Pacific Journal of Mathematics

CHARACTERIZING THE DISTRIBUTIONS OF THREE INDEPENDENT n-DIMENSIONAL RANDOM VARIABLES, X_1, X_2, X_3, HAVING ANALYTIC CHARACTERISTIC FUNCTIONS BY THE JOINT DISTRIBUTION OF $(X_1 + X_3,\ X_2 + X_3)$

PAUL G. MILLER

Vol. 35, No. 2 October 1970

PACIFIC JOURNAL OF MATHEMATICS
Vol. 35, No. 2, 1970

CHARACTERIZING THE DISTRIBUTIONS OF THREE INDEPENDENT n-DIMENSIONAL RENDOM VARIABLES, X_1, X_2, X_3, HAVING ANALYTIC CHARACTERISTIC FUNCTIONS BY THE JOINT DISTRIBUTION OF $(X_1 + X_3,\ X_2 + X_3)$.

PAUL G. MILLER

Kotlarski characterized the distribution of three independent real random variables X_1, X_2, X_3 having nonvanishing characteristic functions by the joint distribution of the 2-dimensional vector $(X_1 + X_3,\ X_2 + X_3)$. In this paper, we shall give a generalization of Kotlarski's result for X_1, X_2, X_3 n-dimensional random variables having analytic characteristic functions which can meet the value zero.

In [3], Kotlarski shows that, for three independent random variables X_1, X_2, X_3, the distribution of $(X_1 + X_3, X_2 + X_3)$ determines the distributions of X_1, X_2 and X_3 up to a change of the location if the characteristic function of the pair $(X_1 + X_3,\ X_2 + X_3)$ does not vanish. Kotlarski also remarks that this result can be generalized in two ways. The statement remains true if the requirement that the pair $(X_1 + X_3, X_2 + X_3)$ has a nonvanishing characteristic function is replaced by the requirement that the random variables, X_1, X_2, X_3, possess analytic characteristic functions. The statement also remains true if X_1, X_2 and X_3 are n-dimensional real random vectors such that the pair $(X_1 + X_3,\ X_2 + X_3)$ has a nonvanishing characteristic function. In this paper, Kotlarski's result is generalized to the case where X_1, X_2, and X_3 are n-dimensional real random vectors possessing analytic characteristic functions.

1. Some notions and lemmas about analytic functions of several complex variables. Let R_n denote n-dimensional real Euclidean space, C_n denote n-dimensional complex Euclidean space, and let $f(t_1, \cdots, t_n)$ be defined on some domain D in C_n. The function f is said to be *analytic at the point* $(t_1^0, \cdots, t_n^0)$ in D if f can be represented by a convergent power series in some neighborhood of $(t_1^0, \cdots, t_n^0)$. The function f is said to be *analytic on the domain* D if it is analytic at every point in D. We now list several lemmas concerning analytic functions of several complex variables; for a discussion of these lemmas and further exposition on this theory, see [2].

LEMMA A. *If $f(t_1, \cdots, t_n)$ and $g(t_1, \cdots, t_n)$ are analytic at the* *point* $(t_1^0, \cdots, t_n^0)$, *and if* $f(t_1^0, \cdots, t_n^0) \neq 0$, *then the quotient* $\frac{g}{f}$ *is also analytic at* $(t_1^0, \cdots, t_n^0)$.

LEMMA B. (*Principle of analytic continuation*). *If f and g are analytic on some domain D in C_n and if $f(t_1, \cdots, t_n) = g(t_1, \cdots, t_n)$ at every point in some subdomain of D, then $f(t_1, \cdots, t_n) = g(t_1, \cdots, t_n)$ at all points of D.*

2. The main theorem and its proof.

THEOREM. *Let X_1, X_2, X_3 be three independent, real, n-dimensional random vectors, and let $Z_1 = X_1 + X_3$, $Z_2 = X_2 + X_3$. If the random vectors X_k possess characteristic functions ϕ_k which are analytic on domains D_k, with $\bar{O} \in D_k$, $(k = 1, 2, 3)$, then the distributions of (Z_1, Z_2) determines the distributions of X_1, X_2 and X_3 up to a change of the location.*

Proof. Let $t = (t_1, t_2, \cdots, t_n)$, $s = (s_1, s_2, \cdots, s_n)$ denote arbitrary points in C_n and $\bar{0} = (0, 0, \cdots, 0)$ denote the origin in C_n; let

$||t|| = \sqrt{|t_1|^2 + |t_2|^2 + \cdots + |t_n|^2}$ and let $t \cdot s = t_1 s_1 + t_2 s_2 + \cdots + t_n s_n$.

Let $\phi_k = Ee^{it \cdot X_k}$, the characteristic function of X_k, be defined on the domain $D_k \in C_n$, $(k = 1, 2, 3)$. Then, letting $\phi(t, s)$ denote the characteristic function of the distribution of the pair (Z_1, Z_2), we have

$$\begin{aligned} \phi(t, s) &= Ee^{i(t \cdot Z_1 + s \cdot Z_2)} \\ &= Ee^{i(t \cdot X_1 + s \cdot X_2 + (t+s) \cdot X_3)} \\ &= Ee^{it \cdot X_1} \, Ee^{is \cdot X_2} \, Ee^{i(t+s) \cdot X_3} \\ &= \phi_1(t) \, \phi_2(s) \, \phi_3(t + s) \end{aligned}$$

where this function is defined on the domain

$$D = \{(t, s) \colon t \in D_1, s \in D_2, (t + s) \in D_3\} \in C_{2n} .$$

Let U_1, U_2, U_3 be three other independent, real, n-dimensional random vectors possessing characteristic functions ψ_1, ψ_2, ψ_3 which are analytic on domains D_1^*, D_2^*, D_3^*. Let $V_1 = U_1 + U_3$, $V_2 = U_2 + U_3$ and let $\psi(t, s) = Ee^{i(t \cdot V_1 + s \cdot V_2)}$. Calculations analogous to those above yield

$$\psi(t, s) = \psi_1(t) \, \psi_2(s) \, \psi_3(t + s)$$

on

$$D^* = \{(t, s) \colon t \in D_1^*, \; s \in D_2^*, \; (t + s) \in D_3^*\} \in C_{2n} .$$

Suppose that the pairs (Z_1, Z_2) and (V_1, V_2) have the same distribution; we shall show that the distributions of X_k and U_k, $(k = 1, 2, 3)$ are equal up to a shift. If the pairs (Z_1, Z_2) and (V_1, V_2) have the same distribution, their characteristic functions are equal so that $D = D^*$ and

$$(1) \qquad \psi_1(t)\ \psi_2(s)\ \psi_3(t+s) = \phi_1(t)\ \phi_2(s)\ \phi_3(t+s)\ .$$

Since each of the functions in equation (1) is analytic and equal to 1 at $\bar{0}$, there exists a domain $D^{**} \in C_{2n}$ of the form

$$\{(t, s) : \sqrt{\|t\|^2 + \|s\|^2} < \alpha, \alpha > 0\}$$

such that, on D^{**}, $|\phi_1(t)| > 1/2$, $|\phi_2(s)| > 1/2$, $|\phi_3(t+s)| > 1/2$ and similar conditions hold for ψ_1, ψ_2, ψ_3. Then on D^{**} equation (1) can be rewritten

$$(2) \qquad \frac{\psi_1(t)}{\phi_1(t)}\ \frac{\psi_2(s)}{\phi_2(s)} = \frac{\phi_3(t+s)}{\psi_3(t+s)}\ .$$

Letting $\chi_1(t) = \psi_1(t)/\phi_1(t)$, $\chi_2(t) = \psi_2(t)/\phi_2(t)$, $\chi_3(t) = \phi_3(t)/\psi_3(t)$, Lemma A asserts that each χ_k, $(k = 1, 2, 3)$, is analytic for $\|t\| < \alpha$. Then on D^{**} equation (2) becomes

$$(3) \qquad \chi_1(t)\ \chi_2(s) = \chi_3(t+s)\ .$$

For $s = \bar{0}$, this equation reduces to $\chi_1(t) = \chi_3(t)$; similarly, setting $t = \bar{0}$ yields $\chi_2(s) = \chi_3(s)$ so that, on D^{**},

$$(4) \qquad \chi_3(t)\ \chi_3(s) = \chi_3(t+s)\ .$$

In [1], it is shown that the only nonzero analytic solutions of (4) are the exponential functions, $e^{c \cdot t}$ where $c \in C_n$.

Therefore, for $\|t\| < \alpha, \psi_3(t) = e^{-c \cdot t}\ \phi_3(t)$; since ψ_3 and ϕ_3 are analytic on D_3, Lemma B asserts that $\psi_3(t) = e^{-c \cdot t}\ \phi_3(t)$ for all $t \in D_3$. Since $\chi_3(t) = \chi_1(t)$ for $\|t\| < \alpha$, $\chi_1(t) = e^{c \cdot t}$ so that $\psi_1(t) = e^{c \cdot t}\ \phi_1(t)$ for $\|t\| < \alpha$. Again, Lemma B asserts that $\psi_1(t) = e^{c \cdot t}\ \phi_1(t)$ for all $t \in D_1$. A similar argument yields $\psi_2(t) = e^{c \cdot t}\ \phi_2(t)$ for all $t \in D_2$.

Since $\phi(-t) = \overline{\phi(t)}$, the conjugate of $\phi(t)$, for any characteristic function ϕ and any $t \in R_n$, it follows that $c = ib$ where $b \in R_n$. Therefore, $\psi_1(t) = e^{ib \cdot t}\ \phi_1(t)$, $\psi_2(t) = e^{ib \cdot t}\ \phi_2(t)$, $\psi_3(t) = e^{-ib \cdot t}\ \phi_3(t)$. From this it follows that the distributions of X_k are equal to those of U_k, $(k = 1, 2, 3)$, up to a change of the location, and the proof is complete.

3. Applications of the theorem. The following two examples show how the theorem can be applied to random vectors X_1, X_2, X_3, of the same dimension, which possess analytic characteristic functions and for which the characteristic function of $(X_1 + X_3, X_2 + X_3)$ assumes the value zero.

Let $X = (X_1, \cdots, X_n)$ denote a random vector; then X has multinomial distribution, $Mu(r; P_1, \cdots, P_n)$, of order r with parameters $P_1, \cdots P_n$, $0 \leqq P_j$, $P_1 + P_2 + \cdots + P_n \leqq 1$, if, for every set of integers

$$\{k_j\colon j = 1, 2, \cdots, n, k_j \geqq 0, \sum_1^n k_i \leqq r\}\ ,$$

$$P(X_1 = k_1, \cdots, X_n = k_n) = \frac{r!\ P_1^{k_1}\cdots P_n^{k_n}\ P_0^{r-k_1\cdots-k_n}}{k_1!\ k_2!\cdots k_n!(r - k_1-\cdots-k_n)!}$$

where $P_0 = 1 - P_1 - P_2 - \cdots - P_n$. The characteristic function of X, $\phi(t_1, \cdots, t_n) = (P_0 + P_1 e^{it_1} + \cdots + P_n e^{it_n})^r$, is clearly an analytic function on C_n. Notice that, for the choice of parameters $P_1 = P_2 = \cdots = P_n = 1/2n$, $P_0 = 1/2$, ϕ has zeros at the points $((2m_1 + 1)\pi, (2m_2 + 1)\pi, \cdots, (2m_n + 1)\pi)$, where $m_1, m_2, \cdots, m_n$ are integers. Let $Mu^*(r_1, r_2, r_3; P_1, P_2, \cdots, P_n)$ denote the joint distribution of the pair (Z_1, Z_2) where $Z_1 = X_1 + X_3$, $Z_2 = X_2 + X_3$ and each X_k, $(k = 1, 2, 3)$ has distribution $Mu(r_k; P_1, \cdots, P_n)$. With these definitions, the above theorem asserts the following result.

COROLLARY 1. *Let X_1, X_2, X_3 be three independent, n-dimensional, random vectors and let $Z_1 = X_1 + X_3$, $Z_2 = X_2 + X_3$. If the pair (Z_1, Z_2) has distribution $Mu^*(r_1, r_2, r_3; P_1, \cdots, P_n)$, then, except for perhaps a change of location, the distribution of X_k is Mu $(r_k; P_1, \cdots, P_n)$, $(k = 1, 2, 3)$.*

As another application of the above theorem, let X be a 2 dimensional real random vector and let us say that X has distribution $U(a)$, $a > 0$, if its distribution has density function

$$f(x, y) = \begin{cases} \dfrac{1}{2a^2} \text{ for } |x| + |y| \leqq a \\ 0 \text{ for } |x| + |y| > a \end{cases}.$$

If X has distribution $U(a)$, its characteristic function

$$\phi_X(t_1, t_2) = \frac{\sin\left[(t_1 + t_2)\dfrac{a}{2}\right]\sin\left[(t_1 - t_2)\dfrac{a}{2}\right]}{a^2\left(\dfrac{t_1 + t_2}{2}\right)\left(\dfrac{t_1 - t_2}{2}\right)}\ ,$$

is an analytic function defined on C_2 with zeros at the points (t_1, t_2) where $(t_1 \pm t_2) = 2\pi/a\ m$, $m = \pm 1, \pm 2, \cdots$. Let $U^*(a_1, a_2, a_3)$ denote the joint distribution of the pair (Z_1, Z_2) where $Z_1 = X_1 + X_3$ and

$Z_2 = X_2 = X_3$ and each X_k has distribution $U(a_k)$, $(k = 1, 2, 3)$. With these definitions, the above theorem asserts the following result.

COROLLARY 2. *Let X_1, X_2, X_3 be three independent 2-dimensional random vectors and let $Z_1 = X_1 + X_3, Z_2 = X_2 + X_3$. If the pair (Z_1, Z_2) has distribution $U^*(a_1, a_2, a_3)$, then, except for perhaps a change of location, the distribution of X_k is $U(a_k)$, $(k = 1, 2, 3)$.*

The author is indebted to Professor Ignacy Kotlarski for suggesting the problem discussed in this paper and for several helpful comments pertaining to its solution.

BIBLIOGRAPHY

1. J. Aczél, *Lectures on Functional Equations and Their Applications*, Academic Press, New York, 1966.

2. H. Cartan, *Elementary Theory of Analytic Functions of One or Several Complex Variables*, Addison-Wesley, Reading, Mass., 1963.

3. Ignacy, Kotlarski, *On characterizing the Gamma and the normal distribution*, Pacific J. Math. **20** (1967), 69-76.

Received February 18, 1970.

OKLAHOMA STATE UNIVERSITY

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Pacific Journal of Mathematics

Vol. 35, No. 2 October, 1970