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**CONCERNING THE DOMAINS OF GENERATORS OF LINEAR
SEMIGROUPS**

J. W. SPELLMANN

CONCERNING THE DOMAINS OF GENERATORS OF LINEAR SEMIGROUPS

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Let S denote a Banach space over the real numbers. Let A denote the infinitesimal generator of a strongly continuous semigroup T of bounded linear transformations on S . It is known that the Riemann integral $\int_a^b T(x) p dx$ is in the domain of A (denoted by $D(A)$) for each p in S and each nonnegative number interval $[a, b]$. This paper develops sufficient conditions on nonnegative continuous functions f and on elements p in S in order that the Riemann integral $\int_a^b T(f(x)) p dx$ be an element of the domain of A .

2. A change of variable technique. A change of variable theorem may sometimes be used to transform

$$\int_a^b T(f(x)) p dx \text{ to } \int_c^d T(x) (f^{-1})'(x) p dx$$

where f^{-1} denotes the inverse of f . This motivates the first theorem.

THEOREM 1. *Suppose $p \in S, 0 \leq c < d$ and h is a real valued function which has a continuous derivative on $[c, d]$. Then*

$$\int_c^d T(x) h(x) p dx$$

is in $D(A)$ and

$$A \int_c^d T(x) h(x) p dx = h(d) T(d) p - h(c) T(c) p - \int_c^d T(x) h'(x) p dx .$$

Proof.

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} [T(\varepsilon) - T(0)] \int_c^d T(x) h(x) p dx \\ &= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \int_{c+\varepsilon}^{d+\varepsilon} T(x) h(x - \varepsilon) p dx - \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \int_c^d T(x) h(x) p dx \\ &= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \int_d^{d+\varepsilon} T(x) h(x - \varepsilon) p dx - \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \int_c^{c+\varepsilon} T(x) h(x) p dx \\ &\quad - \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \int_{c+\varepsilon}^d T(x) [h(x) - h(x - \varepsilon)] p dx \\ &= h(d) T(d) p - h(c) T(c) p - \int_c^d T(x) h'(x) p dx . \end{aligned}$$

The second theorem then follows as an immediate consequence of Theorem 1.

THEOREM 2. *Suppose $p \in S$, $0 \leq a < b$, $0 \leq c < d$ and f is a continuous function from $[a, b]$ to $[0, \infty]$ so that*

(i) *$(f^{-1})''$ is continuous on $[c, d]$ and*

(ii) $\int_a^b T(f(x))p dx = \pm \int_c^d T(x)(f^{-1})'(x)p dx$. *Then $\int_a^b T(f(x))p$ is in $D(A)$ and*

$$A \int_a^b T(f(x))p dx = \pm \left[(f^{-1})'(d)T(d)p - (f^{-1})'(c)T(c)p - \int_c^d T(x)(f^{-1})''(x)p dx \right]$$

EXAMPLE 1. Suppose $0 \leq a < b$, m and k are real numbers so that $m \neq 0$ and $mx + k \geq 0$ for all $x \in [a, b]$. Then $\int_a^b T(mx + k)p dx$ is in $D(A)$ and

$$A \int_a^b T(mx + k)p dx = \frac{1}{m} [T(mb + k)p - T(ma + k)p].$$

It is noted that Theorem 2 says nothing about $\int_a^b T(f(x))p dx$ being in $D(A)$ if $\int_a^b T(f(x))p dx$ does not equal $\pm \int_c^d T(x)(f^{-1})'(x)p dx$ or if $(f^{-1})''$ is not continuous on $[c, d]$. A different approach is considered in the next section which sometimes allows for such exceptions.

3. A closed operator technique. In this section, the restrictions imposed on the function f in the hypothesis of Theorem 2 will be relaxed. In accomplishing this, additional restrictions will be placed on the point p mentioned in Theorem 2. The fact that the infinitesimal generator A of the semigroup T is a closed linear operator implies the next theorem.

THEOREM 3. *Suppose $p \in D(A)$, $0 \leq a < b$ and f is a continuous function from $[a, b]$ to $[0, \infty)$. Then*

$$\int_a^b T(f(x))p dx$$

is in $D(A)$ and

$$A \int_a^b T(f(x))p dx = \int_a^b T(f(x))Ap dx.$$

The fourth theorem follows from Example 1, properties of continuous real valued functions and the fact that the space S is complete.

THEOREM 4. *Suppose $p \in S$, $0 \leq a < b$ and f is a continuous function from $[a, b]$ to $[0, \infty)$. Suppose $\{f_n\}_{n=1}^\infty$ is a sequence of piecewise linear functions, each from $[a, b]$ to $[0, \infty)$, which converge uniformly to f on $[a, b]$. Then $\int_a^b T(f(x))p dx$ is in $D(A)$ whenever $\left\{A \int_a^b T(f_n(x))p dx\right\}_{n=1}^\infty$ is a Cauchy sequence in S . In this case*

$$A \int_a^b T(f(x))p dx = \lim_{n \rightarrow \infty} A \int_a^b T(f_n(x))p dx .$$

In order to develop useful corollaries to Theorem 4, we make the following definitions.

DEFINITION 1. Suppose $K = \{x_j\}_{j=0}^n$ is a partition of $[a, b]$ and f is a continuous real valued function defined on $[a, b]$. Then $[f; K]$ denotes the piecewise linear function defined on $[a, b]$ by the rule

$$[f; K](x) = [f(x_j) - f(x_{j-1})][(x_j - x_{j-1})^{-1}][x - x_{j-1}] + f(x_{j-1})$$

for $x \in [x_{j-1}, x_j]$, $j = 1, 2, \dots, n$.

DEFINITION 2. Suppose $0 < \alpha \leq 1$. Then $\mathcal{A}(\alpha)$ denotes the subset of S which contains p if and only if for each positive number r , there is a positive number $M(r)$ so that

$$\|T(x)p - p\| < x^\alpha M(r)$$

for all $x \in [0, r]$.

It is noted that $D(A) \subseteq \mathcal{A}(\alpha)$ for each $\alpha \in [0, 1]$. However, the next example illustrates that $\mathcal{A}(1)$ may not be a subset of $D(A)$.

EXAMPLE 2. Let S denote the Banach space of real valued functions which are bounded and uniformly continuous on $[0, \infty)$. For each $f \in S$, let

$$\|f\| = \text{lub}_{x \geq 0} |f(x)| .$$

Let T be the strongly continuous linear semigroup defined on S by the rule

$$[T(\beta)f](x) = f(\beta + x)$$

for each pair (β, x) in $[0, \infty) \times [0, \infty)$. Then f is in $D(A)$ if and only if f' is in S .

Let g be the function is S so that

$$g(x) = \begin{cases} 1 - x & \text{if } x \in [0, 1] \\ 0 & \text{if } x \geq 1 \end{cases}.$$

Then g is in $\mathcal{A}(1)$, but g is not in $D(A)$.

DEFINITION 3. Suppose $0 \leq a < b$ and each of $P_1 = \{x_j\}_{j=0}^n$ and $P_2 = \{t_k\}_{k=0}^m$ is a partition of $[a, b]$. The statement the P_2 is a doubling refinement of P_1 means that

- (1) $m = 2n$ and
- (2) $t_{2j} = x_j$ for $j = 0, 1, \dots, n$.

DEFINITION 4. Suppose $0 \leq a < b$, $\alpha \in [0, 1]$, $f: [a, b] \rightarrow [0, \infty)$ and $P = \{P_n\}_{n=1}^\infty = \{\{a_{nk}\}_{k=0}^{2^n-1}\}_{n=1}^\infty$ is a sequence of partitions of $[a, b]$ so that P_{n+1} is doubling refinement of P_n for each positive integer n . The statement that f satisfies condition $S(\alpha)$ relative to P means that

- (1) $\{[f; P_n]\}_{n=1}^\infty$ converges uniformly to f on $[a, b]$,
- (2) $f(a_{n,k+1}) \neq f(a_{n,k})$ for $n = 1, 2, \dots$ and $k = 0, 1, \dots, 2^n - 1$,
- (3) $\sum_{n=1}^\infty \sum_{k=0}^{2^n-1} |\Delta f_{n,k} - \Delta f_{n+1,2k}| |f(a_{n+1,2k+1}) - f(a_{n,k})|^\alpha$ converges and

(4) $\sum_{n=1}^\infty \sum_{k=0}^{2^n-1} |\Delta f_{n,k} - \Delta f_{n+1,2k+1}| |f(a_{n,k+1}) - f(a_{n+1,2k+1})|^\alpha$ converges where $\Delta f_{n,k} = [a_{n,k+1} - a_{n,k}][f(a_{n,k+1}) - f(a_{n,k})]^{-1}$ for n a positive integer, k an integer in the number interval $[0, 2^n - 1]$.

The next theorem is a useful corollary to Theorem 4.

THEOREM 5. Suppose $0 \leq a < b$, $0 < \alpha \leq 1$,

$$P = \{P_n\}_{n=1}^\infty = \{\{a_{nk}\}_{k=0}^{2^n-1}\}_{n=1}^\infty$$

a sequence of partitions of $[a, b]$ so that P_{n+1} is a doubling refinement of P_n for each positive integer n . Suppose $f: [a, b] \rightarrow [0, \infty)$ is continuous and satisfies condition $S(\alpha)$ relative to P . Then if

$$p \in \mathcal{A}(\alpha), \int_a^b T(f(x))p dx$$

is in $D(A)$ and

$$A \int_a^b T(f(x))p dx = \lim_{n \rightarrow \infty} A \int_a^b T([f; P_n](x))p dx.$$

Proof. The proof of Theorem 5 follows from Example 1 and Theorem 4.

The next theorem relaxes conditions on the function f mentioned

in Theorem 2. The conditions imposed on point p , however, will be more restrictive.

THEOREM 6. *Suppose $0 \leq a < b$, $p \in \mathcal{A}(1)$, $f: [a, b] \rightarrow [0, \infty)$ so that*

- (1) f' is continuous on $[a, b]$
- (2) $|f'(x)| > 0$ for all $x \in [a, b]$
- (3) f'' is bounded on $[a, b]$.

Then $\int_a^b T(f(x))pdx$ is in $D(A)$.

Proof. Let $P = \{P_n\}_{n=1}^\infty = \{\{a_{n,k}\}_{k=0}^{2^n}\}_{n=1}^\infty$ be a sequence of partitions of $[a, b]$ so that $a_{n,k} = a + (k2^{-n})(b - a)$. Then P_{n+1} is a doubling refinement of P_n for each positive integer n and $\{[f; P_n]\}_{n=1}^\infty$ converges uniformly to f on $[a, b]$. The mean value theorem and the hypothesis on f imply f satisfies conditive $S(1)$ relative to P . An application of Theorem 4 completes the proof.

It is noted that the same sequence P of partitions used in the proof of Theorem 6 may be used to show that $\int_a^b T(f(x))pdx$ is $D(A)$ whenever $p \in \mathcal{A}(1)$ and f is a nonconstant and nonnegative polynomial whose coefficients are either all positive or all negative.

The next example shows that hypothesis (i) of Theorem 2 is not a necessary condition for $\int_a^b T(f(x))pdx$ to be in $D(A)$.

EXAMPLE 3. Suppose $0 < b$, $\beta > 0$, m is a positive integer, $1 - 1/m < \alpha \leq 1$ and $p \in \mathcal{A}(\alpha)$. Let $f(x) = \beta x^m$ for $x \geq 0$. Then $(f^{-1})''$ is not continuous at 0. However, using the same sequence P as in the proof of Theorem 6, $\int_a^b T(f(x))pdx$ may be shown to be in $D(A)$.

The fourth example will indicate that hypothesis (ii) of Theorem 2 is not necessary for $\int_a^b T(f(x))pdx$ to be in $D(A)$.

EXAMPLE 4. Let C denote Cantor's ternary set (see p. 329 of [3]). For each x in the interval $[0, 1]$, let

$$C_x = \text{lub}(C \cap [0, x]).$$

Let w be the function defined on $[0, 1]$ by the rule

$$w(x) = {}_2(C_x \cdot 2^{-1})$$

where ${}_2(C_x \cdot 2^{-1})$ denotes the binary form of $(C_x \cdot 2^{-1})$. Hille and Tamarkin, in [2], have shown w to be continuous, nondecreasing and to have a zero derivative almost everywhere on $[0, 1]$. Let f be the function so that

$$f(x) = x + w(x) \text{ for } x \in [0, 1].$$

Then f is a strictly increasing function which fails to be absolutely continuous on $[0, 1]$. Thus, one would not expect the second condition of the hypothesis of Theorem 2 to hold. However, $\int_0^1 T(f(x))p$ is in $D(A)$ whenever p is in $\mathcal{A}(1)$. This is seen by using Theorem 4 and proper choice of partitions of $[0, 1]$. Let

$$M_0 = \{0\}, \quad N_0 = \{1\}$$

$$M_1 = \{3.022\dots\}^1$$

$$N_1 = \{3.200\dots\}$$

$$Q_1 = \{3.111\dots\}.$$

For each integer $m \geq 2$, let

$$M_m = \{3.a_1 \dots a_{m-1}022\dots\}$$

$$N_m = \{3.a_1 \dots a_{m-1}200\dots\}$$

$$Q_m = \{3.a_1 \dots a_{m-1}111\dots\}$$

where $a_i \in \{0, 2\}$ for $i = 1, 2, \dots, m-1$.

For each nonnegative integer n , let P_{2n} and P_{2n+1} denote the following partitions of $[0, 1]$.

$$P_{2n} = \left\{ \bigcup_{k=0}^n [M_k \cup N_k] \right\} \cup \left\{ \bigcup_{k=0}^n Q_k \right\}$$

$$P_{2n+1} = P_{2n} \cup Q_{n+1}.$$

Then if $p \in \mathcal{A}(1)$, it may be shown that

$$(1) \quad \left\| A \int_0^1 T([f; P_{2n}](x))p dx - A \int_0^1 T([f; P_{2n+1}](x))p dx \right\| = 0$$

$$(2) \quad \left\| A \int_0^1 T([f; P_{2n+1}](x))p dx - A \int_0^1 T([f; P_{2n+2}](x))p dx \right\| \leq \frac{M^2 2^{n+1}}{2^n + 3^n}.$$

Where M is a number so that

$$(3) \quad Mx \geq \|T(x)p - p\| \quad x \in [0, 2] \text{ and}$$

$$(4) \quad M \geq \|T(x)p\| \quad x \in [0, 2].$$

Thus $\left\{ A \int_0^1 T([f; P_n](x))p dx \right\}_{n=1}^{\infty}$ is a Cauchy sequence in S . Theorem 4 implies $\int_0^1 T(f(x))p dx$ is in $D(A)$ since $\{[f; P_n]\}_{n=1}^{\infty}$ converges uniformly to f on $[a, b]$.

REMARK ON EXAMPLE 4. If $t \in ([0, 1] - C) \cup (\bigcup_{n=0}^{\infty} P_n)$, the above technique may be used to show that $\int_0^t T(f(x))p dx$ is in $D(A)$. This is done by defining the following partitions P'_n of $[0, t]$. Let

¹ (3.0222...) denotes the triadic representation of $1/3$, etc.

$$P'_n = (P_n \cap [0, t]) \cup \{t\}$$

for each nonnegative integer n . If $t \in (C - \bigcup_{n=0}^{\infty} P_n)$ the following theorem may be used to show that $\int_0^t T(f(x))pdx$ is in $D(A)$.

THEOREM 7. *Suppose $0 \leq a < b$, $p \in \mathcal{A}(1)$ and f is a continuous, nonnegative, strictly monotone real valued function defined on $[a, b]$. Then there is a number M so that*

$$\left\| A \int_c^d T([f; P](x))pdx \right\| \leq (d - c)M$$

for each partition P of each subinterval $[c, d]$ of $[a, b]$.

Proof. The proof of Theorem 7 follows from Example 1, the fact that $T(x)p$ is a continuous function of x , on $(0, \infty)$ and the fact that the infinitesimal generator A is linear.

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