$ and $V_{1/2}(f) \subset C(X) - C^*(X)$.

Suppose $X = \bigcup_0^\infty X_i$ where X_i, for $i > 0$, is a component and $X_0 = X - \bigcup_1^\infty X_i$. Define $f(X_i) = i$. Then f is unbounded, and we show that every $F_U \in \tau$ containing f contains a bounded map. Let $F_U \in \tau$ where $U = \bigcup_{i \in \mathscr{A}} A_i \times B_i$ and $f \in F_U$. The set $\{A_i : j \in B_i\}$ covers X_j and since there exists only a finite number of sets of the form $\{A_i : j \in B_i\}$, there exists an M such that for any $n \geqq M$, $\{A_i : n \in B_i\} = \{A_i : j \in B_i\}$ for some $j < M$. Define g such that $g(X_i) = i$ for $i \leqq M$, and for $n > M$, $g(X_n) = j$ where $j < M$ and found as indicated above. Then $g \in F_U$ while $g \in C^*(X)$.

Combining Lemma 7 with Theorem 6 we have the following result.

THEOREM 8. *Let X be locally connected. Then $C^*(X)$ is closed in $C(X)$ with topology τ if and only if $\tau_u \subset \tau$.*

We now turn to the question of necessary and sufficient conditions in order that $\tau \subset \tau_u$. Theorems 3 and 4 indicate that when X is countably compact, then $\tau \subset \tau_u$ on $C(X)$. But the converse of this statement is not true in general, since $C(X)$ might consist of constant maps only and in that event $\tau = \tau_u$ on $C(X)$ regardless of the properties of X.

Before giving an answer to this question, we need a definition.

DEFINITION 9. A completely regular space X is well-separated if for any sequence, $\{x_n\}$, in X having no cluster points, there exists a neighborhood-finite sequence of open sets, $\{U_n\}$, with $x_n \in U_n$ for each n.

S. Naimpally and C. Pareek [4] show in an example that $\tau \not\subset \tau_u$ on $C(X)$ when $X = R$. We wish to show that under certain conditions, when X is not countably compact, then $\tau \not\subset \tau_u$ on $C(X)$. When X is not countably compact, we have the existence of a sequence of distinct points with no cluster points, say $\{x_n\}$. We wish to define a continuous function f on this sequence such that $f(x_n) = 1 - 1/n$ and extend it to all of X in such a way that $f(X) \subset [0, 1)$. For this task, the condition of well-separatedness will be shown to be, in a sense, natural.

THEOREM 10. *If X is well-separated, then $\tau \subset \tau_u$ for $C(X)$ if and only if X is countably compact.*

Proof. Suppose that X is not countably compact and that $\{x_n\}$ is a sequence in X having no cluster points. We can take $\{x_n\}$ to consist of distinct points. There exists a neighborhood-finite sequence, $\{U_n\}$, of open sets with $x_n \in U_n$ for each n, and further we can require that $\bar{U}_n \cap \bar{U}_m = \varnothing$ for $n \neq m$. Choose $F_V \in \tau$ with $V = X \times N_1(0)$. Define f_n on $\bar{U}_n$ such that $f_n(x_n) = 1 - 1/n$ and $f_n(\bar{U}_n - U_n) = 0$ for each $n > 0$, and define f_0 on $c \bigcup_1^\infty U_n$ to be the zero map. Then there exists a unique continuous function f which is an extension of each f_n. Clearly $f \in F_V$, but no $V_\varepsilon(f)$ is contained in F_V.

4. Properties of well-separatedness. In the proof of Theorem 10 we wished to extend a continuous function on a sequence having no cluster points to all of the space. We imposed the condition of well-separatedness. This condition is natural because of the following result. (A subspace $Y \subset X$ is C-embedded in X if each element of $C(Y)$ is the restriction of an element of $C(X)$.)

THEOREM 11. *A completely regular space X is well-separated if and only if every sequence in X having no cluster point is C-embedded in X.*

Proof. Only half of the assertion needs comment. Let $\{x_n\}$ be a sequence of distinct points in X having no cluster points, and define $f(x_n) = n$ for each n. Extend f to a continuous function on all of X. Then $\{f^{-1}(N_{1/3}(n))\}$ is the desired neighborhood-finite collection.

Dugundji [2] defines a completely regular space to be weakly normal if disjoint closed sets, one of which is countable, can be separated by disjoint open sets. Then he shows that in weakly normal spaces, countable compactness and pseudocompactness are equivalent. Even though the class of well-separated spaces is larger than the class of weakly normal spaces, countable compactness and pseudocompactness are equivalent there also.

COROLLARY 12. *If X is well-separated, then countable compactness and pseudocompactness are equivalent.*

Proof. If X is not countably compact, then there exists a sequence of distinct points, $\{x_n\}$, with no cluster points and if $f(x_n) = n$ for each n, then f has a continuous extension to all of X by Theorem 11.

THEOREM 13. *If X is weakly normal, then X is well-separated.*

Proof. Let $\{x_n\}$ be a sequence in X having no cluster points, and choose a sequence of open sets, $\{U_n\}$, *with* $x_n \in U_n$ for each n and such that $U_n \cap U_m = \emptyset$ if $x_n \neq x_m$. Then $\cup \{x_n\}$ and $c \cup U_n$ are disjoint closed sets and there exist disjoint open sets V and W containing $\cup \{x_n\}$ and $c \cup U_n$ respectively. Define $V_n = V \cap U_n$ for each n. Then $\{V_n\}$ is the desired neighborhood-finite collection. For if $x \in U_n$ for some n, then $U_n \cap V_m = \emptyset$ for $x_m \neq x_n$. Further, if $x \notin U_n$ for every n, then $x \in c \cup U_n$ implying that $x \in W$ and $W \cap V_n = \emptyset$ for every n.

Theorem 13 shows that well-separatedness is strictly weaker than normality, since weak normality is strictly weaker than normality. In order to see that well-separatedness is strictly stronger than complete regularity, let $[0, \Omega]$ be the space of ordinals less than or equal to the first uncountable ordinal and $[0, \omega]$ be the space of ordinals less than or equal to the first countable ordinal. Then $[0, \Omega] \times [0, \omega] - \{(\Omega, \omega)\}$ is a completely regular space that is not well-separated. To see this, note that $\{(\Omega, n)\}$ has no cluster points. Choose $\{U_n\}$ with $(\Omega, n) \in U_n$ for each n. Then any sequence $\{z_n\}$ such that $z_n \in U_n$ for each n and the first coordinate of z_n differs from Ω can be seen to cluster. This shows that $\{U_n\}$ is not neighborhood-finite.

We list several more applications of the well-separated property.

THEOREM 14. *Let X be well-separated.*

(a) *If X is not countably compact, then $C(X)$ with topology τ_u is not separable.*

(b) T*he set of units in $C(X)$ with topology τ_u is open if and only if X is countably compact.*

(c) *Scalar multiplication (or multiplication) is continuous in $C(X)$ with topology τ if and only if X is countably compact.*

(d) *The space X is countably compact if and only if $f(X)$ is closed for any $f \in C(X)$.*

Proof. (a) If X is not countably compact, there exists a sequence, $\{x_n\}$, of distinct points with no cluster points. Let $\{f_n\}$ be a countable collection of functions. Define $g(x_n) = f_n(x_n) + 1$ for each n and then extend g to a continuous function on all of X by Theorem 11. Clearly, $V_{1/2}(g)$ contains no f_n.

(b) If X is countably compact and $f \in C(X)$ such that $f > 0$, then by Lemma 2, there exists an $\varepsilon > 0$ such that $V_\varepsilon(f)$ contains only positive functions. If X is not countably compact, choose a sequence, $\{x_n\}$ of distinct points having no cluster points and define $f(x_n) = 1/n$ for $n > 0$, and then extend f to a continuous function on X in such a way that $f > 0$. Then f is a unit while for any $\varepsilon > 0$, $V_\varepsilon(f)$ contains nonunits.

(c) If X is countably compact, $\tau = \tau_u$ and multiplication, as well as scalar multiplication, is continuous in $C(X)$ with topology τ_u. If X is not countably compact, choose a sequence of distinct points $\{x_n\}$ and define $f(x_n) = n$. Extend f to a continuous function on all of X. Let g be the zero map. In order to see that scalar multiplication is not continuous, note that $0 \cdot f = g$, while $N_{1/n}(0) \cdot f \not\subset F_U$ where $U = X \times N_\varepsilon(0)$. (We write $N_{1/n}(0) \cdot f$ for $\{a \cdot f \mid a \in N_{1/n}(0)\}$). The argument for multiplication is almost the same.

(d) If X is not countably compact, then there exists a sequence of distinct points, $\{x_n\}$, which do not cluster. If f is defined so that $f(x_n) = 1/n$ and then extended to a continuous function on X in such a way that $f(X) \subset (0, 1]$, then $f(X)$ is not closed. If X is countably compact, then $f(X)$ is compact and thus closed.

References

1. R. Arens and J. Dugundji, *Topologies for function spaces*, Pacific J. Math. **1** (1951), 5-31.
2. J. Dugundji, *Topology*, Allyn and Bacon, 1966.
3. S. Naimpally, *Graph topology for function spaces*, Trans. Amer. Math. So. **123** (1966), 267-72.
4. S. Naimpally and C. Pareek, *Graph topologies for functions spaces* II, (to appear in Prace Mat).
5. H. Poppe, *Über Graphentopologien für Abbildungsräume*, I, Bull. Acad. Pol. Sci. **15** (1967), 71-80.

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