CHARACTERIZING THE DISTRIBUTIONS OF THREE INDEPENDENT $n$-DIMENSIONAL RANDOM VARIABLES, $X_1$, $X_2$, $X_3$, HAVING ANALYTIC CHARACTERISTIC FUNCTIONS BY THE JOINT DISTRIBUTION OF $(X_1 + X_3$, $X_2 + X_3)$

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CHARACTERIZING THE DISTRIBUTIONS OF THREE INDEPENDENT \( n \)-DIMENSIONAL RANDOM VARIABLES, \( X_1, X_2, X_3 \), HAVING ANALYTIC CHARACTERISTIC FUNCTIONS BY THE JOINT DISTRIBUTION OF \((X_1 + X_3, X_2 + X_3)\).

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Kotlarski characterized the distribution of three independent real random variables \( X_1, X_2, X_3 \) having nonvanishing characteristic functions by the joint distribution of the 2-dimensional vector \((X_1 + X_3, X_2 + X_3)\). In this paper, we shall give a generalization of Kotlarski’s result for \( X_1, X_2, X_3 \) \( n \)-dimensional random variables having analytic characteristic functions which can meet the value zero.

In [3], Kotlarski shows that, for three independent random variables \( X_1, X_2, X_3 \), the distribution of \((X_1 + X_3, X_2 + X_3)\) determines the distributions of \( X_1, X_2 \) and \( X_3 \) up to a change of the location if the characteristic function of the pair \((X_1 + X_3, X_2 + X_3)\) does not vanish. Kotlarski also remarks that this result can be generalized in two ways. The statement remains true if the requirement that the pair \((X_1 + X_3, X_2 + X_3)\) has a nonvanishing characteristic function is replaced by the requirement that the random variables, \( X_1, X_2, X_3 \), possess analytic characteristic functions. The statement also remains true if \( X_1, X_2 \) and \( X_3 \) are \( n \)-dimensional real random vectors such that the pair \((X_1 + X_3, X_2 + X_3)\) has a nonvanishing characteristic function. In this paper, Kotlarski’s result is generalized to the case where \( X_1, X_2, \) and \( X_3 \) are \( n \)-dimensional real random vectors possessing analytic characteristic functions.

1. Some notions and lemmas about analytic functions of several complex variables. Let \( R_n \) denote \( n \)-dimensional real Euclidean space, \( C_n \) denote \( n \)-dimensional complex Euclidean space, and let \( f(t_1, \ldots, t_n) \) be defined on some domain \( D \) in \( C_n \). The function \( f \) is said to be **analytic at the point** \((t_1^0, \ldots, t_n^0)\) in \( D \) if \( f \) can be represented by a convergent power series in some neighborhood of \((t_1^0, \ldots, t_n^0)\). The function \( f \) is said to be **analytic on the domain** \( D \) if it is analytic at every point in \( D \). We now list several lemmas concerning analytic functions of several complex variables; for a discussion of these lemmas and further exposition on this theory, see [2].

**Lemma A.** If \( f(t_1, \ldots, t_n) \) and \( g(t_1, \ldots, t_n) \) are analytic at the
point \((t_1, \cdots, t_n)\), and if \(f(t_1, \cdots, t_n) \neq 0\), then the quotient \(\frac{g}{f}\) is also analytic at \((t_1, \cdots, t_n)\).

**Lemma B.** (Principle of analytic continuation). If \(f\) and \(g\) are analytic on some domain \(D\) in \(\mathbb{C}_n\) and if \(f(t_1, \cdots, t_n) = g(t_1, \cdots, t_n)\) at every point in some subdomain of \(D\), then \(f(t_1, \cdots, t_n) = g(t_1, \cdots, t_n)\) at all points of \(D\).

2. The main theorem and its proof.

**Theorem.** Let \(X_1, X_2, X_3\) be three independent, real, \(n\)-dimensional random vectors, and let \(Z_1 = X_1 + X_3, Z_2 = X_2 + X_3\). If the random vectors \(X_k\) possess characteristic functions \(\phi_k\) which are analytic on domains \(D_k\), with \(\overline{0} \in D_k\), \((k = 1, 2, 3)\), then the distributions of \((Z_1, Z_2)\) determines the distributions of \(X_1, X_2\) and \(X_3\) up to a change of the location.

**Proof.** Let \(t = (t_1, t_2, \cdots, t_n), s = (s_1, s_2, \cdots, s_n)\) denote arbitrary points in \(\mathbb{C}_n\) and \(\overline{0} = (0, 0, \cdots, 0)\) denote the origin in \(\mathbb{C}_n\); let

\[
||t|| = V |t_1|^2 + |t_2|^2 + \cdots + |t_n|^2
\]

and let \(t \cdot s = t_1s_1 + t_2s_2 + \cdots + t_ns_n\).

Let \(\phi_k = \mathbb{E}e^{it \cdot X_k}\), the characteristic function of \(X_k\), be defined on the domain \(D_k \subset \mathbb{C}_n\), \((k = 1, 2, 3)\). Then, letting \(\phi(t, s)\) denote the characteristic function of the distribution of the pair \((Z_1, Z_2)\), we have

\[
\phi(t, s) = \mathbb{E}e^{it \cdot (Z_1 + s \cdot Z_2)}
\]

\[
= \mathbb{E}e^{it \cdot (X_1 + s \cdot X_3 + (t + s) \cdot X_3)}
\]

\[
= \mathbb{E}e^{it \cdot X_1} \mathbb{E}e^{is \cdot X_2} \mathbb{E}e^{i(t + s) \cdot X_3}
\]

\[
= \phi_1(t) \phi_2(s) \phi_3(t + s)
\]

where this function is defined on the domain

\[
D = \{(t, s): t \in D_1, s \in D_2, (t + s) \in D_3\} \subset \mathbb{C}_{2n}.
\]

Let \(U_1, U_2, U_3\) be three other independent, real, \(n\)-dimensional random vectors possessing characteristic functions \(\psi_1, \psi_2, \psi_3\) which are analytic on domains \(D_1^*, D_2^*, D_3^*\). Let \(V_1 = U_1 + U_3, V_2 = U_2 + U_3\) and let \(\psi(t, s) = \mathbb{E}e^{it \cdot (V_1 \cdot (t \cdot t_2))}\). Calculations analogous to those above yield

\[
\psi(t, s) = \psi_1(t) \psi_2(s) \psi_3(t + s)
\]

on

\[
D^* = \{(t, s): t \in D_1^*, s \in D_2^*, (t + s) \in D_3^*\} \subset \mathbb{C}_{2n}.
\]
Suppose that the pairs \((Z_1, Z_2)\) and \((V_1, V_2)\) have the same distribution; we shall show that the distributions of \(X_k\) and \(U_k\), \((k = 1, 2, 3)\) are equal up to a shift. If the pairs \((Z_1, Z_2)\) and \((V_1, V_2)\) have the same distribution, their characteristic functions are equal so that \(D = D^*\) and

\[
\psi_1(t) \psi_2(s) \psi_3(t + s) = \phi_1(t) \phi_2(s) \phi_3(t + s).
\]

Since each of the functions in equation (1) is analytic and equal to 1 at 0, there exists a domain \(D^{**} \subset C_{2n}\) of the form

\[
\{(t, s): \sqrt{||t||^2 + ||s||^2} < \alpha, \alpha > 0\}
\]

such that, on \(D^{**}\), \(|\phi(t)| > 1/2\), \(|\phi_2(s)| > 1/2\), \(|\phi_3(t + s)| > 1/2\) and similar conditions hold for \(\psi_1, \psi_2, \psi_3\). Then on \(D^{**}\) equation (1) can be rewritten

\[
\frac{\psi_1(t)}{\phi_1(t)} \frac{\psi_2(s)}{\phi_2(s)} = \frac{\phi_3(t + s)}{\psi_3(t + s)}.
\]

Letting \(\chi(t) = \psi_1(t)/\phi_1(t)\), \(\chi_2(t) = \psi_2(t)/\phi_2(t)\), \(\chi_3(t) = \phi_3(t)/\psi_3(t)\), Lemma A asserts that each \(\chi_k\), \((k = 1, 2, 3)\), is analytic for \(||t|| < \alpha\). Then on \(D^{**}\) equation (2) becomes

\[
\chi_1(t) \chi_2(s) = \chi_3(t + s).
\]

For \(s = 0\), this equation reduces to \(\chi_1(t) = \chi_3(t)\); similarly, setting \(t = 0\) yields \(\chi_2(s) = \chi_3(s)\) so that, on \(D^{**}\),

\[
\chi_3(t) \chi_3(s) = \chi_3(t + s).
\]

In [1], it is shown that the only nonzero analytic solutions of (4) are the exponential functions, \(e^{ct}\) where \(c \in C_n\).

Therefore, for \(||t|| < \alpha\), \(\psi_3(t) = e^{-ct}\phi_3(t)\); since \(\psi_3\) and \(\phi_3\) are analytic on \(D_3\), Lemma B asserts that \(\psi_3(t) = e^{-ct}\phi_3(t)\) for all \(t \in D_3\). Since \(\chi_3(t) = \chi_1(t)\) for \(||t|| < \alpha\), \(\chi_1(t) = e^{ct}\phi_1(t)\) so that \(\psi_1(t) = e^{ct}\phi_1(t)\) for \(||t|| < \alpha\). Again, Lemma B asserts that \(\psi_1(t) = e^{ct}\phi_1(t)\) for all \(t \in D_1\). A similar argument yields \(\psi_2(t) = e^{ct}\phi_2(t)\) for all \(t \in D_2\).

Since \(\phi(-t) = \overline{\phi(t)}\), the conjugate of \(\phi(t)\), for any characteristic function \(\phi\) and any \(t \in R_n\), it follows that \(c = ib\) where \(b \in R_n\). Therefore, \(\psi_1(t) = e^{ibt}\phi_1(t)\), \(\psi_2(t) = e^{ibt}\phi_2(t)\), \(\psi_3(t) = e^{-ibt}\phi_3(t)\). From this it follows that the distributions of \(X_k\) are equal to those of \(U_k\), \((k = 1, 2, 3)\), up to a change of the location, and the proof is complete.

3. Applications of the theorem. The following two examples show how the theorem can be applied to random vectors \(X_1, X_2, X_3\).
of the same dimension, which possess analytic characteristic functions
and for which the characteristic function of \((X_1 + X_3, X_2 + X_3)\) assumes the value zero.

Let \(X = (X_1, \ldots, X_n)\) denote a random vector; then \(X\) has multivariate distribution, \(Mu(r; P_1, \ldots, P_n)\), of order \(r\) with parameters \(P_1, \ldots, P_n, 0 \leq P_j, P_1 + P_2 + \cdots + P_n \leq 1\), if, for every set of integers

\[ \{k_j: j = 1, 2, \ldots, n, k_j \geq 0, \sum_{i=1}^{n} k_i \leq r \}, \]

\[ P(X_1 = k_1, \ldots, X_n = k_n) = \frac{r! P_1^{k_1} \cdots P_n^{k_n} P_0^{r-k_1-\cdots-k_n}}{k_1! k_2! \cdots k_n! (r - k_1-\cdots-k_n)!} \]

where \(P_0 = 1 - P_1 - P_2 - \cdots - P_n\). The characteristic function of \(X\),
\[ \phi(t_1, \ldots, t_n) = (P_0 + P_1 e^{it_1} + \cdots + P_n e^{it_n})^r, \]
is clearly an analytic function on \(C^n\). Notice that, for the choice of parameters \(P_1 = P_2 = \cdots = P_n = 1/2n, P_0 = 1/2\), \(\phi\) has zeros at the points \((2m_1 + 1) \pi, (2m_2 + 1) \pi, \cdots, (2m_n + 1) \pi)\), where \(m_1, m_2, \ldots, m_n\) are integers.

Let \(Mu^*(r_1, r_2, r_3; P_1, P_2, \ldots, P_n)\) denote the joint distribution of the pair \((Z_1, Z_2)\) where \(Z_1 = X_1 + X_3, Z_2 = X_2 + X_3\) and each \(X_k\), \((k = 1, 2, 3)\) has distribution \(Mu(r_k; P_1, \ldots, P_n)\). With these definitions, the above theorem asserts the following result.

**COROLLARY 1.** Let \(X_1, X_2, X_3\) be three independent, \(n\)-dimensional, random vectors and let \(Z_1 = X_1 + X_3\), \(Z_2 = X_2 + X_3\). If the pair \((Z_1, Z_2)\) has distribution \(Mu^*(r_1, r_2, r_3; P_1, \ldots, P_n)\), then, except for perhaps a change of location, the distribution of \(X_k\) is \(Mu(r_k; P_1, \ldots, P_n)\), \((k = 1, 2, 3)\).

As another application of the above theorem, let \(X\) be a \(2\)-dimensional real random vector and let us say that \(X\) has distribution \(U(a), a > 0\), if its distribution has density function

\[ f(x, y) = \begin{cases} \frac{1}{2a^2} & \text{for } |x| + |y| \leq a \\ 0 & \text{for } |x| + |y| > a \end{cases} \]

If \(X\) has distribution \(U(a)\), its characteristic function

\[ \phi_x(t_1, t_2) = \sin \left[ (t_1 + t_2) \frac{a}{2} \right] \sin \left[ (t_1 - t_2) \frac{a}{2} \right] \]

\[ a^2 (t_1 + t_2) (t_1 - t_2) \]

is an analytic function defined on \(C_2\) with zeros at the points \((t_1, t_2)\) where \((t_1 \pm t_2) = 2\pi/a m, m = \pm 1, \pm 2, \ldots\). Let \(U^*(a_1, a_2, a_3)\) denote the joint distribution of the pair \((Z_1, Z_2)\) where \(Z_1 = X_1 + X_3\) and
$Z_2 = X_2 = X_3$ and each $X_k$ has distribution $U(a_k)$, $(k = 1, 2, 3)$. With these definitions, the above theorem asserts the following result.

**Corollary 2.** Let $X_1, X_2, X_3$ be three independent 2-dimensional random vectors and let $Z_1 = X_1 + X_3, Z_2 = X_2 + X_3$. If the pair $(Z_1, Z_2)$ has distribution $U^*(a_1, a_2, a_3)$, then, except for perhaps a change of location, the distribution of $X_k$ is $U(a_k)$, $(k = 1, 2, 3)$.

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