

Pacific Journal of Mathematics

**THE DERIVED SET OF THE SPECTRUM OF A DISTRIBUTION
FUNCTION**

THEODORE SEIO CHIHARA

THE DERIVED SET OF THE SPECTRUM OF A DISTRIBUTION FUNCTION

T. S. CHIHARA

Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be real sequences with $b_n > 0$, $b_n \rightarrow 0$ ($n \rightarrow \infty$). Let $\{P_n(x)\}_{n=0}^{\infty}$ be the sequence of orthonormal polynomials satisfying the recurrence

$$\begin{aligned} xP_n(x) &= b_{n-1}P_{n-1}(x) + a_nP_n(x) + b_nP_{n+1}(x), \quad (n \geq 0), \\ P_{-1}(x) &= 0, \quad P_0(x) = 1. \end{aligned}$$

Then there is a substantially unique distribution function ψ with respect to which the $P_n(x)$ are orthogonal. This paper verifies a conjecture of D. P. Maki that the set of all limit points of the sequence $\{a_n\}$ is the derived set of the spectrum of ψ .

Let $\{P_n(x)\}$ be a sequence of orthonormal polynomials defined by the recurrence formula

$$(1.1) \quad \begin{aligned} xP_n(x) &= b_{n-1}P_{n-1}(x) + a_nP_n(x) + b_nP_{n+1}(x) \quad (n \geq 0), \\ P_{-1}(x) &= 0, \quad P_0(x) = 1, \quad a_n \text{ real}, \quad b_n > 0. \end{aligned}$$

Then it is well known that there is a bounded, nondecreasing function ψ such that

$$\int_{-\infty}^{\infty} P_m(x)P_n(x)d\psi(x) = \delta_{mn},$$

the spectrum of ψ ,

$$S(\psi) = \{t \mid \psi(t + \varepsilon) - \psi(t - \varepsilon) > 0 \text{ for all } \varepsilon > 0\},$$

being an infinite set.

If we impose the additional hypothesis

$$(1.2) \quad \lim_{n \rightarrow \infty} b_n = 0,$$

then the Hamburger moment problem associated with (1.1) is determined (by Carleman's criterion — see [4, p. 59]) and the distribution function ψ is substantially unique (is uniquely determined up to an arbitrary additive constant at all points of continuity).

In [3], D. P. Maki proved that every (finite) sequential limit point of $\{a_n\}_{n=0}^{\infty}$ is a point in $S(\psi)$ and he conjectured that λ is a limit point of $\{a_n\}$ if and only if λ is a point of the derived set, $S(\psi)'$. Maki's conjecture is correct as will be proved below.

If we denote the smallest and largest limit points (in the extended real number system) of $S(\psi)$ by σ and τ , respectively, then it was proved in [2, Th. 7] that under the hypothesis (1.2),

$$(1.3) \quad \sigma = \liminf_{n \rightarrow \infty} a_n, \quad \tau = \limsup_{n \rightarrow \infty} a_n .$$

It follows that Maki's conjecture will remain valid if we allow infinite limit points also.

2. In the sequel, we will have reference to the J -fraction,

$$(2.1) \quad \frac{1}{|z - a_0|} - \frac{b_0^2}{|z - a_1|} - \frac{b_1^2}{|z - a_2|} - \frac{b_2^2}{|z - a_2|} - \dots .$$

With the hypothesis (1.2), we are dealing with the determinate case so (2.1) converges uniformly on every closed half-plane,

$$\text{Im}(z) \geq \delta > 0 ,$$

to an analytic function F which is not a rational function. ψ can be obtained from F by the Stieltjes inversion formula (see [5, p. 250]) and this shows that if the analytic continuation of F is regular in a region containing a real open interval (a, b) , then ψ is constant on (a, b) .

We will denote the n^{th} convergent of (2.1) by $A_n(z)/B_n(z)$ so that $B_n(z)$ is the monic orthogonal polynomial, $B_n(z) = b_0 b_1 \cdots b_{n-1} P_n(z)$.

We also recall that if the Hamburger moment problem is determined, then (see [4, Corollary 2.6])

$$(2.2) \quad \rho(x) \equiv \left\{ \sum_{n=0}^{\infty} P_n^2(x) \right\}^{-1}$$

vanishes at all points of continuity of ψ and equals the jump of ψ at a point of discontinuity.

3. We now state our main result.

THEOREM. *Let $\lim_{n \rightarrow \infty} b_n = 0$. Then λ is a limit point of the sequence $\{a_n\}_{n=0}^{\infty}$ if and only if λ is a limit point of $S(\psi)$.*

Proof. In view of (1.3), it is sufficient to consider λ finite.

First let λ be a finite limit point of $\{a_n\}$. Then Maki has shown that $\lambda \in S(\psi)$ and also that there is a subsequence $\{P_{n_k}\}$ such that

$$(3.1) \quad \lim_{k \rightarrow \infty} \int_{-\infty}^{\infty} (x - \lambda)^2 P_{n_k}^2(x) d\psi(x) = 0 .$$

We will now show that λ cannot be an isolated point of $S(\psi)$.

Assume λ is isolated. Then ψ has a jump at λ so let $J_\lambda > 0$ denote this jump. It follows that there is an $\varepsilon > 0$ such that $S(\psi)$ contains no points in either of the half-open intervals, $[\lambda - \varepsilon, \lambda)$ and $(\lambda, \lambda + \varepsilon]$. Thus, writing $f_k = P_{n_k}$, we have by a modification of a technique used by Maki,

$$\begin{aligned} \int_{-\infty}^{\infty} f_k^2 d\psi &= \int_{-\infty}^{\lambda-\varepsilon} f_k^2 d\psi + \int_{\lambda+\varepsilon}^{\infty} f_k^2 d\psi + f_k^2(\lambda)J_\lambda = 1, \\ \int_{-\infty}^{\infty} (x - \lambda)^2 f_k^2 d\psi &= \int_{-\infty}^{\lambda-\varepsilon} (x - \lambda)^2 f_k^2 d\psi + \int_{\lambda+\varepsilon}^{\infty} (x - \lambda)^2 f_k^2 d\psi \\ &\geq \varepsilon^2 \left\{ \int_{-\infty}^{\lambda-\varepsilon} f_k^2 d\psi + \int_{\lambda+\varepsilon}^{\infty} f_k^2 d\psi \right\} \\ &= \varepsilon^2 [1 - f_k^2(\lambda)J_\lambda] \geq 0. \end{aligned}$$

Therefore, according to (3.1),

$$\lim_{k \rightarrow \infty} P_{n_k}^2(\lambda) = J_\lambda^{-1} > 0$$

but this contradicts the fact that $\rho(\lambda) = J_\lambda$ (see (2.2)). Thus $\lambda \in S(\psi)'$.

Conversely, let $\lambda \in S(\psi)'$ and assume that λ is not a limit point of $\{a_n\}$. Then there is a $\delta > 0$ and an index N_1 such that $|a_n - \lambda| \geq 2\delta$ for $n \geq N_1$, hence

$$|z - a_n| \geq \delta \text{ for } |z - \lambda| \leq \delta, \quad n \geq N_1.$$

Since by hypothesis, $b_n \rightarrow 0$, there is an index N such that

$$\left| \frac{b_n^2}{(z - a_n)(z - a_{n+1})} \right| \leq \frac{b_n^2}{\delta^2} < \frac{1}{4} \quad \text{for } n \geq N$$

and $z \in D = \{w \mid |w - \lambda| \leq \delta\}$.

It now follows from a Theorem of Worpitzky (see [5, Th.10.1]) that the J -fraction

$$\frac{b_N^2}{|z - a_{N+1}|} - \frac{b_{N+1}^2}{|z - a_{N+2}|} - \frac{b_{N+2}^2}{|z - a_{N+3}|} - \dots$$

converges uniformly on D to an analytic function F_N and, from (2.1), we have

$$F(z) = \frac{A_N(z) - A_{N-1}(z)F_N(z)}{B_N(z) - B_{N-1}(z)F_N(z)}$$

for $z \in D$, z not a zero of $B_N - B_{N-1}F_N$.

Since F_N cannot be a rational function, $B_N - B_{N-1}F_N$ can have at most finitely many zeros in D . That is, F has at most finitely many singularities in D , which in turn implies that ψ has at most

finitely many points of increase in $[\lambda - \delta, \lambda + \delta]$. Thus we again reach a contradiction.

4. REMARKS. 1. W. R. Allaway has shown (private communication) that when $\{a_n\}$ is bounded, then a theorem of Krein [1, pp. 230-231] can be used to prove Maki's conjecture in the case $S(\psi)'$ is finite.

2. Maki's proof shows that (3.1) holds if for some $\{n_k\}$,

$$(4.1) \quad b_{n_{k-1}} \rightarrow 0, b_{n_k} \rightarrow 0, a_{n_k} \rightarrow \lambda \text{ (finite) } (n \rightarrow \infty)$$

so that (4.1) is sufficient for $\lambda \in S(\psi)'$ (assuming a determined moment problem).

3. Maki showed that if A denotes the self-adjoint operator, $Af = xf$, defined on a dense subset of $L^2(\psi)$, then $\sigma(A) \subset S(\psi)$, where $\sigma(A)$ denotes the spectrum of A .

Consideration of the characteristic function for the singleton set $\{\lambda\}$ shows that λ is an eigenvalue of A if and only if λ is a point of discontinuity of ψ . That is, $P\sigma(A) = D(\psi)$, $C\sigma(A) \subset S(\psi) \setminus D(\psi)$, where $P\sigma(A)$, $C\sigma(A)$ denote the point and continuous spectra of A and $D(\psi)$ is the set of jump points of ψ .

On the other hand, if $\lambda \in S(\psi) \setminus D(\psi)$ (λ finite), then there is a measurable function f_n with support in $[\lambda - 1/n, \lambda + 1/n]$ and with $\|f_n\| = 1$. Then

$$\|(A - \lambda)f_n\|^2 \leq n^{-2} \int_{\lambda-1/n}^{\lambda+1/n} f_n^2 d\psi = n^{-2}$$

so that $\lambda \in \sigma(A)$. It follows that $C\sigma(A) = S(\psi) \setminus D(\psi)$.

REFERENCES

1. N. I. Ahiezer and M. Krein, *Some Questions in the Theory of Moments*. Transl. Math. Monographs, Vol. 2, Amer. Math. Soc., 1962.
2. T. S. Chihara, *Chain sequences and orthogonal polynomials*, Trans. Amer. Math. Soc. **104** (1962), 1-16.
3. D. P. Maki, *A note on recursively defined orthogonal polynomials*, Pacific J. Math. **28** (1969), 611-613.
4. J. Shohat and J. Tamarkin, *The Problem of Moments*, Math. Surveys No. 1, Amer. Math. Soc., 1943, 1950.
5. H. S. Wall, *Analytic Theory of Continued Fractions*, van Nostrand, New York, 1948.

Received March 9, 1970.

UNIVERSITY OF ALBERTA

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON
Stanford University
Stanford, California 94305

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

RICHARD PIERCE
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLE

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
TRW SYSTEMS
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Pacific Journal of Mathematics

Vol. 35, No. 3

November, 1970

John D. Arrison and Michael Rich, <i>On nearly commutative degree one algebras</i>	533
Bruce Alan Barnes, <i>Algebras with minimal left ideals which are Hilbert spaces</i>	537
Robert F. Brown, <i>An elementary proof of the uniqueness of the fixed point index</i>	549
Ronn L. Carpenter, <i>Principal ideals in F-algebras</i>	559
Chen Chung Chang and Yiannis (John) Nicolas Moschovakis, <i>The Suslin-Kleene theorem for V_κ with cofinality $(\kappa) = \omega$</i>	565
Theodore Seio Chihara, <i>The derived set of the spectrum of a distribution function</i>	571
Tae Geun Cho, <i>On the Choquet boundary for a nonclosed subspace of $C(S)$</i>	575
Richard Brian Darst, <i>The Lebesgue decomposition, Radon-Nikodym derivative, conditional expectation, and martingale convergence for lattices of sets</i>	581
David E. Fields, <i>Dimension theory in power series rings</i>	601
Michael Lawrence Fredman, <i>Congruence formulas obtained by counting irreducibles</i>	613
John Eric Gilbert, <i>On the ideal structure of some algebras of analytic functions</i>	625
G. Goss and Giovanni Viglino, <i>Some topological properties weaker than compactness</i>	635
George Grätzer and J. Sichler, <i>On the endomorphism semigroup (and category) of bounded lattices</i>	639
R. C. Lacher, <i>Cell-like mappings. II</i>	649
Shiva Narain Lal, <i>On a theorem of M. Izumi and S. Izumi</i>	661
Howard Barrow Lambert, <i>Differential mappings on a vector space</i>	669
Richard G. Levin and Takayuki Tamura, <i>Notes on commutative power joined semigroups</i>	673
Robert Edward Lewand and Kevin Mor McCrimmon, <i>Macdonald's theorem for quadratic Jordan algebras</i>	681
J. A. Marti, <i>On some types of completeness in topological vector spaces</i>	707
Walter J. Meyer, <i>Characterization of the Steiner point</i>	717
Saad H. Mohamed, <i>Rings whose homomorphic images are q-rings</i>	727
Thomas V. O'Brien and William Lawrence Reddy, <i>Each compact orientable surface of positive genus admits an expansive homeomorphism</i>	737
Robert James Plemmons and M. T. West, <i>On the semigroup of binary relations</i>	743
Calvin R. Putnam, <i>Unbounded inverses of hyponormal operators</i>	755
William T. Reid, <i>Some remarks on special disconjugacy criteria for differential systems</i>	763
C. Ambrose Rogers, <i>The convex generation of convex Borel sets in euclidean space</i>	773
S. Saran, <i>A general theorem for bilinear generating functions</i>	783
S. W. Smith, <i>Cone relationships of biorthogonal systems</i>	787
Wolmer Vasconcelos, <i>On commutative endomorphism rings</i>	795
Vernon Emil Zander, <i>Products of finitely additive set functions from Orlicz spaces</i>	799
G. Sankaranarayanan and C. Suyambulingom, <i>Correction to: "Some renewal theorems concerning a sequence of correlated random variables"</i>	805
Joseph Zaks, <i>Correction to: "Trivially extending decompositions of E^n"</i>	805
Dong Hoon Lee, <i>Correction to: "The adjoint group of Lie groups"</i>	805
James Edward Ward, <i>Correction to: "Two-groups and Jordan algebras"</i>	806