

# Pacific Journal of Mathematics

**ON THE NUMBER OF FINITELY GENERATED  $\mathcal{O}$ -GROUPS**

DOUGLAS B. SMITH

## ON THE NUMBER OF FINITELY GENERATED 0-GROUPS

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Let  $K$  be a class of relational systems of a fixed similarity type,  $\aleph$  an infinite cardinal. A system  $\mathfrak{A}$  of cardinality  $\aleph$  is  $(\aleph, K)$ -weakly universal if each system in  $K$  of cardinality at most  $\aleph$  is isomorphically embeddable in  $\mathfrak{A}$ . The object of this note is to construct  $2^{\aleph_0}$  nonisomorphic finitely generated 0-groups and hence answer in the negative the following problem attributed to B. H. Neumann. Is there a group which is  $(\aleph_0, K_1)$ -weakly universal, where  $K_1$  is the class of 0-groups?

If  $\mathfrak{A}$  is  $(\aleph, K)$ -weakly universal and also a member of  $K$ , then  $\mathfrak{A}$  is  $(\aleph, K)$ -universal. It is known that  $(\aleph, K)$ -universal systems exist for many classes  $K$  and cardinals  $\aleph$ . In particular, Morley and Vaught established a useful condition for the existence of  $(\aleph, K)$ -universal systems for  $K$  an elementary class,  $\aleph$  an appropriate cardinal (see [7]). However there are no theorems of wide applicability concerning the existence of  $(\aleph_0, K)$ -universal systems; here the structure of the systems in  $K$  must be carefully analyzed. To illustrate this, consider the classes  $K_1$  of 0-groups;  $K_2$  of abelian 0-groups (i.e., torsion free abelian groups);  $K_3$  of ordered groups (i.e., groups of type  $\langle H, \cdot, \leq \rangle$  where  $\langle H, \cdot \rangle$  is an 0-group linearly ordered by  $\leq$ );  $K_4$  of abelian ordered groups. By applying the results in [7], (assuming the generalized continuum hypothesis), it is easily seen that there exists an  $(\aleph, K_i)$ -universal system for all  $\aleph > \aleph_0$  and  $i = 1, 2, 3$ , or 4.

The situation for  $\aleph = \aleph_0$  is more complicated. There is an  $(\aleph_0, K_2)$ -universal group (see [1, p. 64]). However, there is no ordered group which is  $(\aleph_0, K_4)$ -weakly universal and hence there is no  $(\aleph_0, K_3)$ -universal group. This follows readily from the fact that the free abelian group on two generators has  $2^{\aleph_0}$  nonisomorphic orders (see [2, p. 50]). Theorem 2, which establishes the nonexistence of a group which is  $(\aleph_0, K_1)$ -weakly universal, solves a problem of B. H. Neumann (see [2, p. 211, Problem 17]).

1. Definitions. An 0-group is a group  $G$  for which there exists a linear ordering relation  $\leq$  on  $G$  satisfying the following condition:

$a \leq b$  implies  $cad \leq cbd$  for all  $a, b, c, d \in G$ . For a group  $G$  the commutator of  $x$  and  $y$  in  $G$  is denoted  $[x, y] = x^{-1}y^{-1}xy$ ; for subsets  $A$  and  $B$  of  $G$ ,  $[A, B]$  is the subgroup of  $G$  generated by  $\{[a, b] : a \in A, b \in B\}$ ;  $G' = [G, G]$ ;  $G'' = [G', G']$ . Let  $F$  be the free

group generated by a set  $X$ ; a set  $R$  of equations of the form  $w_1 = w_2$ , where  $w_1$  and  $w_2$  are words in  $F$ , is a set of relations in  $X$ . A group  $G$  generated by the set  $X$  is given by a set  $R$  of defining relations if the following conditions are satisfied.

(i)  $R$  is a set of relations in  $X$ .

(ii) Let  $\varphi_G$  be the unique homomorphism from  $F$  onto  $G$  which extends the identity map on  $X$ . Then the kernel of  $\varphi_G$  is the normal subgroup of  $F$  generated by  $\{w_1 w_2^{-1} : w_1 = w_2 \in R\}$ .

2. **Finitely generated 0-groups.** In [4], P. Hall constructs  $2^{\aleph_0}$  nonisomorphic finitely generated groups  $H$  each having torsion-free center and satisfying the condition  $[H'', H] = 1$ . We will show that these groups are also 0-groups.

LEMMA 1. (*B. H. Neumann*). *Let  $G$  be an 0-group generated by a set  $X$  and given by a set  $R$  of defining relations; let  $H$  be a group generated by the set  $\{a\} \cup X$  where  $a \notin X$ , with the relations  $R$  and  $[a^{-n} b a^n, b'] = 1$  for all  $b, b' \in X, n = 1, 2, 3, \dots$  as a set of defining relations. Then  $H$  is an 0-group.*

*Proof.* See [6, pp. 10-11].

The next lemma is a slight variant of von Dyck's Theorem (see [5, p. 130]).

LEMMA 2. *Let  $G$  be a group generated by a set  $X$ , given by a set  $R$  of defining relations; let  $H$  be a group generated by  $X$ , given by the set  $R \cup S$  of defining relations. Then  $H$  is isomorphic to  $G/N$  where  $N$  is the normal subgroup of  $G$  generated by*

$$\{\varphi_G(w_1 w_2^{-1}) : w_1 = w_2 \in S\}.$$

THEOREM 1. *There exist  $2^{\aleph_0}$  nonisomorphic finitely generated 0-groups.*

*Proof.* In his construction, P. Hall used a group  $G$  satisfying the following conditions:

(1)  $G$  is generated by the set  $\{a, b\}$ . For notational convenience we will write  $b = b_0$  and

$$b_i = a^{-i} b a^i \quad i = 0, \pm 1, \pm 2, \dots$$

$G$  is given by the defining relations

$$[[b_i, b_j], b_k] = 1 \quad i, j, k = 0, \pm 1, \pm 2, \dots$$

$$[b_j, b_i] = [b_{j+k}, b_{i+k}] \quad i, j, k = 0, \pm 1, \pm 2, \dots$$

and  $i < j$ .

(2) the center  $Z$  of  $G$  is free abelian with generators

$$\{[b_i, b] : i = 1, 2, 3, \dots\}.$$

Let  $C$  be a denumerable torsion-free abelian group. Appealing to [4] (p. 433), we find that there is a set  $\{H_\iota : \iota < 2^{\aleph_0}\}$  of nonisomorphic groups satisfying the following conditions:

(3) the center  $C_\iota$  of  $H_\iota$  is isomorphic to  $C$  and  $H_\iota/C_\iota$  is isomorphic to  $G/Z$ ;

(4)  $[H_\iota'', H_\iota] = 1$  and each  $H_\iota$  is generated by two elements.

As is known (see [3, p. 94]), a group  $H$  is an 0-group if both its center  $C$  and the factor group  $H/C$  are 0-groups. But by (3), each  $C_\iota$  is an 0-group and  $H_\iota/C_\iota$  is isomorphic to  $G/Z$ . Hence, to verify that each  $H_\iota$  is an 0-group it suffices to show that  $G/Z$  is an 0-group.

Let  $B$  be a group generated by the set  $\{a, b\}$  and given by the defining relations occurring in (1) and the relations

$$(5) \quad [b_k, b_0] = 1 \quad \text{for } k = 1, 2, 3, \dots.$$

By Lemma 2,  $B$  is isomorphic to  $G/N$ , where  $N$  is the normal subgroup of  $G$  generated by

$$\{\mathcal{P}_G[b_k, b_0] : k = 1, 2, 3, \dots\} = \{[b_k, b_0] \in G : k = 1, 2, 3, \dots\}.$$

Applying (2), we have  $N = Z$  and  $B$  is isomorphic to  $G/Z$ . Furthermore,  $B$  is given by the defining relations (5) alone. For if we assume  $j > 0$  and use (5), then we have:

$$\begin{aligned} (a^j b a^{-j}) b &= (a^j b a^{-j}) b (a^j a^{-j}) \\ &= a^j (b (a^{-j} b a^j) a^{-j}) \\ &= a^j ((a^{-j} b a^j) b a^{-j}) \\ &= b (a^j b a^{-j}). \end{aligned}$$

Thus,

$$[b_k, b_0] = 1 \quad \text{for } k = 0, \pm 1, \pm 2, \dots.$$

A similar computation yields

$$[b_i, b_j] = 1 \quad \text{for } i, j = 0, \pm 1, \pm 2, \dots.$$

Hence the relations in (1) hold trivially.

Since the free group with generating set  $\{b\}$  is an 0-group, we can infer from Lemma 1 that the group  $B$  which is generated by

$\{a, b\}$  and given by the defining relations (5) is an 0-group; i.e.  $G/Z$  is an 0-group.

Since a countable group has only countably many finitely generated subgroups, we obtain our conclusion :

**THEOREM 2.** *There does not exist a group which is  $(\aleph_0, K_1)$ -weakly universal.*

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