

# Pacific Journal of Mathematics

**PRINCIPAL IDEALS IN  $F$ -ALGEBRAS**

RONN L. CARPENTER

## PRINCIPAL IDEALS IN $F$ -ALGEBRAS

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Suppose  $B$  is a commutative Banach algebra with unit. Gleason has proved that if  $I$  is a finitely generated maximal ideal in  $B$ , then there is an open neighborhood  $U$  of  $I$  in the spectrum of  $B$  such that  $U$  is homeomorphic in a natural way to an analytic variety and the Gelfand transforms of elements of  $B$  are analytic on this variety. In this paper it is shown that this result remains valid for principal ideals in uniform  $F$ -algebras with locally compact spectra. From this it follows that if  $A$  is an  $F$ -algebra of complex valued continuous functions on its spectrum satisfying (1) the spectrum of  $A$  is locally compact and has no isolated points, and (2) every closed maximal ideal in  $A$  is principal, then the spectrum of  $A$  can be given the structure of a Riemann surface in such a way that  $A$  can be identified with a closed subalgebra of the algebra of all functions which are analytic on the spectrum of  $A$ . Finally an example is given which shows that neither Gleason's result nor the characterization described in the preceding sentence extends to nonuniform algebras.

2. Lemmas and theorems. We assume throughout this paper that  $A$  is a uniform commutative  $F$ -algebra with unit, and that the spectrum  $X$  of  $A$  is locally compact and has no isolated points. We identify  $A$  and  $\hat{A}$  and regard  $A$  as a compact open closed subalgebra of  $C(X)$  the algebra of all continuous functions on  $X$ . Since  $A$  is an  $F$ -algebra, the space  $X$  is hemi-compact. We fix an ascending sequence  $\{X_n\}$  of compact  $A$ -convex subsets of  $X$  such that each compact subset of  $X$  is contained in some  $X_n$ . (A subset  $E$  of  $X$  is said to be  $A$ -convex if for each  $x$  in  $X - E$  there is an element  $f$  of  $A$  such that  $|f(x)| > \sup \{|f(y)| : y \in E\}$ .) Denote by  $B_n$  the completion of the algebra  $A|X_n$  with respect to the supremum norm on  $X_n$ . Then  $A = \lim \text{inv } B_n$ ,  $X = \cup X_n$ , and  $\text{Spec } B_n = X_n$  for each  $n$ . Define  $\pi_n$  to be the natural homomorphism of  $A$  into  $B_n$ . In the case under consideration  $\pi_n$  is the restriction map defined by  $f \rightarrow f|X_n$  for each  $f$  in  $A$ . Finally for a subset  $E$  of  $X$  and  $f$  in  $C(X)$  we let  $|f|_E = \sup \{|f(y)| : y \in E\}$ .

In [3] Michael defines a strong topological divisor of zero to be an element  $f$  of  $A$  such that the map  $g \rightarrow gf$  ( $g \in A$ ) is not a homeomorphism of  $A$  into  $A$ .

**LEMMA 2.1.** *Suppose  $x$  is a point of  $X$  and  $f$  is an element of  $A$  such that  $\ker(x) = Af$ . Then  $f$  is not a strong topological divisor*

of zero.

*Proof.* Suppose  $hf = 0$  for some  $h \in A$ . Since  $\text{hull}(f) = \{x\}$ ,  $h$  must be identically zero on  $X - \{x\}$ . Since  $\{x\}$  is not isolated and  $h$  is continuous,  $h$  must be identically zero on  $X$ . Therefore, the map  $g \rightarrow gf$ ,  $g \in A$  is one-to-one.

Since  $Af$  is closed in  $A$ , it is an  $F$ -algebra. Hence, the map  $g \rightarrow gf$  is a one-to-one continuous linear map of the  $F$ -algebra  $A$  onto the  $F$ -algebra  $Af$ . The open mapping theorem (see [1], p. 55) implies that the map  $g \rightarrow gf$  is a homeomorphism of  $A$  onto  $Af$ . Therefore,  $f$  is not a strong topological divisor of zero.

For a Banach algebra  $B$  we use  $\partial B$  to denote the Šilov boundary of  $B$ . The next lemma is a generalization of Corollary 3.3.7 in [4].

**LEMMA 2.2.** *Suppose  $f \in A$  and  $\text{hull}(f)$  is compact. Then  $f$  is not a strong topological divisor of zero in  $A$  if, and only if, there is an integer  $n$  such that for  $j \geq n$ ,  $\text{hull}(f) \cap \partial B_j = \emptyset$ .*

*Proof.* Suppose that for every positive integer  $n$  there is a positive integer  $j$  such that  $j \geq n$  and  $\text{hull}(f) \cap \partial B_j \neq \emptyset$ . Then, without loss of generality we can assume that  $\text{hull}(f) \cap \partial B_j \neq \emptyset$  for  $j = 1, 2, \dots$ . For each integer  $j$  and open neighborhood  $U$  of  $\text{hull}(f)$  we can choose an element  $g(j, U)$  of  $A$  such that  $|g(j, U)|_{\partial B_j} = 1$  and  $|g(j, U)|_{X_j - U} < j^{-1}$ .

Order the pairs  $(j, U)$  consisting of a positive integer  $j$  and an open neighborhood  $U$  of  $\text{hull}(f)$  by  $(j, U) \geq (i, V)$  if, and only if,  $j \geq i$  and  $U$  is contained in  $V$ . With this ordering  $\{g(j, U)\}$  is a net in  $A$ .

Fix a positive integer  $k$  and an  $\varepsilon > 0$ . Choose an open neighborhood  $V$  of  $\text{hull}(f)$  such that  $|f|_V < \varepsilon$ . Then for  $j \geq \max(k, \varepsilon^{-1}|f|_{x_k})$  and  $U$  contained in  $V$  we have,  $|g(j, U)f|_{x_k} < \varepsilon$ . Therefore  $\lim_{(j, U)} |g(j, U)f|_{x_k} = 0$ . Since  $k$  was arbitrary we have that  $g(j, U)f$  converges to zero in  $A$ .

Since  $\text{hull}(f)$  is compact and  $X$  is locally compact there is an open neighborhood  $W$  of  $\text{hull}(f)$  which is pre-compact. Since  $W$  is pre-compact there is an integer  $n$  such that  $W$  is contained in  $X_j$  for  $j \geq n$ . Then for  $(j, U) \geq (1, W)$  we have  $1 \leq |g(j, U)|_W \leq |g(j, U)|_{x_n}$ . Hence,  $\{g(j, U)\}$  does not converge to zero in  $A$ . Therefore  $f$  is a strong topological divisor of zero in  $A$ .

Suppose there is an integer  $n$  such that for  $j \geq n$  we have  $\text{hull}(f) \cap \partial B_j = \emptyset$ . Assume  $\{g_i\}$  is a sequence in  $A$  such that  $\lim_i (g_i f) = 0$ . Fix a positive integer  $k$  greater than  $n$ . Let  $\delta =$

$\min \{|f(x): x \in \partial B_k\}$ . Since  $\text{hull}(f) \cap \partial B_k = \emptyset$  we have that  $\delta$  is greater than zero. Now  $\delta |g_i|_{x_k} = \delta |g_i|_{\partial B_k} \leq |g_i f|_{x_k}$ . Combining this estimate on  $|g_i|_{x_k}$  with  $\lim_i |g_i f|_{x_k} = 0$  we have that  $\lim_i |g_i|_{x_k} = 0$ . Since the only restriction on  $k$  was that  $k$  be greater than  $n$  we have that  $\{g_i\}$  converges to zero in  $A$ . Therefore  $f$  is not a strong topological divisor of zero.

**LEMMA 2.3.** *If  $x \in X$  and  $f \in A$  such that  $\ker(x) = Af$ , then there is an integer  $n$  such that  $x$  is not an element of  $\partial B_j$  for  $j \geq n$ .*

*Proof.* Lemma 2.1 implies that  $f$  is not a strong topological divisor of zero in  $A$ . Since  $Af = \ker(x)$  we have that  $\text{hull}(f) = \{x\}$ . Lemma 2.2 implies there is an integer  $n$  such that  $x \notin \partial B_j$  for  $j \geq n$ .

Recall that  $\pi_n$  is the natural projection of  $A$  into  $B_n$ .

**LEMMA 2.4.** *If  $x \in X$  and  $f \in A$  such that  $\ker(x) = Af$ , then there is an integer  $n$  such that  $B_j \pi_j(f)$  is a maximal ideal in  $B_j$  for  $j \geq n$ .*

*Proof.* Lemma 2.3 implies that there is an integer  $n$  such that  $x$  is not contained in  $\partial B_j$  for  $j \geq n$ . We may assume, without loss of generality, that  $x$  is contained in  $X_j$  for  $j \geq n$ .

Fix  $j \geq n$ . Since  $\text{hull}(f) \cap \partial B_j = \emptyset$ , we have that  $B_j \pi_j(f)$  is closed in  $B_j$ . Since  $Af$  is a closed maximal ideal in  $A$  we have that  $A = Af + C$ . Hence  $\pi_j(A) \pi_j(f) + C$  is dense in  $B_j$ . Therefore  $B_j \pi_j(f) + C$  is dense in  $B_j$ . Since  $B_j \pi_j(f)$  is closed in  $B_j$  we have that  $B_j = B_j \pi_j(f) + C$ . This proves that  $B_j \pi_j(f)$  is a maximal ideal in  $B_j$ .

The following lemma appeared in [5]. Consequently we will merely sketch a proof.

**LEMMA 2.5.** *If  $x$  is a point of  $X$  and  $x$  is isolated in each  $X_n$  which contains it, then  $x$  is isolated in  $X$ .*

*Proof.* Suppose  $x$  is isolated in each  $X_n$  which contains it. We use Šilov's idempotent theorem on the Banach algebras  $B_n$  and the fact that the only idempotent in the radical of a Banach algebra is zero to obtain an idempotent  $e$  in  $A$  such that  $e(x) = 1$  and  $e = 0$  elsewhere on  $X$ .

**THEOREM 2.6.** *Suppose that  $A$  is a commutative uniform F-algebra with unit and that the spectrum  $X$  of  $A$  is locally compact.*

If  $x$  is a nonisolated point of  $X$  and  $\ker(x) = Af$  for some  $f \in A$ , then there is an open subset  $U$  of  $X$  such that  $x \in U$ ,  $f$  maps  $U$  homeomorphically onto an open disc  $\Delta$  in  $\mathbb{C}$ , and  $gf^{-1}$  is analytic on  $\Delta$  for each  $g \in A$ .

*Proof.* Lemma 2.4 allows us to choose an integer  $n_1$ , such that  $B_j \pi_j(f)$  is a maximal ideal in  $B_j$  for  $j \geq n_1$ . Since  $x$  is not isolated we can use Lemma 2.5 to obtain an integer  $n_2$  such that  $x$  is not isolated in  $X_j$  for  $j \geq n_2$ . Since  $X$  is locally compact  $x$  has a precompact open neighborhood  $W$ . There is an integer  $n_3$  such that  $W$  is contained in  $X_j$  for  $j \geq n_3$ .

Fix an integer  $k$  such that  $k \geq \max(n_1, n_2, n_3)$ . Then  $B_k \pi_k(f)$  is a nonisolated maximal ideal in  $B_k$ . Gleason proved in [2] that there is an open neighborhood  $V_1$  of  $x$  in  $X_k$  and a disc  $\Delta'$  about the origin in  $\mathbb{C}$  such that  $\pi_k(f)$  maps  $V_1$  homeomorphically onto  $\Delta'$ , and  $g\pi_k(f)^{-1}$  is analytic on  $\Delta'$  for each  $g \in B_k$ . Set  $V_2 = V_1 \cap W$  where  $W$  is the open set defined in the previous paragraph. Let  $\Delta$  be an open disc centered at the origin of  $\mathbb{C}$  such that  $\Delta$  is contained in  $f(V_2)$ . Set  $U = f^{-1}(\Delta) \cap V_2$ . Then  $U$  is an open neighborhood of  $x$  in  $X$ ,  $f$  maps  $U$  homeomorphically onto  $\Delta$ , and  $gf^{-1}$  is analytic on  $\Delta$  for each  $g \in A$ .

**COROLLARY 2.7.** *Suppose that  $A$  is a commutative uniform  $F$ -algebra with unit, and that the spectrum  $X$  of  $A$  is locally compact and has no isolated points. If every closed maximal ideal in  $A$  is principal, then  $X$  can be given the structure of a Riemann surface in such a way that  $A$  is topologically isomorphic to a closed subalgebra of  $\text{Hol}(X)$ .*

*Proof.* It follows from Theorem 2.6 that for each point  $x$  of  $X$  we can choose  $f_x \in A$  and an open set  $U_x$  containing  $x$  such that  $f_x$  maps  $U_x$  topologically onto the open unit disc in  $\mathbb{C}$  and  $gf_x^{-1}$  is analytic on the open unit disc for each  $g \in A$ . Let  $x$  and  $y$  be elements of  $X$  such that  $U_x \cap U_y \neq \emptyset$ . Then  $f_x \circ f_y^{-1}$  is an analytic map of  $f_y(U_x \cap U_y)$  onto  $f_x(U_x \cap U_y)$ . Therefore the set  $\{f_x: x \in X\}$  is a set of local coordinates for  $X$  with respect to which  $A \subset \text{Hol}(X)$ .

**EXAMPLE.** We give an example of a nonuniform  $F$ -algebra  $A$  such that (1) every maximal ideal in  $A$  is principal, and (2)  $A$  has no analytic structure. Thus the restriction to uniform algebras in Theorem 2.6 and Corollary 2.7 is essential. We note that the example shows that Gleason's result does not extend to nonuniform  $F$ -algebras.

Let  $A = C^\infty(\mathbb{R})$  with seminorms  $\{\|\cdot\|_n\}$  defined by  $\|f\|_n = \sum_{i=0}^n (1/i!) \max\{|f^{(i)}(x)|: x \in [-n, n]\}$ . With respect to these seminorms  $A$  is an  $F$ -algebra. We list below some of the properties of  $A$ .

- (1)  $A$  is (singly) generated by the function  $f$  defined by  $f(x) = x$  for each  $x \in R$ .
- (2)  $A$  is semisimple.
- (3)  $\text{Spec } A = R$ ; hence,  $\text{Spec } A$  is locally compact and connected, but  $\text{Spec } A$  contains no discs.
- (4) Every closed maximal ideal in  $A$  is principal.

Properties 1, 2, and 3 are clear. There are several ways to prove property 4. The most elementary is to show that for any  $g$  in  $A$  such that  $g(0) = 0$  the function  $h$  defined by

$$h(x) = \begin{cases} x^{-1}g(x), & \text{if } x \neq 0 \\ g^{(1)}(0), & \text{if } x = 0 \end{cases}$$

is in  $A$ . This can be accomplished by an induction argument using l'Hospital's rule and the mean value theorem. Once we have shown  $h$  is in  $A$  we have  $g = hf$ . This shows that the maximal ideal consisting of all functions which vanish at zero is  $Af$ . Clearly, a similar argument establishes that all closed maximal ideals are principal.

**3. Remarks.** We have proved that Gleason's result extends to principal ideals in uniform  $F$ -algebras with locally compact spectra. We have also produced an example to show that it does not extend even to principal ideals in nonuniform  $F$ -algebras. The obvious question is: does Gleason's result remain valid for finitely generated ideals in uniform commutative  $F$ -algebras with unit? At the present time we do not know the answer to this question.

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