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This paper applies a few results on quadratic forms in Hilbert space and the theory of focal points from a paper by Hestenes to a linear control problem with a constraint equation. The abnormality inherent in this problem allows focal intervals to exist. The main results are, after assuming the strengthened Clebsch condition, the following: (1) The signature is equal to the sum of the focal points on the open interval, (2) The focal points are the discontinuous points of rank and abnormality of the conjugate base matrix, and (3) The dimension of a maximal linear space of broken transversal extremal arcs is less than or equal to $n - a$, where a is the abnormality of the problem.

The theory of focal points in the calculus of variations arises in the study of the second variation. It is an accounting of the zeros of the solutions of the Euler-Lagrange equations subject to boundary conditions. The number of zeros is equal to the signature, i.e., the number of negative terms in the associated quadratic form. It is also equal to the number of negative characteristic values of the associated boundary value problem.

The results of this paper were proved earlier by Hazard [2] using calculus of variations methods; the author of this paper approaches a control problem of the same sort by using Hilbert space results from a paper by Hestenes [3]. Earlier work on focal point theory can be found in Birkhoff and Hestenes [1] and Morse [5].

Our problem is the study of the focal points of the quadratic form

$$J(x) = b^*Fb + \int_{t^0}^{t^1} 2\omega(t, x, u)dt ,$$

where

$$2\omega(t, x, u) = x^*P(t)x + x^*Q(t)u + u^*Q^*(t)x + u^*R(t)u ,$$

relative to a one-parameter family $\mathscr{E}(\lambda)$ $(t^0 \leqq \lambda \leqq t^1)$ of subspaces of arcs satisfying the linear control equation

$$\dot{x} = Ax + Bu \qquad (t^0 \leqq t \leqq \lambda) \tag{1.1}$$

a linear constraint equation

$$Mx + Nu = 0 \qquad (t^0 \leqq t \leqq \lambda) \tag{1.2}$$

and the boundary conditions

$$x(t^0) = Cb \tag{1.3}$$

$$x(t) = 0,\ u(t) = 0 \qquad (\lambda \leqq t \leqq t^1)\ . \tag{1.4}$$

The matrices $A, P = P^*$ and Q are square-integrable, B, M, N and $R = R^*$ are essentially bounded and measurable, and C, D and $F = F^*$ are constant matrices. The matrix N in addition is assumed to have the inverse of NN^* existing and essentially bounded. Stated in another way, there exists a positive number h such that at almost all t on $t^0 \leqq t \leqq t^1$,

$$\pi^* N(t) N^*(t) \pi \geqq h \pi^* \pi$$

for every π in E^m (a unitary m-dimensional space). An important assumption the strengthened Clebsch condition will be made later.

2. Preliminary remarks. Let $\mathscr{A}$ be a Hilbert space over the complex or real field. A continuous linear functional $L(x)$ on $\mathscr{A}$ is called a *linear form* and a continuous Hermitian quadratic functional $Q(x)$ on $\mathscr{A}$ is called a *quadratic form*. The asscciated bilinear form will be denoted by $Q(x, y)$. A quadratic form $K(x)$ is said to be *compact* on $\mathscr{A}$ if $K(x_q) \to K(x_0)$ whenever x_q converges weakly to x_0. For any quadratic form $Q(x)$ on $\mathscr{A}$, two vectors x and y are said to be *Q-orthogonal* whenever $Q(x, y) = 0$. If $\mathscr{B}$ is a subspace of A, then the *Q-orthogonal complement* of $\mathscr{B}$, symbolically $\mathscr{B}^Q$, are the vectors in $\mathscr{A}$ Q-orthogonal to $\mathscr{B}$. And the space $\mathscr{B}_0 = \mathscr{B} \cap \mathscr{B}^Q$ is called the *Q-null space*; the vectors in $\mathscr{B}_0$ are called *Q-null vectors*. If $\mathscr{B}_0$ consists only of the null vector, then $Q(x)$ is said to be *nondegenerate* on $\mathscr{B}$. The *relative nullity of* $\mathscr{B}$ relative to $\mathscr{A}$, symbolically $rn(\mathscr{B})$, is defined to be the dimension of a linear subspace $\mathscr{C}$ such that $\mathscr{B}_0 = \mathscr{C} + \mathscr{B} \cap \mathscr{A}_0$, $\mathscr{C} \cap \mathscr{A}_0 = 0$.

The signature (or index) of a quadratic form $Q(x)$ on $\mathscr{A}$, denoted by s or $s(\mathscr{A})$, is the dimension of a maximal linear subspace $\mathscr{B}$ of $\mathscr{A}$ on which $Q(x)$ is negative. The nullity of $Q(x)$ on $\mathscr{A}$, denoted by n or $n(\mathscr{A})$, is the dimension of the Q-null space of $\mathscr{A}$. The signature and nullity are well-defined [3]. The quadratic form $J(x)$ in this paper shall have finite signature and nullity because it will be shown to be an *elliptic form*, i.e., a quadratic form that can be decomposed into a difference $D(x) - K(x)$, where $D(x)$ is a positive definite quadratic form and $K(x)$ is a compact quadratic form.

A quadratic form $Q(x)$ is said to be *nonsingular* on a linear subspace $\mathscr{B}$ of $\mathscr{A}$ if given a linear form $L(x)$ on $\mathscr{B}$ there is a unique vector y in $\mathscr{B}$ such that $L(x) = Q(x, y)$ for all x in $\mathscr{B}$. A *quasi-nonsingular* quadratic form $Q(x)$ on $\mathscr{A}$ is a quadratic form nonsingular on each closed linear subspace on which it is nondegenerate.

3. The Euler-Lagrange equations. The Hilbert space $\mathscr{H}$ of this paper is composed of vectors of the form

$$x : x(t), u(t), b \quad (t^0 \leqq t \leqq t^1) ,$$

where $x(t)$ is an n-dimensional square-integrable vector, $u(t)$ is a q-dimensional square-integrable vector, and b is an r-dimensional scalar vector. The inner product is defined to be

$$(x, y) = c^*b + \int_{t_0}^{t^1} [y^*(t)x(t) + v^*(t)u(t)]dt ,$$

where $y: y(t), v(t)$, c is also an element of $\mathscr{H}$.

The one-parameter family $\mathscr{C}(\lambda)$ $(t^0 \leqq \lambda \leqq t^1)$ described above are subspaces of $\mathscr{H}$ and the quadratic form $J(x)$ is defined for elements x of $\mathscr{H}$ such that $x(t)$ is absolutely continuous.

THEOREM 3.1. *Let $x \in \mathscr{H}$ with $x(t)$ absolutely continuous on $t^0 \leqq t \leqq \lambda$. Then x is J-orthogonal to $\mathscr{C}(\lambda)$ if and only if there exist an absolutely continuous vector $p(t)$ $(t^0 \leqq t \leqq \lambda)$ and a square-integrable vector $\mu(t)$ $(t^0 \leqq t \leqq \lambda)$ such that*

$$\dot{p} + A^*p + M^*\mu = \omega_x \tag{3.1}$$

$$B^*p + N^*\mu = \omega_u \qquad (t^0 \leqq t \leqq \lambda) \tag{3.2}$$

$$Fb - C^*p(t^0) = 0 . \tag{3.3}$$

For a proof see Mikami [6].

Equations (3.1), (3.2) together with (1.1), (1.2) shall be called the *Euler-Lagrange equations* and equation (3.3) is the *transversality condition*. An element x of $\mathscr{H}$ that satisfies the Euler-Lagrange equations on the subinterval $t^0 \leqq t \leqq \lambda$ and the transversality condition is called a *transversal extremal arc* on $t^0 \leqq t \leqq \lambda$.

4. The strengthened Clebsch condition. Let the symbol $\mathscr{B}$ designate the subspace of $\mathscr{H}$ whose elements satisfy the equations

$$\dot{x} = Ax + Bu, Mx + Nu = 0, x(t^0) = Cb .$$

We wish to prove that the quadratic form $J(x)$ is elliptic on $\mathscr{B}$ if and only if the *strengthened Clebsch condition* holds: There exist positive numbers h_0, h_1 such that at almost all points t on $t^0 \leqq t \leqq t^1$,

$$\pi^*R(t)\pi + h_1\pi^*N^*(t)N(t)\pi \geqq h_0\pi^*\pi$$

for all π in E^q. This is equivalent to, as the reader can verify: There exist positive numbers h_2, h_3 such that at almost all t on $t^0 \leqq t \leqq t^1$,

$$\pi^*R(t)\pi \geqq h_2\pi^*\pi$$

whenever $\pi^*N^*(t)N(t)\pi < h_3\pi^*\pi$. If the matrices $R(t)$ and $N(t)$ are continuous, then the strengthened Clebsch condition is equivalent to: There exists a positive number h_4 such that for all t on $t^0 \leqq t \leqq t^1$,

$$\pi^*R(t)\pi \geqq h_4\pi^*\pi$$

whenever $N(t)\pi = 0$.

Let $\mathscr{A}$ be the subspace of $\mathscr{H}$ whose elements satisfy the equations

$$\dot{x} = Ax + Bu, x(t^0) = Cb .$$

The following lemma describes weak convergence in $\mathscr{A}$.

LEMMA 4.1. *If $\{x_m\}$ $(m = 1, 2, \cdots)$ is a sequence in $\mathscr{A}$ converging weakly to x_0 in $\mathscr{A}$, then $x_m(t) \to x_0(t)$ uniformly on $t^0 \leqq t \leqq t^1$.*

The proof is an application of the well-known characterization of weak convergence in $L_2(n)$.

Define

$$S(x) = b^*b + \int_{t^0}^{t^1} u^*Rudt \qquad (x \in \mathscr{A}) .$$

Then $S(x)$ is a positive definite quadratic form on $\mathscr{A}$ if and only if the *strengthened Legendre condition* holds, i.e., there exists a positive number h_5 such that $\pi^*R(t)\pi \geqq h_5\pi^*\pi$ a.e. on $t^0 \leqq t \leqq t^1$ for all $\pi \in E^q$. For a proof see [3]. And one can easily show that

$$K(x) = \int_{t^0}^{t^1} (x^*Px + x^*Qu + u^*Q^*x)dt$$

is a compact quadratic form on $\mathscr{A}$. We summarize these facts as

LEMMA 4.2. *The quadratic form $J(x)$ is elliptic on $\mathscr{A}$ if and only if the strengthened Legendre condition holds.*

Let $H(x)$ be a quadratic form on $\mathscr{A}$ defined by

$$H(x) = \int_{t^0}^{t^1} (Mx + Nu)^*(Mx + Nu)dt .$$

THEOREM 4.3. *If the strengthened Clebsch condition holds, then $J(x)$ is elliptic on $\mathscr{B}$.*

Proof. By Lemma 4.2, $J(x) + h_1H(x)$ is elliptic on $\mathscr{A}$, and since $H(x) = 0$ on $\mathscr{B}$, $J(x)$ is elliptic on $\mathscr{B}$.

To prove the converse of Theorem 4.3, we begin with

LEMMA 4.4. *The quadratic form $H(x)$ is nonnegative and quasi-nonsingular on $\mathscr{A}$.*

Proof. We will utilize a theorem from Hestenes [3, p. 555] which states that a nonnegative quadratic form $Q(x)$ on an arbitrary Hilbert space $\mathscr{E}$ is quasi-nonsingular if and only if it is positive definite on the orthogonal complement of the Q-null space $\mathscr{E}_0$.

Augment the matrices M and N so that

$$Y = \begin{bmatrix} M \\ O \end{bmatrix}, \quad Z = \begin{bmatrix} N \\ L \end{bmatrix}$$

are square matrices having $Z^{-1}(t)$ existing a.e. and essentially bounded on $t^0 \leqq t \leqq t^1$. Then

$$D(x, y) = c^*b + \int_{t^0}^{t^1} (Yy + Zv)^*(Yx + Zu)dt$$

can be shown to be an inner product on $\mathscr{A}$. We wish to show that $H(x) = D(x)$ on the D-orthogonal complement of $\mathscr{B}$. For arbitrary vectors c in E^r and $w(t)$ in $L_2(q - m)$ (if $q = m$, omit $w(t)$), there exists a vector y in $\mathscr{B}$ such that

$$My + Nv = 0, \; Lv = w, \; y(t^0) = Cc \; .$$

Since $D(x, y) = 0$, we have

$$c^*b + \int_{t^0}^{t^1} w^*Ludt = 0$$

and so $b = 0$ and $Lu = 0$. Thus $D(x) = H(x)$ on $\mathscr{B}^D$. Observe that $\mathscr{B}$ is the H-null space of $\mathscr{A}$ and so in view of the theorem mentioned earlier, $H(x)$ is quasi-nonsingular on $\mathscr{A}$.

THEOREM 4.5. *If $J(x)$ is elliptic on $\mathscr{B}$ then the strengthened Clebsch condition holds.*

Proof. The quadratic form $H(x)$ is nonnegative and quasi-nonsingular on $\mathscr{A}$ and $J(x)$ is elliptic on $\mathscr{B}$, the H-null space of $\mathscr{A}$. A theorem of Hestenes [3, p. 531] asserts the existence of a positive number s such that $J(x) + sH(x)$ is elliptic on $\mathscr{A}$. By Lemma 4.2 this is equivalent to the strengthened Clebsch condition.

Let us define a *nonsingularity* condition as follows: The inverse of the matrix

$$R_1 = \begin{bmatrix} R & -N^* \\ N & 0 \end{bmatrix}$$

exists a.e. on $t^0 \leqq t \leqq t^1$ and has essentially bounded elements. We will prove that the nonsingularity condition holds when the strengthened Clebsch condition is assumed.

LEMMA 4.6. *If the strengthened Clebsch condition holds, then there exists a positive number h_6 such that for almost all t on $t^0 \leqq t \leqq t^1$,*

$$\pi^*(R^2 + c^2N^*N)\pi \geqq h_6\pi^*\pi$$

for all $c \geqq 1$ and all π in E^q.

Proof. Let I be a measureable set of almost all points of $t^0 \leqq t \leqq t^1$ such that if $t \in I$, then $\pi^*(R(t)\pi \geqq h_2\pi^*\pi$ whenever

$$\pi^*N^*(t)N(t)\pi < h_3\pi^*\pi \ .$$

Suppose for each integer n, there exists $t_n \in I$, $\pi_n{}^*\pi_n = 1$, such that

$$\pi_n^*(R^2(t_n) + N^*(t_n)N(t_n))\pi_n < 1/n \ .$$

So for all n large enough,

$$\pi_n^*N^*(t_n)N(t_n)\pi_n < h_3$$
$$\pi_n^*R^2(t_n)\pi_n < h_2^2 \ .$$

Thus

$$h_2^2 > \pi_n^*R^2(t_n)\pi_n \geqq (\pi_n^*R(t_n)\pi_n)^2 \geqq h_2^2 \ ,$$

a contradiction.

THEOREM 4.7. *If the strengthened Clebsch condition holds, then the nonsingularity condition holds.*

Proof. Let

$$R_c = \begin{bmatrix} R & -cN^* \\ cN & 0 \end{bmatrix}, \qquad c \geqq 1 \ .$$

We wish to show that $R_c^*R_c$ is positive definite a.e. on $t^0 \leqq t \leqq t^1$ for some number c large enough. So let $w \in E^q$, $z \in E^m$ be arbitrary vectors. Then

$$[w \quad z]^*R_c^*R_c\begin{bmatrix} w \\ z \end{bmatrix} \geqq h_6|w|^2 - 2ck|w||z| + c^2h|z|^2$$

where k is a scalar constant. By choosing c large enough, $R_c^*R_c$ is positive definite. The determinant of R_c is bounded away from zero and since

$$|\det R_c| = c^{2m}|\det R_1| ,$$

so is the determinant of R_1.

5. **Abnormality.** Let the control equation $\dot{x} = Ax + Bu$ and the constraint equation $Mx + Nu = 0$ be given. With respect to these equations, there is inherent the notion of abnormality. For the moment define the order of abnormality of a subinterval $t^2 \leqq t \leqq t^3$ of $t^0 \leqq t \leqq t^1$ as the number of linearly independent solutions $\{p, \mu\}$, $p(t)$ absolutely continuous and $\mu(t)$ square integrable, of

$$\begin{aligned} &\dot{p} + A^*p + M^*\mu = 0 \\ &B^*p + N^*\mu = 0 \end{aligned} \tag{5.1}$$

a.e. on $t^2 \leqq t \leqq t^3$. Because of the assumption that the inverse of NN^* exists and is essentially bounded, we can write

$$\mu = -(NN^*)^{-1}NB^*p .$$

Equation (5.1) then reduce to the form

$$\begin{aligned} &p + A_1^*p = 0 \\ &B_1^*p = 0 , \end{aligned} \tag{5.2}$$

and here we refer the reader to [4]. It follows from (5.2) that the order of abnormality is at most n.

Later the order of abnormality will be redefined for the problem on hand. Abnormality is important in this paper because it allows nontrivial solutions of the Euler-Lagrange equations to have $x(t) = 0$, $u(t) = 0$ on a subinterval of $t^0 \leqq t \leqq t^1$.

6. **Focal points.** The strengthened Clebsch condition is assumed for the rest of this paper. Hence the quadratic form $J(x)$ is elliptic on $\mathscr{B}$ and on any closed subspace of $\mathscr{B}$.

Let $\mathscr{F}$ be the subspace of arcs x in $\mathscr{B}$ that satisfy the Euler-Lagrange equations and the transversality condition. Such arcs shall be called *focal arcs*. If a focal arc x has $x(t) = 0$ on a subinterval I of $t^0 \leqq t \leqq t^1$ then $u(t) = 0$ on I also. Now for a fixed λ, $t^0 \leqq \lambda \leqq t^1$, the space of focal arcs $\mathscr{F}$ can be decomposed into a direct sum of two subspaces: one that has $x(\lambda) = 0$ and one that doesn't. The former subspace can be further decomposed into a direct sum of three subspaces: one that has $x(t)$ vanishing on $t^0 \leqq t \leqq \lambda$, one that has $x(t)$ vanishing on some subinterval $\lambda \leqq t \leqq \delta, \lambda < \delta$, and one that does not have a combination of these two properties. If the last subspace is denoted by $\mathscr{F}(\lambda)$, then $\mathscr{F}(\lambda)$ is a maximal subspace in $\mathscr{F}$ whose elements x have $x(\lambda) = 0$ and x is not a linear combination of arcs y

that have $y(t) = 0$ on $t^0 \leqq t \leqq \lambda$ and arcs z that have $z(t) = 0$ on some $\lambda \leqq t \leqq \delta, \lambda < \delta$. It is the dimension of the subspace $\mathscr{F}(\lambda)$ that interests us.

Let

$$\mathscr{C} = \{x \in \mathscr{B} : x(t^1) = 0\} .$$

Now a one-parameter family of subspaces $\mathscr{C}(\lambda)$ $(t^0 \leqq \lambda \leqq t^1)$, which we shall call a *resolution* of $\mathscr{C}$, has been defined earlier. Let $s(\lambda)$ and $n(\lambda)$ denote the signature and nullity of $J(x)$ on $\mathscr{C}(\lambda)$. The signature is a monotonic nondecreasing function of λ. A result of Hestenes [3] states that $s(\lambda) = s(\lambda - 0)$ and that the dimension of $\mathscr{F}(\lambda)$ is equal to $s(\lambda + 0) - s(\lambda)$, i.e., the "jump" at a discontinuity of $s(\lambda)$. So let us define a *focal point* as a point λ $(t^0 \leqq \lambda < t^1)$ such that $\mathscr{F}(\lambda)$ is not the null space and define the *order* of the focal point λ as the dimension of $\mathscr{F}(\lambda)$. In other words λ is a focal point if and only if there exists a focal arc $x \neq 0$ such that $x(t)$ vanishes at λ and x is not a linear sum of arcs that have $y(t) = 0$ on $t^0 \leqq t \leqq \lambda$ and arcs that have $z(t) = 0$ on some $\lambda < t \leqq \delta$. Later we shall refer a focal point as a focal point of the end point, for example, λ is a focal point of t^0 in the above case.

In the special case where the matrix $C = 0$, i.e., $x(t^0) = 0$, the word "conjugate" is used instead of "focal".

The above facts can be summarized in the following:

THEOREM 6.1. *The signature of $J(x)$ on $\mathscr{C}$ is equal to the sum of the order of the focal points λ, $t^0 < \lambda < t^1$.*

Theorem 6.1 was proved by Birkhoff and Hestenes [1] for the case $\dot{x} = u$ and by Hazard [2] for the case $\dot{x} = u$, $Mx + Nu = 0$. Hazard used focal intervals, i.e., intervals on which a focal arc vanishes. Our focal point is the right endpoint of Hazard's focal interval.

We have described the notion of abnormality, relating it to the existence of focal intervals. For example, if the problem were *normal*, i.e., the order of abnormality equals zero for all subintervals of $t^0 \leqq t \leqq t^1$, then the focal intervals collapse into points. In fact if the order of abnormality is the same for all subintervals, then again the focal intervals collapse into points.

7. **Conjugate bases.** In a simple variational problem, for example, one in which the arcs satisfy

$$\dot{x} = u, b = 0, x(t^0) = 0, x(t^1) = 0 ,$$

and the strengthened Legendre condition holds, it is well-known that

there exists a conjugate base of the left end point t^0, $x_\alpha(t)$ $(\alpha = 1, \cdots, n)$, such that the zeros of the determinant of the matrix

$$(x_\alpha^i(t)) \qquad (t^0 < t < t^1;\ i, \alpha = 1, \cdots, n) \tag{7.1}$$

are the conjugate points of t^0 and that the order of a conjugate point λ, $t^0 < \lambda < t^1$, is equal to n minus the rank $(x_\alpha^i(\lambda))$. In our control problem the determinant of the matrix (7.1) would be zero on focal intervals and even possibly identically zero. In short the determinant does not always give the focal points; however, the rank and the abnormality together provide the location of the focal points.

Let us begin by assuming that the matrix $[F, C^*]$ has full rank r. And let us define a *focal extremal* (relative to t^0) to be a system of vectors

$$x\colon x(t), u(t), b, p(t), \mu(t) \qquad (t^0 \leqq t \leqq t^1)$$

satisfying the Euler-Lagrange equations, the end condition (1.3), and the transversality condition. It follows easily that if x and y are focal extremals then

$$\frac{d}{dt}(p^*y - x^*q) = 0 .$$

So we say that two focal extremals are *conjugate* to each other if

$$p^*(t)y(t) - x^*(t)q(t) = 0 \qquad (t^0 \leqq t \leqq t^1) .$$

THEOREM 7.1. *There exists a linearly independent set of mutually conjugate focal extremals, x_α $(\alpha = 1, \cdots, n)$, such that every focal extremal can be written as a finite linear combination of the x_α.*

The proof is a simple exercise in differential equation theory.

The set described in Theorem 7.1 shall be called a *conjugate base* (of t^0).

THEOREM 7.2. *Let*

$$x_\alpha\colon x_\alpha(t), u_\alpha(t), b_\alpha, p_\alpha(t), \mu_\alpha(t) \qquad (t^0 \leqq t \leqq t^1;\ \alpha = 1, \cdots, n)$$

be a set of linearly independent mutually conjugate vectors satisfying the Euler-Lagrange equations and having $a_\alpha x_\alpha(t^0) = 0$ (summed over α) whenever $a_\alpha b_\alpha = 0$ (a_α are scalars). Then there exist matrices C and F such that $F = F^*$, rank $[F, C^*] = r$, $x_\alpha(t^0) = Cb_\alpha$, and

$$Fb_\alpha - C^*P_\alpha(t^0) = 0 .$$

Proof. Without loss of generality assume

$$b_1 = [10 \cdots 0]^*$$
$$\cdots$$
$$b_\gamma = [0 \cdots 010 \cdots 0]^* \text{ (1 in } \gamma\text{-th position)}$$
$$b_{\gamma+1} = \cdots = b_n = 0 .$$

Let the matrix C be the columns of $[x_1(t^0), \cdots, x_\gamma(t^0), 0, \cdots, 0]$. Then $x_\alpha(t^0) = Cb_\alpha$ $(\alpha = 1, \cdots, n)$.

Let F be the matrix

$$F = \begin{bmatrix} x_\alpha^*(t^0)p_\beta(t^0) & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & 1 \end{bmatrix} \qquad (\alpha, \beta \leqq \gamma) .$$

Then $F = F^*$ because the x_α are mutually conjugate. The remaining relations are easily verified.

Let $t^0 < \lambda < t^1$. The order of abnormality $a(\lambda)$ of this problem is defined to be the number of linearly independent solutions $\{p, \mu, b\}$ of the equations

$$\begin{aligned} &\dot{p} + A^*p + M^*\mu = 0 \\ &B^*p + N^*\mu = 0 \qquad (t^0 \leqq t \leqq \lambda) \\ &Cb = 0, Fb - C^*p(t^0) = 0 . \end{aligned}$$

By referring back to § 5 we see that the order of abnormality $a(\lambda)$ is a nonincreasing function of λ and continuous from the left. It is at most n.

THEOREM 7.3. *If x_α $(\alpha = 1, \cdots, n)$ is a conjugate base and if $d(t)$ is the rank of the matrix $x_\alpha^i(t)$ $(t^0 \leqq t \leqq t^1; i, \alpha = 1, \cdots, n)$, then the focal points of t^0 are the points λ, $t^0 < \lambda < t^1$ at which*

$$d(\lambda + 0) + a(\lambda + 0) - d(\lambda) - a(\lambda) > 0 . \tag{7.2}$$

Moreover if λ is such a focal point, then the order of the focal point equals the left-hand side of (7.2). *The nullity of $J(x)$ on $\mathscr{C}$ is equal to $n - d(t^1)$.*

The proof follows from the definitions of a focal point and abnormality.

8. Separated end conditions. This section deals with a subspace having separated end conditions. The control parameter b is composed of two component vectors

$$b: b_0^\sigma, b_1^\tau \qquad (\sigma = 1, \cdots, r: \tau = 1, \cdots, s) .$$

Let $\mathscr{D}$ denote the vectors x in $\mathscr{H}$ that satisfy

$$\dot{x} = Ax + Bu,\ Mx + Nu = 0$$
$$x(t^0) = C_0 b_0,\ x(t^1) = C_1 b_1 .$$

Let

$$\mathscr{E}(\lambda) = \{x \in \mathscr{D}: b_1 = 0,\ x(t) = 0,\ u(t) = 0 \text{ on } \lambda \leqq t \leqq t^1\}$$
$$\mathscr{D}(\lambda) = \{x \in \mathscr{D}: b_0 = 0,\ x(t) = 0,\ u(t) = 0 \text{ on } t^0 \leqq t \leqq \lambda\} \qquad (t^0 \leqq \lambda \leqq t^1)$$

be resolutions of $\mathscr{E}(t^1)$ and $\mathscr{D}(t^0)$, respectively. The quadratic form $J(x)$ is written

$$J(x) = b_0^* F_0 b_0 + b_1^* F_1 b_1 + \int_{t^0}^{t^1} 2\omega(t, x, u)dt ,$$

where $F_0 = F_0^*$ and $F_1 = F_1^*$. Again assume

$$\text{rank } [F_0, C_0^*] = r,\ \text{rank } [F_1, C_1^*] = s .$$

Let $t^0 < \lambda < t^1$. For convenience let

$$\mathscr{E}(\lambda) = [\mathscr{E}(\lambda) + \mathscr{D}(\lambda)]^J = \mathscr{E}(\lambda)^J \cap \mathscr{D}(\lambda)^J$$

(J-orthogonality relative to $\mathscr{D}$). An arc x is in $\mathscr{E}(\lambda)$ if and only if x is a transversal extremal arc on $t^0 \leqq t \leqq \lambda$ and a transversal extremal arc on $\lambda \leqq t \leqq t^1$. If the vector $p(t)$ has a discontinuity at λ, then we say that x is a *broken transversal extremal arc*. A result of [3, p. 565] is

$$s(\mathscr{D}) = s[\mathscr{E}(\lambda)] + s[\mathscr{D}(\lambda)] + k(\lambda) \tag{8.1.}$$

where

$$k(\lambda) = s[\mathscr{E}(\lambda)] + rn[\mathscr{E}(\lambda) + \mathscr{D}(\lambda)] .$$

The number $k(\lambda)$ is the dimension of a maximal linear space of broken transversal extremal arcs having $J(x) \leqq 0$. Note that $k(\lambda)$ is constant except at focal points of t^0 or t^1 by virtue of (8.1).

In normal problems it is known that $k(\lambda) \leqq n$. This is also true for problems with the order of abnormality not equal to zero. Let a be the maximal number of linearly independent solutions $\{p, \mu, b_0, b_1\}$ of the equations

$$\dot{p} + A^* p + M^* \mu = 0$$
$$B^* p + N^* \mu = 0 \qquad (t^0 \leqq t \leqq t^1) \tag{8.2}$$

$$F_1 b_1 + C_1^* p(t^1) = 0$$
$$F_0 b_0 - C_0^* p(t^0) = 0 \tag{8.3}$$
$$C_1 b_1 = 0,\ C_0 b_0 = 0 .$$

We call the number a the order of abnormality of the problem.

THEOREM 8.1.

$$0 \leqq k(\lambda) \leqq n - a \qquad (t^0 < \lambda < t^1) \tag{8.4}$$

$$s(\mathscr{D}) = s(\mathscr{D}(t^0)) + k(t^0 + 0) \tag{8.5}$$

$$s(\mathscr{D}) = s(\mathscr{E}(t^1)) + k(t^1 - 0) \tag{8.6}$$

Proof. Let $\{p_\alpha, \mu_\alpha, b_{0\alpha}, b_{1\alpha}\}$ $(\alpha = 1, \cdots, a)$ be linearly independent solutions of (8.2)-(8.3). Fix $t^0 < \lambda < t^1$ and abbreviate

$$e = rn[\mathscr{E}(\lambda) + \mathscr{D}(\lambda)] .$$

Then there exist J-null vectors x_β of $\mathscr{E}(\lambda) + \mathscr{D}(\lambda)$ and vectors p_β $(\beta = 1, \cdots, e)$ such that $x_\beta(\lambda) = 0$ and the set of vectors

$$p_\alpha(\lambda),\ p_\beta(\lambda - 0) - p_\beta(\lambda + 0) \qquad (\alpha = 1, \cdots, a;\ \beta = 1, \cdots, e)$$

are linearly independent. If not, then there would exist a vector x equal to a finite linear combination of the x_β and a vector p continuous at λ, a contradiction. Since the vectors x_β are J-orthogonal to $\mathscr{E}(\lambda)$ it can be verified by computation that

$$p_\alpha(\lambda)^* y(\lambda) = 0 \qquad (\alpha = 1, \cdots, a)$$

$$[p_\beta(\lambda - 0) - p_\beta(\lambda + 0)]^* y(\lambda) = 0 \qquad (\beta = 1, \cdots, e)$$

for each y in $\mathscr{E}(\lambda)$. This means there are $n - a - e$ linearly independent $y(\lambda)$ and therefore the number of linearly independent vectors in $\mathscr{E}(\lambda)$ whose linear combinations do not vanish at λ is less than or equal to $n - a - e$. Since the vectors in $\mathscr{E}(\lambda)$ that have $y(\lambda) = 0$ have $J(y) = 0$, we conclude that $s[\mathscr{E}(\lambda)] \leqq n - a - e$. Finally $k(\lambda) \leqq (n - a - e) + e = n - a$.

To prove (8.5) choose λ in (8.1) near t^0 so that no focal point of t° or t^1 is in $t^0 < t \leqq \lambda$. Equation (8.5) follows readily.

COROLLARY 8.2. *If s_0 and s_1 respectively denote the number of focal points of t^0 and t^1 on a given interval I, then the following relations hold:*

$$s_0 - s_1 = k(t^0 + 0) - k(t^1 - 0) \text{ if } I = (t^0, t^1)$$
$$s_1 - s_1 = k(t^0 + 0) - k(t^1 + 0) \text{ if } I = (t^0, t^1]$$
$$s_0 - s_1 = k(t^0 - 0) - k(t^1 - 0) \text{ if } I = [t^0, t^1)$$
$$s_0 - s_1 = k(t^0 - 0) = k(t^1 + 0) \text{ if } I = [t^0, t^2] .$$

In each case $|s_0 - s_1| \leqq n - a$.

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