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**CHARACTERIZING THE DISTRIBUTIONS OF THREE
INDEPENDENT n -DIMENSIONAL RANDOM VARIABLES,
 X_1, X_2, X_3 , HAVING ANALYTIC CHARACTERISTIC
FUNCTIONS BY THE JOINT DISTRIBUTION OF
 $(X_1 + X_3, X_2 + X_3)$**

PAUL G. MILLER

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Kotlarski characterized the distribution of three independent real random variables X_1, X_2, X_3 having nonvanishing characteristic functions by the joint distribution of the 2-dimensional vector $(X_1 + X_3, X_2 + X_3)$. In this paper, we shall give a generalization of Kotlarski's result for X_1, X_2, X_3 n -dimensional random variables having analytic characteristic functions which can meet the value zero.

In [3], Kotlarski shows that, for three independent random variables X_1, X_2, X_3 , the distribution of $(X_1 + X_3, X_2 + X_3)$ determines the distributions of X_1, X_2 and X_3 up to a change of the location if the characteristic function of the pair $(X_1 + X_3, X_2 + X_3)$ does not vanish. Kotlarski also remarks that this result can be generalized in two ways. The statement remains true if the requirement that the pair $(X_1 + X_3, X_2 + X_3)$ has a nonvanishing characteristic function is replaced by the requirement that the random variables, X_1, X_2, X_3 , possess analytic characteristic functions. The statement also remains true if X_1, X_2 and X_3 are n -dimensional real random vectors such that the pair $(X_1 + X_3, X_2 + X_3)$ has a nonvanishing characteristic function. In this paper, Kotlarski's result is generalized to the case where X_1, X_2 , and X_3 are n -dimensional real random vectors possessing analytic characteristic functions.

1. Some notions and lemmas about analytic functions of several complex variables. Let R_n denote n -dimensional real Euclidean space, C_n denote n -dimensional complex Euclidean space, and let $f(t_1, \dots, t_n)$ be defined on some domain D in C_n . The function f is said to be *analytic at the point* (t_1^0, \dots, t_n^0) in D if f can be represented by a convergent power series in some neighborhood of (t_1^0, \dots, t_n^0) . The function f is said to be *analytic on the domain* D if it is analytic at every point in D . We now list several lemmas concerning analytic functions of several complex variables; for a discussion of these lemmas and further exposition on this theory, see [2].

LEMMA A. *If $f(t_1, \dots, t_n)$ and $g(t_1, \dots, t_n)$ are analytic at the*

point (t_1^0, \dots, t_n^0) , and if $f(t_1^0, \dots, t_n^0) \neq 0$, then the quotient $\frac{g}{f}$ is also analytic at (t_1^0, \dots, t_n^0) .

LEMMA B. (*Principle of analytic continuation*). If f and g are analytic on some domain D in C_n and if $f(t_1, \dots, t_n) = g(t_1, \dots, t_n)$ at every point in some subdomain of D , then $f(t_1, \dots, t_n) = g(t_1, \dots, t_n)$ at all points of D .

2. The main theorem and its proof.

THEOREM. Let X_1, X_2, X_3 be three independent, real, n -dimensional random vectors, and let $Z_1 = X_1 + X_3, Z_2 = X_2 + X_3$. If the random vectors X_k possess characteristic functions ϕ_k which are analytic on domains D_k , with $\bar{0} \in D_k, (k = 1, 2, 3)$, then the distributions of (Z_1, Z_2) determines the distributions of X_1, X_2 and X_3 up to a change of the location.

Proof. Let $t = (t_1, t_2, \dots, t_n), s = (s_1, s_2, \dots, s_n)$ denote arbitrary points in C_n and $\bar{0} = (0, 0, \dots, 0)$ denote the origin in C_n ; let

$$\|t\| = \sqrt{|t_1|^2 + |t_2|^2 + \dots + |t_n|^2}$$

and let $t \cdot s = t_1s_1 + t_2s_2 + \dots + t_ns_n$.

Let $\phi_k = Ee^{it \cdot X_k}$, the characteristic function of X_k , be defined on the domain $D_k \in C_n, (k = 1, 2, 3)$. Then, letting $\phi(t, s)$ denote the characteristic function of the distribution of the pair (Z_1, Z_2) , we have

$$\begin{aligned} \phi(t, s) &= Ee^{i(t \cdot Z_1 + s \cdot Z_2)} \\ &= Ee^{i(t \cdot X_1 + s \cdot X_2 + (t+s) \cdot X_3)} \\ &= Ee^{it \cdot X_1} Ee^{is \cdot X_2} Ee^{i(t+s) \cdot X_3} \\ &= \phi_1(t) \phi_2(s) \phi_3(t+s) \end{aligned}$$

where this function is defined on the domain

$$D = \{(t, s) : t \in D_1, s \in D_2, (t + s) \in D_3\} \in C_{2n} .$$

Let U_1, U_2, U_3 be three other independent, real, n -dimensional random vectors possessing characteristic functions ψ_1, ψ_2, ψ_3 which are analytic on domains D_1^*, D_2^*, D_3^* . Let $V_1 = U_1 + U_3, V_2 = U_2 + U_3$ and let $\psi(t, s) = Ee^{i(t \cdot V_1 + s \cdot V_2)}$. Calculations analogous to those above yield

$$\psi(t, s) = \psi_1(t) \psi_2(s) \psi_3(t + s)$$

on

$$D^* = \{(t, s) : t \in D_1^*, s \in D_2^*, (t + s) \in D_3^*\} \in C_{2n} .$$

Suppose that the pairs (Z_1, Z_2) and (V_1, V_2) have the same distribution; we shall show that the distributions of X_k and U_k , ($k = 1, 2, 3$) are equal up to a shift. If the pairs (Z_1, Z_2) and (V_1, V_2) have the same distribution, their characteristic functions are equal so that $D = D^*$ and

$$(1) \quad \psi_1(t) \psi_2(s) \psi_3(t + s) = \phi_1(t) \phi_2(s) \phi_3(t + s) .$$

Since each of the functions in equation (1) is analytic and equal to 1 at $\bar{0}$, there exists a domain $D^{**} \in C_{2n}$ of the form

$$\{(t, s) : \sqrt{\|t\|^2 + \|s\|^2} < \alpha, \alpha > 0\}$$

such that, on D^{**} , $|\phi_1(t)| > 1/2$, $|\phi_2(s)| > 1/2$, $|\phi_3(t + s)| > 1/2$ and similar conditions hold for ψ_1, ψ_2, ψ_3 . Then on D^{**} equation (1) can be rewritten

$$(2) \quad \frac{\psi_1(t)}{\phi_1(t)} \frac{\psi_2(s)}{\phi_2(s)} = \frac{\phi_3(t + s)}{\psi_3(t + s)} .$$

Letting $\chi_1(t) = \psi_1(t)/\phi_1(t)$, $\chi_2(t) = \psi_2(t)/\phi_2(t)$, $\chi_3(t) = \phi_3(t)/\psi_3(t)$, Lemma A asserts that each χ_k , ($k = 1, 2, 3$), is analytic for $\|t\| < \alpha$. Then on D^{**} equation (2) becomes

$$(3) \quad \chi_1(t) \chi_2(s) = \chi_3(t + s) .$$

For $s = \bar{0}$, this equation reduces to $\chi_1(t) = \chi_3(t)$; similarly, setting $t = \bar{0}$ yields $\chi_2(s) = \chi_3(s)$ so that, on D^{**} ,

$$(4) \quad \chi_3(t) \chi_3(s) = \chi_3(t + s) .$$

In [1], it is shown that the only nonzero analytic solutions of (4) are the exponential functions, $e^{c \cdot t}$ where $c \in C_n$.

Therefore, for $\|t\| < \alpha$, $\psi_3(t) = e^{-c \cdot t} \phi_3(t)$; since ψ_3 and ϕ_3 are analytic on D_3 , Lemma B asserts that $\psi_3(t) = e^{-c \cdot t} \phi_3(t)$ for all $t \in D_3$. Since $\chi_3(t) = \chi_1(t)$ for $\|t\| < \alpha$, $\chi_1(t) = e^{c \cdot t}$ so that $\psi_1(t) = e^{c \cdot t} \phi_1(t)$ for $\|t\| < \alpha$. Again, Lemma B asserts that $\psi_1(t) = e^{c \cdot t} \phi_1(t)$ for all $t \in D_1$. A similar argument yields $\psi_2(t) = e^{c \cdot t} \phi_2(t)$ for all $t \in D_2$.

Since $\phi(-t) = \overline{\phi(t)}$, the conjugate of $\phi(t)$, for any characteristic function ϕ and any $t \in R_n$, it follows that $c = ib$ where $b \in R_n$. Therefore, $\psi_1(t) = e^{ib \cdot t} \phi_1(t)$, $\psi_2(t) = e^{ib \cdot t} \phi_2(t)$, $\psi_3(t) = e^{-ib \cdot t} \phi_3(t)$. From this it follows that the distributions of X_k are equal to those of U_k , ($k = 1, 2, 3$), up to a change of the location, and the proof is complete.

3. Applications of the theorem. The following two examples show how the theorem can be applied to random vectors X_1, X_2, X_3 ,

of the same dimension, which possess analytic characteristic functions and for which the characteristic function of $(X_1 + X_3, X_2 + X_3)$ assumes the value zero.

Let $X = (X_1, \dots, X_n)$ denote a random vector; then X has multinomial distribution, $Mu(r; P_1, \dots, P_n)$, of order r with parameters $P_1, \dots, P_n, 0 \leq P_j, P_1 + P_2 + \dots + P_n \leq 1$, if, for every set of integers

$$\{k_j: j = 1, 2, \dots, n, k_j \geq 0, \sum_1^n k_i \leq r\},$$

$$P(X_1 = k_1, \dots, X_n = k_n) = \frac{r! P_1^{k_1} \dots P_n^{k_n} P_0^{r-k_1-\dots-k_n}}{k_1! k_2! \dots k_n! (r - k_1 - \dots - k_n)!}$$

where $P_0 = 1 - P_1 - P_2 - \dots - P_n$. The characteristic function of X , $\phi(t_1, \dots, t_n) = (P_0 + P_1 e^{it_1} + \dots + P_n e^{it_n})^r$, is clearly an analytic function on C_n . Notice that, for the choice of parameters $P_1 = P_2 = \dots = P_n = 1/2n, P_0 = 1/2$, ϕ has zeros at the points $((2m_1 + 1)\pi, (2m_2 + 1)\pi, \dots, (2m_n + 1)\pi)$, where m_1, m_2, \dots, m_n are integers. Let $Mu^*(r_1, r_2, r_3; P_1, P_2, \dots, P_n)$ denote the joint distribution of the pair (Z_1, Z_2) where $Z_1 = X_1 + X_3, Z_2 = X_2 + X_3$ and each $X_k, (k = 1, 2, 3)$ has distribution $Mu(r_k; P_1, \dots, P_n)$. With these definitions, the above theorem asserts the following result.

COROLLARY 1. *Let X_1, X_2, X_3 be three independent, n -dimensional, random vectors and let $Z_1 = X_1 + X_3, Z_2 = X_2 + X_3$. If the pair (Z_1, Z_2) has distribution $Mu^*(r_1, r_2, r_3; P_1, \dots, P_n)$, then, except for perhaps a change of location, the distribution of X_k is $Mu(r_k; P_1, \dots, P_n), (k = 1, 2, 3)$.*

As another application of the above theorem, let X be a 2 dimensional real random vector and let us say that X has distribution $U(a), a > 0$, if its distribution has density function

$$f(x, y) = \begin{cases} \frac{1}{2a^2} & \text{for } |x| + |y| \leq a \\ 0 & \text{for } |x| + |y| > a \end{cases}.$$

If X has distribution $U(a)$, its characteristic function

$$\phi_x(t_1, t_2) = \frac{\sin\left[(t_1 + t_2)\frac{a}{2}\right] \sin\left[(t_1 - t_2)\frac{a}{2}\right]}{a^2 \left(\frac{t_1 + t_2}{2}\right) \left(\frac{t_1 - t_2}{2}\right)},$$

is an analytic function defined on C_2 with zeros at the points (t_1, t_2) where $(t_1 \pm t_2) = 2\pi/a m, m = \pm 1, \pm 2, \dots$. Let $U^*(a_1, a_2, a_3)$ denote the joint distribution of the pair (Z_1, Z_2) where $Z_1 = X_1 + X_3$ and

$Z_2 = X_2 = X_3$ and each X_k has distribution $U(a_k)$, ($k = 1, 2, 3$). With these definitions, the above theorem asserts the following result.

COROLLARY 2. *Let X_1, X_2, X_3 be three independent 2-dimensional random vectors and let $Z_1 = X_1 + X_3, Z_2 = X_2 + X_3$. If the pair (Z_1, Z_2) has distribution $U^*(a_1, a_2, a_3)$, then, except for perhaps a change of location, the distribution of X_k is $U(a_k)$, ($k = 1, 2, 3$).*

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