SELF-ADJOINT DIFFERENTIAL OPERATORS

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Let $\mathcal{H}$ denote the Hilbert space of square summable analytic functions on the unit disk, and consider the formal differential operator

$$L = \sum_{i=0}^{n} p_{i} D^{i}$$

where the $p_{i}$ are in $\mathcal{H}$. This paper is devoted to a study of symmetric operators in $\mathcal{H}$ arising from $L$. A characterization of those $L$ which give rise to symmetric operators $S$ is obtained, and the question of when such an $S$ is self-adjoint or admits of a self-adjoint extension is considered. If $A$ is a self-adjoint extension of $S$ and $E(\lambda)$ the associated resolution of the identity, the projection $E_{\lambda}$ corresponding to the interval $\Delta = (a, b]$ is shown to be an integral operator whose kernel can be expressed in terms of a basis of solutions for the equation $(L - \lambda)u = 0$ and a spectral matrix.

Let $\mathcal{A}$ denote the space of functions analytic on the unit disk and $\mathcal{H}$ the subspace of square summable functions in $\mathcal{A}$ with inner product

$$(f, g) = \int_{|z|<1} f(z)\overline{g(z)}dzd\gamma.$$ 

Then $\mathcal{H}$ is a Hilbert space with the reproducing property, i.e., for each $z$ there exists a unique element $K_{z}$ of $\mathcal{H}$ such that

$$f(z) = (f, K_{z}).$$

Moreover, if the sequence $\{f_{n}\}$ converges to $f$ in norm, $f_{n}(z)$ converges to $f(z)$ uniformly on compact subsets of the disk. A complete orthonormal set for $\mathcal{H}$ is provided by the normalized powers of $z$,

$$e_{n}(z) = [(n + 1)/\pi]^{1/2}z^{n}, \quad n = 0, 1, \cdots.$$ 

From this it follows that $\mathcal{H}$ is identical with the space of power series $\sum_{n=0}^{\infty} a_{n}z^{n}$ which satisfy

$$(1.1) \quad \sum_{n=0}^{\infty} |a_{n}|^{2}/(n + 1) < \infty.$$ 

Consider the formal differential operator

$$L = p_{n}D^{n} + \cdots + p_{1}D + p_{0},$$

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where \( D = d/dz \) and the \( p_i \) are in \( \mathcal{A} \). For \( f \) in \( \mathcal{H} \) the element \( Lf \) is in \( \mathcal{A} \), but not necessarily in \( \mathcal{H} \). To see this we take \( L = d/dz \) and \( f(z) = \sum_{n=1}^{\infty} n^{-1/2} z^n \), from (1.1) it follows that \( f \) is in \( \mathcal{H} \) but \( Lf \) is not. In order to consider \( L \) as an operator in \( \mathcal{H} \) we must restrict the class of functions on which \( L \) acts in some suitable manner. Since our concern is with densely defined operators it is only natural to demand that powers of \( z \) be mapped into \( \mathcal{H} \). This requires some restrictions on the coefficients of \( L \). As an example consider the operator \( L = pD \) where \( p(z) = \sum_{n=0}^{\infty} (n+1)z^n \).

We have \( Le_k(z) = k(k+1)^{1/2}\pi^{-1/2} \sum_{n=k}^{\infty} (n-k)^{1/2} z^n \), and hence \( Le_k \notin \mathcal{H} \). A sufficient condition for the \( Le_k \) to be in \( \mathcal{H} \) is that the coefficients \( p_i \) be in \( \mathcal{H} \).

Let \( L = \sum_{i=0}^{\infty} p_i D^i \), where the \( p_i \) are in \( \mathcal{H} \), and let \( \mathcal{D}_0 \) denote the span of the \( e_k \) and \( \mathcal{D} \) the set of all \( f \) in \( \mathcal{H} \) for which \( Lf \) is in \( \mathcal{H} \). We now define the operators \( T_0 \) and \( T \) as follows.

\[
T_0 f = Lf \quad f \in \mathcal{D}_0 ,
\]
\[
T f = Lf \quad f \in \mathcal{D} .
\]

**Theorem 1.1.** \( T_0 \) and \( T \) are densely defined operators with range in \( \mathcal{H} \), \( T_0 \subseteq T \), and \( T \) is closed.

*Proof.* We first show that \( T \) is closed. Let \( \{f_n\} \) be a sequence of functions in \( \mathcal{D} \) such that \( f_n \rightarrow f \) and \( Tf_n \rightarrow g \), hence \( f_n(z) \) and \( Lf_n(z) \) converge uniformly on compact subsets to \( f(z) \) and \( g(z) \) respectively. But \( Lf_n(z) \) also converges to \( Lf(z) \). Hence \( Lf(z) = g(z) \), \( |z| < 1 \), so \( Tf \in \mathcal{H} \) and \( Tf = g \).

Since \( \mathcal{D}_0 \) is dense in \( \mathcal{H} \) and \( T_0 f = Tf \) for \( f \in \mathcal{D}_0 \cap \mathcal{D} \) it suffices to show that the \( e_j \) are in \( \mathcal{D} \). Since \( Le_j = \sum_{i=0}^{\infty} p_i D^i e_j \) and \( p_i D^i e_j \) is either zero or of the form \( p_i e_k \) for some nonnegative integer \( k \), it suffices to show that \( p_i e_k \in \mathcal{H} \). Let \( p_i = \sum_{j=0}^{\infty} a_j e_j \), a simple computation yields

\[
e_k e_j = [(k+1)\pi]^{1/2}[(j+1)/(j+k+1)]^{1/2} e_{j+k} ,
\]
and consequently,

\[
\|e_k p_i\| \leq [(k+1)\pi] \|p_i\| < \infty .
\]

\( T_0 \) and \( T \) are respectively the minimal and maximal operators in \( \mathcal{H} \) associated with the formal differential operator \( L \). We now proceed to study the class of formal differential operators for which \( T_0 \) is symmetric.

It is clear that the operator \( T_0 \) associated with the formal differential operator \( L \) is symmetric if and only if
We shall refer to those formal operators satisfying (1.2) as formally symmetric. As an example we have the real Euler operator
\[ L = \sum_{i=0}^{n} a_i z^i D^i , \]
a_i real. Then \( L e_j = p(j) e_j \) where \( p \) is the characteristic polynomial
\[ p(x) = a_0 + a_1 x + \cdots + a_n x(x-1) \cdots (x-n+1) . \]
Since \( p(j) = \overline{p(j)} \), \( L \) is formally symmetric. A characterization of formally symmetric \( L \) in terms of the coefficients \( p_i \) is given in the next section. We now proceed to the consideration of the adjoint operators \( T_0^* \) and \( T^* \). In what follows we shall make use of the result that if \( L \) is formally symmetric of order \( n \), then the coefficients \( p_i \) are polynomials of degree at most \( n + i \), \( i = 0, 1, \ldots, n \). A proof of this is given in Theorem 2.2.

**Theorem 1.2.** If \( T_0 \) is symmetric, \( T_0^* = T \) and \( T^* \subseteq T \). The closure of \( T_0, S \), is self adjoint if and only if \( S = T \).

**Proof.** By Theorem 2.2 the coefficients \( p_i \) are polynomials of degree at most \( n + i \). This implies that \( T_0 \) maps \( \mathcal{D}_0 \) into itself. In particular,
\begin{align*}
L e_m &= \sum_{i=0}^{m} a_i e_i , \quad 0 \leq m \leq n , \\
L e_{m+j} &= \sum_{i=j}^{2n+j} a_i e_i , \quad j = 1, 2, \ldots .
\end{align*}
Using this we show that \( T_0^* \subseteq T \). Let \( g = \sum_{i=0}^{\infty} a_i e_i \) and \( g^* = \sum_{j=0}^{\infty} b_j e_j \) be in the graph of \( T_0^* \) and consider the sequence \( \{ g_p \} \) in \( \mathcal{D}_0 \) defined as \( g_p = \sum_{j=0}^{p} a_j e_j \). Since \( g_p \to g \) we have \((T_0 e_k, g_p) \to (T_0 e_k, g) = (e_k, g^*)\). Hence \((e_k, T_0 g_p) \to (e_k, g^*)\). Now \( Lg \) is in \( \mathcal{D} \) and \( T_0 g_p \) converges to \( Lg \) uniformly on compact subsets. Since the \( e_j \) are just the normalized powers of \( z \), the power series expansion of \( Lg \) can be written as \( \sum_{j=0}^{\infty} c_j e_j(z) \). Since \( Lg_p(z) = \sum_{j=0}^{p} a_j L e_j(z) \) converges uniformly to \( \sum_{j=0}^{\infty} c_j e_j(z) \), it follows from (1.3) that \( Lg_p \) has the same coefficient of \( e_m \) as does \( Lg \) for \( p > n + m + 1 \). Hence \((e_m, T_0 g_p) = \bar{c}_m \) for \( p > n + m + 1 \) and since \((e_m, T_0 g_p) \to (e_m, g^*)\) we have \( c_m = b_m \). Therefore \( g^* = Lg \), so that \( g \in \mathcal{D} \) and \( g^* = Tg \).

To show that \( T \subseteq T_0^* \) it will suffice to show that \((T_0 e_m, g) = (e_m, Tg)\) for all \( g \) in \( \mathcal{D} \) and \( m = 0, 1, \ldots \). Let \( g = \sum_{i=0}^{\infty} a_i e_i \) be in \( \mathcal{D} \) and \( g_p \) as before. Since \( T_0 \) is symmetric and \( g_p \to g \) we have \((e_m, T_0 g_p) = (T_0 e_m, g_p) \to (T_0 e_m, g)\). By precisely the same argument
as before \((e_m, T_0 g_p) = (e_m, T g)\) for \(p > n + m + 1\), from which it follows that \((e_m, T g) = (T_0 e_m, g)\) and \(T_0^* = T\). Since \(T_0 \subseteq T\), \(T^* \subseteq T_0^* = T\).

The closure \(S\) of the symmetric operator \(T_0\) is given by \(T_0^{**} = T^* \subseteq T\). Since \(T\) is closed \(T^{**} = T\), from which it follows that \(S^* = T\). Hence \(S = T\) implies \(S = S^*\). Conversely if \(S\) is self-adjoint we have \(S = T^* = S^* = T\).

A sufficient condition for \(T\) to be self-adjoint is given by the following theorem.

**Theorem 1.3.** For \(f = \sum_{i=0}^\infty \alpha_i e_j\) set \(f_m = \sum_{j=0}^m \alpha_j e_j\). If \(\sup_{m} ||T f_m|| < \infty\) for each \(f\) in \(\mathcal{D}\), then \(S\) is self-adjoint.

**Proof.** Since \(T^* \subseteq T\), \(T\) symmetric implies \(T = T^*\) and hence \(S = S^*\). We show that \((T f, g) - (f, T g)\) vanishes for all \(f, g\) in \(\mathcal{D}\). If \(L\) is of order \(n\) we have \((T f_m, g_p) = (T f, g_p)\) for \(m > n + p + 1\). Using this fact and the symmetry of \(T_0\) we obtain

\[
(T f, g_k) = (T f_{kn+n+1}, g_{kn}) = (f_{kn+n+1}, T g_{kn})
\]

\[
= (f_{kn-n-1}, T g_{kn}) + (f_{kn+n+1} - f_{kn-n-1}, T g_{kn})
\]

\[
= (f_{kn-n-1}, T g) + (f_{kn+n+1} - f_{kn-n-1}, T g_{kn})
\]

\(k = 1, 2, \ldots\).

Therefore,

\[
(T f, g) - (f, T g) = \lim_{k \to \infty} (f_{kn+n+1} - f_{kn-n-1}, T g_{kn}).
\]

Since the \(T g_{kn}\) are bounded in norm this implies \((T f, g) - (f, T g) = 0\).

**Corollary.** If \(L\) is a formally symmetric Euler operator, then \(S\) is self-adjoint.

**Proof.** For \(f = \sum_{i=0}^\infty b_i e_j\) in \(\mathcal{D}\), \(T f\) and \(T f_m\) are given by \(\sum_{i=0}^\infty p(j) b_i e_j\) and \(\sum_{i=0}^m p(j) b_i e_j^i\) respectively, where \(p(x)\) is the characteristic polynomial for \(L\). Hence

\[
||T f_m||^2 = \sum_{j=0}^m p(j)^2 ||b_j||^2 \leq ||T f||^2,
\]

and the result follows.

2. Formal considerations. The formal operator \(L = \sum_{i=0}^n p_i D^i\) is formally symmetric if

\[(L e_n, e_m) = (e_n, L e_m), n, m = 0, 1, \ldots .\]

To obtain a characterization of the formally symmetric operators
in terms of their coefficients we first determine the action of $L$ on $e_k$.

**Lemma 2.1.** Let $L = \sum_{i=0}^{n} p_i D^i$ where $p_i(z) = \sum_{k=0}^{\infty} a_k(i) z^k$. Then $L e_i = \sum_{j=0}^{\infty} c_{ij} e_j$ where

$$
c_{ij} = A(i, j) \sum_{k=0}^{n} B(i, k) a_{j-k+i}(k), \quad i, j = 0, 1, \ldots
$$

(2.1)

$$
A(i, j) = [(i + 1)/(j + 1)]^{i/2},
B(i, k) = i!/i - k! \quad i \geq k
= 0 \quad i < k.
$$

**Proof.** Consider the elementary operators $L_{pq} = z^p D^q$, $p, q = 0, 1, \ldots$. A simple calculation yields

$$
L_{pq} e_m = B(m, q) A(m, m + p - q) e_{m+p-q}.
$$

Now consider $L e_m$ (as an element of $\mathcal{A}$),

$$
L e_m(z) = \sum_{i=0}^{n} \sum_{k=0}^{\infty} a_k(i) L_{ki} e_m(z)
$$

$$
= \sum_{i=0}^{n} \sum_{k=0}^{\infty} a_{k-m+i}(i) B(m, i) A(m, k) e_k(z)
$$

$$
= \sum_{k=0}^{\infty} c_{mk} e_k(z) \quad |z| < 1.
$$

But $e_k(z)$ is just a multiple of $z^k$, therefore it follows from the uniqueness of power series representation of elements of $\mathcal{A}$, that \( \sum_{k=0}^{\infty} c_{mk} e_k \) converges to $T e_m$ in $\mathcal{A}$.

It follows that $L$ is formally symmetric if and only if the coefficients $a_k(\zeta), \zeta, k = 0, 1, \ldots$, satisfy the linear system

(2.2)

$$
c_{ij} = \overline{c_{ji}}, \quad i, j = 0, 1, \ldots.
$$

The following provides a simplification of the system (2.2).

**Theorem 2.2.** If $L = \sum_{i=0}^{n} p_i D^i$ is formally symmetric the $p_i$ are polynomials of degree at most $n + i$.

**Proof.** Consider $c_{n+p,0}$ for $p \geq 1$. Since $j - n - p < 0$ for $p \geq 1$ and $j = 0, \ldots, n$, $a_{j-n-p}(j) = 0$. Consequently $c_{n+p,0} = \overline{c_{0,n+p}}$ reduces to $A(0, n + p)a_{n+p}(0) = 0$, $p \geq 1$, and $p_0$ is of degree at most $n$. We now proceed inductively. Consider

(2.3)

$$
c_{n+p,k+1} = \overline{c_{k+1,n+p}}, \quad p \geq k + 2.
$$
Since \( k + 1 + j - n - p < 0 \) for \( p \geq k + 2 \) and \( j = 0, \ldots, n \), (2.3) reduces to

\[
A(k + 1, n + p) \sum_{j=0}^{k} B(k + 1, j) a_{n+p+j-k-1}(j) = 0, \quad p \geq k + 2.
\]

Since \( n + p + j - k - 1 \geq n + j + 1 \), it follows from the inductive hypothesis that \( a_{n+p+j-k-1}(j) = 0 \) for \( j = 0, \ldots, k \), and hence

\[
A(k + 1, n + p)(k + 1)! a_{n+p}(k + 1) = 0, \quad p \geq k + 2.
\]

Therefore degree \( p_{k+1} \leq n + k + 1 \).

This result allows a considerable simplification of the system (2.2). For each nonnegative integer \( p \) consider the subsystem \( S_p \) of (2.2)

\[
c_{i,i+p} = \bar{c}_{i+p,i}, \quad i = 0, 1, \ldots.
\]

Since the equation \( c_{ij} = \bar{c}_{ji} \) appears only in \( S_{i-j} \), we have a partition of (2.2). Since the \( p_i \) are polynomial of degree at most \( n + i \),

\[
a_{i+p}(\ell) = 0 \quad p > n, \quad \ell = 0, \ldots, n,
\]

from which it follows that \( S_p \) is trivial for \( p > n \). From (2.1) we see that \( a_{\ell}(i) \) appears only in \( S_{\ell-i} \). Hence (2.2) is equivalent to the \( n + 1 \) systems,

\[
S_p: c_{i,i+p} = \bar{c}_{i+p,i}, \quad i = 0, 1, \ldots,
\]

where the \( a_{i+p}(j) \) appear only in \( S_p \). Using (2.1) this becomes

\[
S_p: \sum_{k=0}^{n} a_{p+k}(k) B(i,k) = \sum_{k=p}^{n} \bar{a}_{k-p}(k) B(i+p,k) A^2(i+p,i).
\]

**Theorem 2.3.** The system \( S_p \) is satisfied if and only if

\[
j! a_{j+p}(j) = R^j_i, \quad j = 0, 1, \ldots, n,
\]

where \( R^j_i = \sum_{k=p}^{n} \bar{a}_{k-p}(k) B(i+p,k) A^2(i+p,i) \), and the \( R^j_i \) are obtained recursively by

\[
R^j_i = R^{j-1}_{i+1} - R^{j-1}_i.
\]

**Proof.** For fixed \( p \) denote the left and right hand sides of the \( i \)th member of \( S_p \) by \( L^i_i \) and \( R^i_i \) respectively. We now employ a reduction scheme. Form the sequence of systems \( \{L^i_i = R^i_i\}, \{L^i_i = R^i_i\}, \ldots \), where

\[
L^{i+1}_i = L^{i+1}_{i+1} - L^i_i, \quad R^{i+1}_i = R^{i+1}_{i+1} - R^i_i, \quad i, j = 0, 1, \ldots.
\]
By induction on $j$ it can be shown that

$$L_i^j = \sum_{k=0}^{n} a_{k+p}(k) B(i, k - j) P_j(k)$$

where $P_j(k) = k(k - 1) \cdots (k - j + 1)$. Consequently, $L_i^j = j! a_{i+p}(j)$ and the necessity follows.

For the sufficiency we use the fact that for a given system of linear equations, $L_j^i = R_j^i$, $j = 0 \ldots, n$, there exists a unique set of linear systems $\{\hat{L}_i^j = \hat{R}_i^j\}, \ldots, \{\hat{L}_i^n = \hat{R}_i^n\}$ which have the properties $P1$ thru $P3$.

$$P1 \ \hat{L}_i^j = \hat{L}_i^{j-1} - \hat{L}_i^{j-1} \quad j = 1, \ldots, n$$

$$\hat{R}_i^j = \hat{R}_i^{j-1} - \hat{R}_i^{j-1} \quad i = 0, 1, \ldots$$

$$P2 \ \hat{L}_i^0 = L_i^0, \hat{R}_i^0 = R_i^0 \quad j = 0, \ldots, n$$

$$P3 \ \hat{L}_i^n = L_i^n, \hat{R}_i^n = R_i^n \quad i = 0, 1, \ldots$$

This set is constructed in the following manner.

The system $\{\hat{L}_i^j = \hat{R}_i^j\}$ is defined by $P3$. To satisfy $P1$ and $P2$ we define the system $\{\hat{L}_i^{n-1} = \hat{L}_i^{n-1}\}$ inductively by $\hat{L}_i^{n-1} = L_i^{n-1}$, $\hat{R}_i^{n-1} = R_i^{n-1}$, $\hat{L}_i^{n-1} = \hat{L}_i^{n-1} + L_i^n$, and $\hat{R}_i^{n-1} = \hat{R}_i^{n-1} + R_i^n$. Similarly we define the system $\{\hat{L}_i^{n-2} = \hat{R}_i^{n-2}\}$ through $\{\hat{L}_i^n = \hat{R}_i^n\}$ by means of the equations

$$\hat{L}_i^{n-2} = L_i^{n-2}, \hat{R}_i^{n-2} = R_i^{n-2}$$

$$\hat{L}_i^{n-2} = \hat{L}_i^{n-2} + \hat{L}_i^{n-3}, \hat{R}_i^{n-2} = \hat{R}_i^{n-2} + \hat{R}_i^{n-1}$$

$$\hat{L}_i^0 = L_i^0, \hat{R}_i^0 = R_i^0$$

$$\hat{L}_i^0 = \hat{L}_i^0 + \hat{L}_i^1, \hat{R}_i^0 = \hat{R}_i^0 + \hat{R}_i^1$$

From the method of construction the systems $\{\hat{L}_i^j = \hat{R}_i^j\}$ thru $\{\hat{L}_i^n = \hat{R}_i^n\}$ are the unique systems satisfying $P1$ thru $P3$.

Since $P_j(k)$ vanishes for $0 \leq k \leq j - 1$ it follows that $L_i^j = 0$ for $j > n$ and all $i$. Moreover, for $j = n$ we have $L_i^n = n! a_{n+p}(n)$, a constant independent of $i$. From (2.4) we see that $R_i^n = \sum_{k=p}^{n} \tilde{a}_{k-p}(k) C_k(i)$, where the $C_k(i)$ are polynomials in $i$ of degree $k$. Hence $R_i^n = R_i^{n+1} - R_i^n$ can be written in the form $\sum_{k=p}^{n} \tilde{a}_{k-p}(k) C_k(i)$, where the $C_k(i)$ are of degree $k - 1$. Continuing in this manner we obtain

$$R_i^j = 0 \quad j > n \quad i = 0, 1, \ldots$$

$$R_i^n = R_i^n \quad i = 0, 1, \ldots$$

Hence the systems $\{L_i^j = R_i^j\}$ $j = 0, \ldots, n$ satisfy $P1$ thru $P3$ where $L_i^j = R_i^j$ corresponds to the $L_j^i = R_j^i$ and the system $\{\hat{L}_i^j = \hat{R}_i^j\}$ corresponds to the system $S_p$. This yields the sufficiency.

This theorem provides an algorithm for determining all formally
symmetric operators of a given order. As an application we give the general formally symmetric first order operator. Use of 2.5 for \( p = 0 \) and 1 yields

\[
L = (cz^2 + ax + \bar{c})d/dz + (2cz + b),
\]

where \( a \) and \( b \) are real.

3. Self-adjoint extensions. The operator \( S \) has another characterization which will be of use in the study of self-adjoint extensions. For \( f \) and \( g \) in \( \mathcal{D} \) consider the bilinear form

\[
\langle fg \rangle = (Lf, g) - (f, Lg),
\]

and let \( \mathcal{D} \) be the set of those \( f \) in \( \mathcal{D} \) for which \( \langle fg \rangle = 0 \) for all \( g \) in \( \mathcal{D} \). Since \( S = T^* \) and \( \mathcal{D}(T^*) = \mathcal{D} \), \( S \) has domain \( \mathcal{D} \).

Let \( \mathcal{D}^+ \) and \( \mathcal{D}^- \) denote the set of all solutions of the equations \( Lu = iu \) and \( Lu = -iu \) respectively, which are in \( \mathcal{H} \). It is known from the general theory of Hilbert space [3, p. 1227-1230] that

\[
\mathcal{D} = \mathcal{D} + \mathcal{D}^+ + \mathcal{D}^-,
\]

and every \( f \in \mathcal{D} \) has the unique representation

\[
f = \tilde{f} + f^+ + f^- , \quad (\tilde{f} \in \mathcal{D}, f^+ \in \mathcal{D}^+, f^- \in \mathcal{D}^-).\]

Let the dimensions of \( \mathcal{D}^+ \) and \( \mathcal{D}^- \) be \( m^+ \) and \( m^- \) respectively. Clearly, \( m^+ \) and \( m^- \) cannot exceed the order of \( L \). These integers are referred to as the deficiency indices of \( S \), and \( S \) has self-adjoint extensions if and only if \( m^+ = m^- \). Moreover \( S \) is itself self-adjoint if and only if \( m^+ = m^- = 0 \).

We assume that \( m^+ = m^- = m \) and seek to characterize all self-adjoint extensions of \( S \). Von Neumann has shown that the self-adjoint extensions of \( S \) are in a one-to-one correspondence with the unitary operators \( U \) of \( \mathcal{D}^+ \) onto \( \mathcal{D}^- \). Corresponding to any such \( U \) there exists a self-adjoint extension \( A \) of \( S \) whose domain is the set of all \( f \in \mathcal{D} \) which are of the form

\[
f = \tilde{f} + (I - U)f^+, \quad (f \in \mathcal{D}, f^+ \in \mathcal{D}^+),
\]

where \( I \) is the identity operator on \( \mathcal{D}^+ \). Conversely every such \( A \) has a domain of this type.

We now introduce the notion of abstract boundary conditions and indicate how the domain of any self-adjoint extension of \( S \) can be obtained. A boundary condition is a condition on \( f \in \mathcal{D} \) of the form

\[
\langle fh \rangle = 0 ,
\]
where \( h \) is a fixed function in \( \mathcal{D} \). The conditions

\[
\langle fh_j \rangle = 0, \quad j = 1, \ldots, n,
\]

are said to be linearly independent if the only set of complex numbers \( \alpha_1, \ldots, \alpha_n \) for which

\[
\sum_{j=1}^{n} \alpha_j \langle fh_j \rangle = 0
\]

identically in \( f \in \mathcal{D} \) is \( \alpha_1 = \cdots = \alpha_n = 0 \). A set of \( n \) linearly independent boundary conditions \( \langle fh_j \rangle = 0, \ j = 1, \ldots, n, \) is said to be self-adjoint if \( \langle h_jh_k \rangle = 0, \ j, k = 1, \ldots, n. \)

The following theorem follows directly from the proof of Theorem 3 in the paper of Coddington [1].

**Theorem 3.1.** If \( A \) is a self-adjoint extension of \( S \) with domain \( \mathcal{D}_A \), then there exists a set of \( m \) self-adjoint boundary conditions,

\[
\langle fh_j \rangle = 0, \quad j = 1, \ldots, m,
\]

such that \( \mathcal{D}_A \) is the set of all \( f \in \mathcal{D} \) satisfying these conditions. Conversely, if (3.3) is a set of \( m \) self-adjoint boundary conditions, there exists a self-adjoint extension \( A \) of \( S \) whose domain is the set of all \( f \in \mathcal{D} \) satisfying (3.3)

Let \( \phi_1, \cdots, \phi_m \) and \( \psi_1, \cdots, \psi_m \) be orthonormal sets for \( \mathcal{D}_+ \) and \( \mathcal{D}_- \) respectively and \( (u_{jk}) \) a unitary matrix representing \( U \), then the \( h_j \) are given by

\[
h_j = \phi_j - \sum_{k=1}^{m} u_{jk} \psi_k, \quad j = 1, \ldots, m.
\]

Let \( A \) be a self-adjoint operator associated with \( L \) and \( E(\lambda) \) the corresponding resolution of the identity. We shall show the projection \( E_j \) corresponding to \( \Delta = (a, b] \) can be expressed as an integral operator with a kernel given in terms of a basis of solutions for \( Lu - \lambda u = 0 \) and a certain spectral matrix. Our work was inspired by the treatment of E. A. Coddington [2] of the case when \( A \) arises from a formal differential operator in the space \( L_2(I) \), \( I \) an open interval. We begin by showing that the resolvent operator of \( A \),

\[
R(\zeta) = (A - \zeta)^{-1}, \quad \text{Im}(\zeta) \neq 0,
\]

is an integral operator with a nice kernel.
THEOREM 3.2. $R(\zeta)$ is an integral operator with kernel $K$,
\begin{equation}
R(\zeta)f(z) = \int_{|w|<1} K(z, w, \zeta)f(w)dudv, \ f \in \mathcal{H}.
\end{equation}

$K$ is jointly analytic in $z$, $\bar{w}$, and $\zeta$ on the region $|z|<1$, $|w|<1$, $\operatorname{Im}(\zeta) \neq 0$.

Moreover, $K(z, w, \zeta) = \overline{K(w, z, \zeta)}$ and
\begin{equation}
(L - \zeta)K(w, z, \zeta) = K_z(w), \text{ for fixed } z \text{ and } \zeta.
\end{equation}

Proof. Since $R(\zeta)f(z) = (R(\zeta)f, K_\zeta)$ and $R^*(\zeta) = R(\overline{\zeta})$, it follows that (3.1) holds with $K(z, w, \zeta) = R(\zeta)K(w)$. Hence $K$ is analytic in $\overline{w}$ for fixed $z$ and $\zeta$. That $K(z, w, \zeta) = \overline{K(w, z, \zeta)}$ can be seen from the following computations,
\[
K(z, w, \zeta) = (R(\zeta)K_z, K_w) = (\overline{K_z}, R(\zeta)K_w) = \overline{K(w, z, \zeta)}.
\]

Hence $K(z, w, \zeta)$ is analytic in $z$ for fixed $w$ and $\zeta$. It follows from the analyticity of $R(\zeta)$ for $\operatorname{Im}(\zeta) \neq 0$ that $K(z, w, \zeta) = (R(\zeta)K_z, K_w)$ is analytic in $\zeta$ for fixed $z$ and $w$ on any region for which $\operatorname{Im}(\zeta) \neq 0$. Since analyticity in each of the variables separately implies joint analyticity it only remains to verify (3.6). This follows from the fact that $K(w, z, \zeta) = \overline{K(z, w, \zeta)} = R(\zeta)K_\zeta(w)$.

We now split the kernel $K(z, w, \zeta)$ into two parts one of which satisfies the homogeneous equation $(L - \zeta)u = 0$. Since the coefficients of $L$ are polynomials, $p_n$ has at most a finite number of zeros in the unit disk. Introducing radial branchcuts at these zeros, we obtain the region $\tilde{D}$, simply connected relative to $D$, in which $p_n$ never vanishes. Let $z_0 \in \tilde{D}$, it follows from standard theorems that there exists a basis of solutions for the equation $(L - \zeta)\psi = 0$ such that:

1. $\psi_i(\zeta)$, $i = 1, \ldots, n$, are single-valued analytic functions on $\tilde{D}$
2. $\psi_i^{(j-1)}(z_0, \zeta) = \delta_{ij}$, $i, j = 1, \ldots, n$
3. $\psi_i(w, \zeta)$, $i = 1, \ldots, n$, is entire in $\zeta$ for each $w \in \tilde{D}$

THEOREM 3.3. The kernel $K(z, w, \zeta)$ has the representation
\begin{equation}
K(z, w, \zeta) = \sum_{i,j=1}^n \psi_{ij}(\zeta)\phi_i(z, \zeta)\overline{\phi_j(w, \zeta)} + G(z, w, \zeta),
\end{equation}
where $G(z, w, \zeta)$ is entire in $\zeta$ for fixed $z$ and $w$.

Proof. For fixed $z \in \tilde{D}$ and $\operatorname{Im}(\zeta) \neq 0$ it follows from (3.6) that
(3.8) \[ K(w, z, \overline{z}) = \sum_{j=1}^{n} \psi_j(z, \epsilon) \overline{\phi_j(w, \overline{z})} + \Omega(z, w, \overline{z}) , \]

where \( \Omega(z, w, \overline{z}) \) is the particular solution furnished by the variation of parameters method and is entire in \( \overline{z} \) for fixed \( z, w \). Moreover,

(3.9) \[ \frac{\partial^{i-1}}{\partial w^{i-1}} \Omega(z, z_0, \overline{z}) = 0 , \quad i = 1, \ldots, n . \]

Now consider the differential equation \((L_z - \epsilon)K(z, w, \epsilon) = K_w(z)\), where \( L_z \) denotes the fact that \( L \) is applied with respect to \( z \). Differentiating with respect to \( \bar{w} \) and making use of the symmetry of \( K \) we obtain

\[ (L_z - \epsilon) \frac{\partial^{i-1}}{\partial \bar{w}^{i-1}} K(w, z, \overline{z}) = \frac{\partial^{i-1}}{\partial \bar{w}^{i-1}} K_w(z) , \quad j = 1, \ldots, n . \]

Using (3.8), (3.9) and the relationships

\[ \phi_i^{(j-1)}(z_0, \epsilon) = \delta_{ij} \]

we obtain

\[ (L_z - \epsilon) \overline{\psi_j(z, \epsilon)} = \frac{\partial^{i-1}}{\partial \bar{w}^{i-1}} K_{z_0}(z) . \]

Variation of parameters yields

(3.10) \[ \psi_j(z, \epsilon) = \sum_{i=1}^{n} \overline{\psi_{ij}(\epsilon)} \overline{\phi_i(z, \epsilon)} + \Omega_j(z, \epsilon) , \quad j = 1, \ldots, n , \]

where the \( \Omega_j(z, \epsilon) \) are entire in \( \epsilon \) for fixed \( z \) and satisfy

(3.11) \[ \frac{\partial^{i-1}}{\partial z^{i-1}} \Omega_j(z_0, \epsilon) = 0 , \quad i, j = 1, \ldots, n . \]

It follows from (3.8) and (3.10) that (3.7) holds where

\[ G(z, w, \epsilon) = \overline{\Omega(z, w, \overline{\epsilon})} + \sum_{j=1}^{n} \Omega_j(z, \epsilon) \overline{\phi_j(w, \overline{\epsilon})} \]

is entire in \( \epsilon \) for each \( z, w \in \bar{D} \).

Concerning the matrix \( \psi = (\psi_{ij}) \) we have the following.

**Theorem 3.4.** The matrix \( \psi \) is analytic for \( \text{Im} \epsilon \neq 0 \), \( \psi^*(\epsilon) = \psi(\overline{\epsilon}) \), and \( \text{Im} \psi(\epsilon)/\text{Im} \epsilon \geq 0 \), where \( \text{Im} \psi = (\psi - \psi^*)/2i \).

**Proof.** It follows from (3.9) and (3.10) that
(3.12) \[ \psi_{ij}(\zeta) = \frac{\partial^{i+j-2}}{\partial z^{i-1}\partial \bar{w}^{j-1}} K(z_0, z_0, \zeta) , \quad i, j = 1, \ldots, n , \]
and hence \( \psi \) is analytic for \( \text{Im} (\zeta) \neq 0 \). Using (3.12) and the symmetry of \( K \) we obtain \( \psi_{ij}(\zeta) = \overline{\psi_{ji}(\zeta)} \).

In order to demonstrate the positivity of \( \text{Im} \psi(\zeta)/\text{Im} (\zeta) \geq 0 \) we consider the functionals \( \xi_k \) defined by \[ \xi_k(f) = f^{(k-1)}(z_0) , \quad f \in \mathcal{H} , \quad k = 1, \ldots, n . \]

Since convergence in \( \mathcal{H} \) implies uniform convergence on compact subsets, the \( \xi_k \) are bounded linear functional on \( \mathcal{H} \). Consequently there exist functions \( K_1, \ldots, K_n \) in \( \mathcal{H} \) for which \[ f^{(k-1)}(z_0) = (f, K_k) \]
all \( f \) in \( \mathcal{H} \). Let \( \xi_1, \ldots, \xi_n \) be any set of \( n \) complex numbers and consider the function \( f = \sum_{k=1}^{n} \xi_k K_k \). The inner product \( (R(\zeta)f, f) = \sum_{k=1}^{n} \xi_k \overline{\xi}_k (R(\zeta)K_k, K_k) \). Now \( R(\zeta)K_i(z) = (K_i, K_{i\zeta}) \), where \( K_{i\zeta}(w) = K(z, w, \zeta) = K(w, z, \zeta) \). Consequently,
\[ R(\zeta)K_i(z) = \frac{\partial^{i-1}}{\partial w^{i-1}} K(z_0, z, \zeta) , \]
and
\[ (R(\zeta)K_i, K_j) = \frac{\partial^{i+j-2}}{\partial z^{i-1}\partial \bar{z}^{j-1}} K(z_0, z_0, \zeta) = \psi_{ji}(\zeta) . \]

Using the resolvent equation it is not hard to see that
\[ \text{Im} (R(\zeta)f, f)/\text{Im} (\zeta) = \| R(\zeta)f \|^2 \geq 0 \]
and hence
\[ \sum_{i,j=1}^{n} \frac{\text{Im} \psi_{ji}(\zeta) \xi_i \overline{\xi}_j}{\text{Im} (\zeta)} \geq 0 . \]

This completes the proof.

It is shown in [2] that Theorem 3.4 implies the existence of a spectral matrix \( \rho \) for the resolvent \( R \).

**Theorem 3.5.** The matrix \( \rho \) defined by
\[ \rho(\lambda) = \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_0^i \text{Im} (\nu + i\varepsilon) d\nu \]
exists, is nondecreasing, and is of bounded variation on any finite interval.
We now consider the projections $E_\Delta$ corresponding to the interval $\Delta = (a, b]$. It follows from the proof of Theorem 3.2, that $E_\Delta$ is an integral operator with kernel $e_\Delta(z, w) = E_\Delta K_w(w)$. The following theorem shows how $e_\Delta(z, w)$ can be described in terms of the basis $\phi_1, \cdots, \phi_n$ and the spectral matrix given by Theorem 3.5.

**Theorem 3.6.** If $a$ and $b$ are continuity points of $E$ then

$$e_\Delta(z, w) = \sum_{i,j=1}^n \phi_i(z, \nu)\overline{\phi_j(w, \nu)}d\rho_{ij}(\nu),$$

where $\rho = (\rho_{ij})$ is the spectral matrix given by Theorem 3.5.

**Proof.** The idea is to use the inversion formula

$$(E_\Delta f, g) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\gamma} ((R(\nu + i\varepsilon)f, g) - (R(\nu - i\varepsilon)f, g))d\nu,$$

for all $f$ and $g$ in $\mathcal{S}$, $a$ and $b$ continuity points of $E_\Delta$. Since $E_\Delta$ is self-adjoint $e_\Delta(z, w) = (E_\Delta K_w, K_z)$ and hence

$$e_\Delta(z, w) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\gamma} (K(\nu + i\varepsilon)K_w, K_z) - (K(\nu - i\varepsilon)K_w, K_z))d\nu.$$

For $z, w \in \partial D$, this becomes

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\gamma} \sum_{i,j=1}^n \psi_{ij}(\nu + i\varepsilon)\overline{\phi_i(z, \nu + i\varepsilon)}\phi_j(w, \nu - i\varepsilon)$$

$$- \psi_{ij}(\nu - i\varepsilon)\overline{\phi_i(z, \nu - i\varepsilon)}\phi_j(w, \nu + i\varepsilon)d\nu$$

$$+ \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\gamma} G(z, w, \nu + i\varepsilon) - G(z, w, \nu - i\varepsilon)d\nu.$$

Since $G(z, w, \gamma)$ is entire in $\gamma$ the later integral tends to zero as $\varepsilon \to 0$.

We now rewrite the first integrand as

$$\sum_{i,j=1}^n \left[\psi_{ij}(\nu + i\varepsilon) - \psi_{ij}(\nu - i\varepsilon)\right]\phi_i(z, \nu)\overline{\phi_j(w, \nu)} +$$

$$\sum_{i,j=1}^n \psi_{ij}(\nu + i\varepsilon)[\phi_i(z, \nu + i\varepsilon)\overline{\phi_j(w, \nu - i\varepsilon)} - \phi_i(z, \nu)\overline{\phi_j(w, \nu)}] +$$

$$\sum_{i,j=1}^n \psi_{ij}(\nu - i\varepsilon)[\phi_i(z, \nu)\overline{\phi_j(w, \nu)} - \phi_i(z, \nu - i\varepsilon)\overline{\phi_j(w, \nu + i\varepsilon)}],$$

and denote the three sums by $I_1(\nu, \varepsilon)$, $I_2(\nu, \varepsilon)$, and $I_3(\nu, \varepsilon)$ respectively. Consider $I_i(\nu, \varepsilon)$,
\[ \lim_{\varepsilon \to +0} \frac{1}{2\pi i} \int_{\Delta} I_i(\nu, \varepsilon) d\nu = \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_{\Delta} \sum_{i,j=1}^{n} \text{Im} \psi_{ij}(\nu + i\varepsilon)\phi_i(z, \nu)\overline{\phi_j(w, \nu)} d\nu. \]

Now
\[ \rho(\lambda) = \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_{\Delta} \text{Im} (\psi(\nu + i\varepsilon) d\nu \]
and it follows from a theorem of Helly that
\[ \lim_{\varepsilon \to +0} \frac{1}{2\pi i} \int_{\Delta} I_i(\nu, \varepsilon) d\nu = \int_{\Delta} \sum_{i,j=1}^{n} \phi_i(z, \nu)\overline{\phi_j(w, \nu)} d\rho_{ij}(\nu). \]

As is shown in [2] we have the following estimate
\[ \sum_{i,j=1}^{n} \int_{\Delta} |\psi_{ij}(\nu \pm i\varepsilon)| d\nu = O\left( \log \frac{1}{\varepsilon} \right) \quad (\varepsilon \to +0). \]

Since the \( \phi_i(z, \nu) \) are entire in \( \nu \) for fixed \( z \) there exists a constant \( M > 0 \) such that for \( \varepsilon \) sufficiently small
\[ |\phi_i(z, \nu + i\varepsilon)\overline{\phi_j(w, \nu - i\varepsilon)} - \phi_i(z, \nu)\overline{\phi_j(w, \nu)}| < M\varepsilon \]
for all \( \nu \in \Delta. \)

Combining (3.15) and (3.16) we see that
\[ \frac{1}{\pi} \int_{\Delta} I_2(\nu, \varepsilon) d\nu = O\left( \varepsilon \log \frac{1}{\varepsilon} \right) \quad (\varepsilon \to +0), \]
which tends to zero as \( \varepsilon \to +0. \) A similar result holds for
\[ \frac{1}{\pi} \int_{\Delta} I_3(\nu, \varepsilon) d\nu. \]

Consequently we have
\[ e_d(z, w) = \int_{\Delta} \sum_{i,j=1}^{n} \phi_i(z, \nu)\overline{\phi_j(w, \nu)} d\rho_{ij}(\nu). \]

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