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RITT'S QUESTION ON THE WRONSKIAN

DAVID G. MEAD AND B. D. MCLEMORE

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Among the questions for investigation at the end of his Colloquium Publication, *Differential Algebra*, J. F. Ritt suggested the study of special differential ideals, in particular those generated by the Wronskians. In this paper we obtain a test for an element to be a member of a certain (algebraic) ideal, and apply this result to the differential ideal generated by the second order Wronskian.

Let $y_i, z_j, i, j \in \{0, 1, 2, \dots\}$ be independent indeterminants over a field F . We work in the ring $R = F[y_i, z_j]$. Let (a, b) , with a and b integers satisfying $0 \leq a < b$, represent the determinant

$$\begin{vmatrix} y_a & z_a \\ y_b & z_b \end{vmatrix}$$

and call $a + b$ the weight of this determinant. If F is of characteristic zero and $y_i(z_i)$ is considered to be the i^{th} derivative of $y(z)$, then $W = (0, 1)$ is the Wronskian of y and z . Using the Wronskian as a model, we consider ideals

$$I_t = (W_0, W_1, \dots, W_t),$$

where W_i is any fixed linear combination with nonzero coefficients in F , of all determinants of weight $i + 1$. For $P \in R$ we obtain a constructive procedure to determine if $P \in I = I_0 \cup I_1 \cup I_2 \cup \dots$. In fact, the procedure can be applied directly to polynomials in expressions $P(a_1, b_1) \dots P(a_n, b_n)$. This work is similar to that of Levi [3] for the differential ideals $[y^n]$ and $[uv]$ as well as [1], [2], [4], [5], and [6]. Our results are a generalization, for $n = 2$, of those in [1] to a general ring.

It is known ([1]) that the exponent of $\{I\}$ with respect to I is infinite. We will see that if $P \in \{I\}$ then $P \cdot Q \in I$ if Q is a power product of sufficient degree in y_i, z_j with small i and j , while if $P \notin I$ then $P \cdot Q \notin I$ for all power products Q if i and j are large. In §2 we obtain a particular basis for R as a vector space over F , a subset of which provides a basis of R modulo I . This leads directly to canonical forms for elements of R and a constructive test for an element of R to be in I . (Although it is known ([7], p. 34) that the Wronskian is zero if and only if y and z are linearly dependent, the Ritt-Randenbush Theorem of Zeros ([7], p. 27) informs us that one cannot distinguish by zeros, elements which are in $\{I\}$ from those in I . Thus

a test for membership in I cannot be stated in terms of solutions.)

1. Ordering. We order m -tuples, $X = (x_1, \dots, x_m)$, with each x_i a rational number, lexicographically, and say that $X' = (x'_1, \dots, x'_m)$ is higher than X if $x_1 < x'_1$ or $x_i = x'_i$ for $i \leq h - 1$ and $x_h < x'_h$.

We consider elements of $R - F$, called δ -terms, which are expressed in the form

$$P = y_{i_1} \cdots y_{i_k} z_{j_1} \cdots z_{j_l} (a_1, b_1) \cdots (a_n, b_n)$$

and let $S = \{i_1, \dots, i_k, j_1, \dots, j_l, a_1, b_1, \dots, a_n, b_n\}$ be the set of subscripts of P , $k + n = \deg_y P$, $l + n = \deg_z P$. Comparing only elements with the same set of subscripts, the same degree in y , and the same degree in z , we partially order R by

$$(n + 1)^{-1}, a_1 + b_1, a_2 + b_2, \dots, a_n + b_n, i_1, \dots, i_k, b_1, \dots, b_n$$

where we assume $a_1 + b_1 \leq a_2 + b_2 \leq \dots \leq a_n + b_n$ and $i_1 \leq i_2 \leq \dots \leq i_k$. (We also assume $a_i < b_i$ for all i .) It is clear that this is indeed a partial ordering and that if $P > P'$ then $PQ > P'Q$ for all $Q \neq 0$.

We say that the δ -term P is *replaceable* if

$$P = \sum c_i Q_i \text{ with } c_i \in F$$

where each Q_i is a δ -term comparable with P and lower than P (in the ordering just described). If for each Q_i the difference with P occurs before b_1 , we say that P is *s-replaceable*.

2. Basis.

DEFINITION. The δ -term P is called a λ -term if

- (1) $n = 0$ or $a_1 \leq a_2 \leq \dots \leq a_n$ and $b_1 \leq b_2 \leq \dots \leq b_n$;
- (2) $i_1 \leq \dots \leq i_k \leq j_1 \leq \dots \leq j_l$;
- (3) $a_n \leq i_1$ and $a_n \leq j_1$.

In this section we show that the set of λ -terms is a basis of R .

LEMMA 1. If P is a δ -term which fails to satisfy (1) of the definition of a λ -term, then P is *s-replaceable*.

Proof. Assume $a_1 < a_2$ and $b_2 < b_1$, and consider the fourth order determinant

$$D = \begin{vmatrix} y_{a_1} & y_{b_1} & y_{a_2} & y_{b_2} \\ z_{a_1} & z_{b_1} & z_{a_2} & z_{b_2} \\ 0 & y_{b_1} & y_{a_2} & y_{b_2} \\ 0 & z_{b_1} & z_{a_2} & z_{b_2} \end{vmatrix}.$$

Subtracting the third row from the first, the fourth from the second and then expanding by minors of the first two rows, we see that $D = 0$. Expanding D (in the original form) by minors of the first two rows and using $D = 0$ we find:

$$(a_1, b_1)(a_2, b_2) = (a_1, a_2)(b_1, b_2) + (a_1, b_2)(a_2, b_1).$$

Now, since $a_1 < a_2 < b_2 < b_1$, it follows that $a_1 + a_2$ and $a_1 + b_2$ are both less than $a_1 + b_1$ and $a_2 + b_2$. Thus each product on the right side of the equation is lower than $(a_1, b_1)(a_2, b_2)$. It follows that P is s -replaceable.

LEMMA 2. *If P is a δ -term which fails to satisfy (2) of the definition of a λ -term, then P is s -replaceable.*

Proof. Assume $i_k > j_1$ and let $a = i_k, b = j_1$. Note that $y_a z_b = -(b, a) + y_b z_a$, and each term on the right is lower than $y_a z_b$. It follows that P is s -replaceable.

LEMMA 3. *If P is a δ -term which fails to satisfy (3) of the definition of a λ -term, then P is s -replaceable.*

Proof. Assume $i_1 < a_n$ and consider the third order determinant

$$D = \begin{vmatrix} y_c & y_a & y_b \\ y_c & y_a & y_b \\ z_c & z_a & z_b \end{vmatrix}$$

where $c = i_1, a = a_n$, and $b = b_n$. Expanding D by minors of the first row and using $D = 0$, we find

$$y_c(a, b) = y_a(c, b) - y_b(c, a).$$

Again, since $c < a$, each term on the right is lower than P and it follows that P is s -replaceable. (The other case $j_1 < a_n$ is treated similarly.)

The three lemmas show that if P is a δ -term which is not a λ -term, then P is replaceable. Since the number of δ -terms with a fixed set of subscripts is finite, this replacement process must terminate. Thus we have proved

THEOREM 1. *The λ -terms span R .*

We now complete the proof that the λ -terms are a basis of R .

THEOREM 2. *The λ -terms are linearly independent over F .*

Proof. Assume the λ -terms are dependent and let

$$(1) \quad \sum c_i P_i = 0$$

where the P_i are λ -terms and $c_i \in F$, with some $c_i \neq 0$. It is clear that we may assume that each P_i has the same set of subscripts, S , and the same degree, d , in y . Let d be minimal; that is, we assume the λ -terms with degree in y less than d are linearly independent. (Clearly, the λ -terms of degree zero in y are independent.) We rewrite (1) in the form

$$(2) \quad \sum c_i P_i = c_0 P_0$$

where for each P_i on the left the number of determinants in P_i is positive, while P_0 is a power product of y 's and z 's. Of all the terms on the left with $c_i \neq 0$, let $b = \max b_i$ where (a_i, b_i) is the determinant of minimum weight in P_i . We note that for all i , $a_i = a = \text{minimum number in } S$.

In (2), let $y_i = y_a$ and $z_i = z_a$ for $i < b$. If, by this substitution, P_i becomes \bar{P}_i , we see that although some \bar{P}_i may be zero, not all of them are. Also each \bar{P}_i which is not zero is a λ -term, and has (a, b) as the determinant of lowest weight. Then, with $\bar{P}_i = (a, b)\bar{Q}_i$ we have

$$(2) \quad (a, b) \sum c_i \bar{Q}_i = c_0 \bar{P}_0$$

and if $T = \sum c_i \bar{Q}_i$,

$$(3) \quad (a, b)T = c_0 \bar{P}_0.$$

But on the left side of (3) is the expression $y_i z_a T$ which cannot appear on the right since $a < b$ and \bar{P}_0 is a λ -term. Thus $T = 0$. But $T = \sum c_i \bar{Q}_i$, some $c_i \bar{Q}_i \neq 0$, and each nonzero \bar{Q}_i is a λ -term of degree $d - 1$ in y . However, d was the minimum degree in y for which λ -terms were dependent. This contradiction completes the proof of Theorem 2, and also concludes the proof that the λ -terms are a basis of R .

3. Canonical forms.

DEFINITION. Let P be a λ -term. P is called a β -term if:

- (1) $a_1 > 0$
- (2) $a_i < a_{i+1}$ for all i
- (3) $b_i < b_{i+1}$ for all i .

LEMMA 4. *If the λ -term P is not a β -term, then P is replaceable, modulo I*

Proof. If $a_1 = 0$, expand $P(a_1, b_1)^{-1}W_{b_1-1} \equiv 0 \pmod{I}$ and solve for P . Similarly, if $a_{k-1} = a_k$, or if $b_k = b_{k+1}$, expand $P(a_k, b_k)^{-1}W_h \equiv 0 \pmod{I}$ where $h = a_k + b_k - 1$ and solve for P . In each case it is easy to see that every λ -term obtained is lower than P , and, since every term which is not a λ -term is s -replaceable, it follows that P is itself replaceable. Again, because there are a finite number of λ -terms with a given set of subscripts, this process must terminate. Thus we have proved half of

THEOREM 3. *Every element in R is expressible as a linear combination, with coefficients in F , of a finite number of distinct terms*

$$(*) \quad PW_a W_b \cdots W_r$$

where P is a β -term or 1. This expression, which may be of degree zero in the W 's, is unique.

Proof. For each term A of the form $(*)$ we will obtain the highest λ -term, B , in the expression for A as a linear combination of λ -terms. The correspondence $A \rightarrow B$ is one-to-one, hence no linear combination of terms A of the form $(*)$ can vanish, since the highest B cannot cancel.

Let A be a fixed term of the form $(*)$. With our standard notation for P , and with $V_i = a_i + b_i$, we define a determinant C_h for every W_h in $(*)$. If $S = 1, n = 0$, or $h + 1 < V_1$, let $C_h = (0, h + 1)$. If $V_k \leq h + 1 < V_{k+1}$, let $C_h = (a_k, h + 1 - a_k)$. Finally, if $V_n \leq h + 1$, let $C_h = (a_n, h + 1 - a_n)$. It is easy to see that $B = PC_a C_b \cdots C_r$ has the properties described above and this completes the proof of the theorem.

COROLLARY 1. *The β -terms form a basis of $R \pmod{I}$.*

COROLLARY 2. *A necessary and sufficient condition for an element of R to be in I is that none of the terms $(*)$ of its canonical form is of degree zero in the W 's.*

COROLLARY 3. *If P is a β -term of degrees d_1 and d_2 in y and z respectively, and of degree n in 2^{nd} order determinants, then the weight of $P \geq n(d_1 + d_2 + 2 - n)$.*

Proof. The β -term of minimal weight and the desired degrees is $y_n^{d_1-n} z_n^{d_2-n} (1, 2)(2, 3) \cdots (n, n + 1)$.

An equivalent statement of Corollary 3 is

COROLLARY 3'. *If P is a λ -term of degree d_1 and d_2 in y and z respectively and of degree n in 2^{nd} order determinants and the weight of $P < n(d_1 + d_2 + 2 - n)$, then $P \in I$.*

COROLLARY 4. *If P is a λ -term of degree n in determinants, and*

(a) *Q is a power product in $y, y_1, \dots, y_{n-1}, z, z_1, \dots, z_{n-1}$, and the degree of Q is large enough, then $P \cdot Q \in I$.*

(b) *Q is a power product in y_i and z_j , with $i, j \geq n$, then $PQ \in I$ if and only if $P \in I$.*

REFERENCES

1. A. P. Hillman, D. W. Forsland, and G. J. Giaccai, *Determinantal ideals with applications to differential algebra*. The University of New Mexico Technical Report, No. 143, August, 1967.
2. A. P. Hillman, D. G. Mead, K. B. O'Keefe, and E. S. O'Keefe, *Ideals generated by products*, Proc. Amer. Math. Soc. **17** (1966), 717-719. MR 33 #5622.
3. H. Levi, *On the structure of differential polynomials and on their theory of ideals*, Trans. Amer. Math. Soc. **51** (1942), 532-568. MR 3, 264.
4. D. G. Mead, *Differential ideals*, Proc. Amer. Math. Soc. **6** (1955), 420-432. MR 17, 123.
5. ———, *A necessary and sufficient condition for membership in $[uv]$* , Proc. Amer. Math. Soc. **17** (1966), 470-473. MR 33 #5623.
6. K. B. O'Keefe, and E. S. O'Keefe, *The differential ideal $[uv]$* , Proc. Amer. Math. Soc. **17** (1966), 750-757. MR 33 #5624.
7. J. F. Ritt, *Differential Algebra*, Amer. Math. Soc. Colloquium Publications, vol. 33, New York, 1950. MR 12,7.

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