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THE DERIVED SET OF THE SPECTRUM OF A DISTRIBUTION FUNCTION

THEODORE SEIO CHIHARA

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T. S. CHIHARA

Let $\{a_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty$ be real sequences with $b_n > 0$, $b_n \rightarrow 0$ ($n \rightarrow \infty$). Let $\{P_n(x)\}_{n=0}^\infty$ be the sequence of orthonormal polynomials satisfying the recurrence

$$\begin{aligned} xP_n(x) &= b_{n-1}P_{n-1}(x) + a_nP_n(x) + b_nP_{n+1}(x), \quad (n \geq 0), \\ P_{-1}(x) &= 0, \quad P_0(x) = 1. \end{aligned}$$

Then there is a substantially unique distribution function ψ with respect to which the $P_n(x)$ are orthogonal. This paper verifies a conjecture of D. P. Maki that the set of all limit points of the sequence $\{a_n\}$ is the derived set of the spectrum of ψ .

Let $\{P_n(x)\}$ be a sequence of orthonormal polynomials defined by the recurrence formula

$$(1.1) \quad \begin{aligned} xP_n(x) &= b_{n-1}P_{n-1}(x) + a_nP_n(x) + b_nP_{n+1}(x) \quad (n \geq 0), \\ P_{-1}(x) &= 0, \quad P_0(x) = 1, \quad a_n \text{ real}, \quad b_n > 0. \end{aligned}$$

Then it is well known that there is a bounded, nondecreasing function ψ such that

$$\int_{-\infty}^{\infty} P_m(x)P_n(x)d\psi(x) = \delta_{mn},$$

the spectrum of ψ ,

$$S(\psi) = \{t \mid \psi(t + \varepsilon) - \psi(t - \varepsilon) > 0 \text{ for all } \varepsilon > 0\},$$

being an infinite set.

If we impose the additional hypothesis

$$(1.2) \quad \lim_{n \rightarrow \infty} b_n = 0,$$

then the Hamburger moment problem associated with (1.1) is determined (by Carleman's criterion — see [4, p. 59]) and the distribution function ψ is substantially unique (is uniquely determined up to an arbitrary additive constant at all points of continuity).

In [3], D. P. Maki proved that every (finite) sequential limit point of $\{a_n\}_{n=0}^\infty$ is a point in $S(\psi)$ and he conjectured that λ is a limit point of $\{a_n\}$ if and only if λ is a point of the derived set, $S(\psi)'$. Maki's conjecture is correct as will be proved below.

If we denote the smallest and largest limit points (in the extended real number system) of $S(\psi)$ by σ and τ , respectively, then it was proved in [2, Th. 7] that under the hypothesis (1.2),

$$(1.3) \quad \sigma = \liminf_{n \rightarrow \infty} a_n, \quad \tau = \limsup_{n \rightarrow \infty} a_n.$$

It follows that Maki's conjecture will remain valid if we allow infinite limit points also.

2. In the sequel, we will have reference to the J -fraction,

$$(2.1) \quad \frac{1}{|z - a_0|} - \frac{b_0^2}{|z - a_1|} - \frac{b_1^2}{|z - a_2|} - \frac{b_2^2}{|z - a_2|} - \dots.$$

With the hypothesis (1.2), we are dealing with the determinate case so (2.1) converges uniformly on every closed half-plane,

$$\operatorname{Im}(z) \geq \delta > 0,$$

to an analytic function F which is not a rational function. ψ can be obtained from F by the Stieltjes inversion formula (see [5, p. 250]) and this shows that if the analytic continuation of F is regular in a region containing a real open interval (a, b) , then ψ is constant on (a, b) .

We will denote the n^{th} convergent of (2.1) by $A_n(z)/B_n(z)$ so that $B_n(z)$ is the monic orthogonal polynomial, $B_n(z) = b_0 b_1 \cdots b_{n-1} P_n(z)$.

We also recall that if the Hamburger moment problem is determined, then (see [4, Corollary 2.6])

$$(2.2) \quad \rho(x) \equiv \left\{ \sum_{n=0}^{\infty} P_n^2(x) \right\}^{-1}$$

vanishes at all points of continuity of ψ and equals the jump of ψ at a point of discontinuity.

3. We now state our main result.

THEOREM. *Let $\lim_{n \rightarrow \infty} b_n = 0$. Then λ is a limit point of the sequence $\{a_n\}_{n=0}^{\infty}$ if and only if λ is a limit point of $S(\psi)$.*

Proof. In view of (1.3), it is sufficient to consider λ finite.

First let λ be a finite limit point of $\{a_n\}$. Then Maki has shown that $\lambda \in S(\psi)$ and also that there is a subsequence $\{P_{n_k}\}$ such that

$$(3.1) \quad \lim_{k \rightarrow \infty} \int_{-\infty}^{\infty} (x - \lambda)^2 P_{n_k}^2(x) d\psi(x) = 0.$$

We will now show that λ cannot be an isolated point of $S(\psi)$.

Assume λ is isolated. Then ψ has a jump at λ so let $J_\lambda > 0$ denote this jump. It follows that there is an $\varepsilon > 0$ such that $S(\psi)$ contains no points in either of the half-open intervals, $[\lambda - \varepsilon, \lambda)$ and $(\lambda, \lambda + \varepsilon]$. Thus, writing $f_k = P_{n_k}$, we have by a modification of a technique used by Maki,

$$\begin{aligned} \int_{-\infty}^{\infty} f_k^2 d\psi &= \int_{-\infty}^{\lambda-\varepsilon} f_k^2 d\psi + \int_{\lambda+\varepsilon}^{\infty} f_k^2 d\psi + f_k^2(\lambda)J_\lambda = 1, \\ \int_{-\infty}^{\infty} (x - \lambda)^2 f_k^2 d\psi &= \int_{-\infty}^{\lambda-\varepsilon} (x - \lambda)^2 f_k^2 d\psi + \int_{\lambda+\varepsilon}^{\infty} (x - \lambda)^2 f_k^2 d\psi \\ &\geq \varepsilon^2 \left\{ \int_{-\infty}^{\lambda-\varepsilon} f_k^2 d\psi + \int_{\lambda+\varepsilon}^{\infty} f_k^2 d\psi \right\} \\ &= \varepsilon^2 [1 - f_k^2(\lambda)J_\lambda] \geq 0. \end{aligned}$$

Therefore, according to (3.1),

$$\lim_{k \rightarrow \infty} P_{n_k}^2(\lambda) = J_\lambda^{-1} > 0$$

but this contradicts the fact that $\rho(\lambda) = J_\lambda$ (see (2.2)). Thus $\lambda \in S(\psi)'$.

Conversely, let $\lambda \in S(\psi)'$ and assume that λ is not a limit point of $\{a_n\}$. Then there is a $\delta > 0$ and an index N_1 such that $|a_n - \lambda| \geq 2\delta$ for $n \geq N_1$, hence

$$|z - a_n| \geq \delta \text{ for } |z - \lambda| \leq \delta, \quad n \geq N_1.$$

Since by hypothesis, $b_n \rightarrow 0$, there is an index N such that

$$\left| \frac{b_n^2}{(z - a_n)(z - a_{n+1})} \right| \leq \frac{b_n^2}{\delta^2} < \frac{1}{4} \quad \text{for } n \geq N$$

and $z \in D = \{w \mid |w - \lambda| \leq \delta\}$.

It now follows from a Theorem of Worpitzky (see [5, Th.10.1]) that the J -fraction

$$\frac{b_N^2}{|z - a_{N+1}|} - \frac{b_{N+1}^2}{|z - a_{N+2}|} - \frac{b_{N+2}^2}{|z - a_{N+3}|} - \dots$$

converges uniformly on D to an analytic function F_N and, from (2.1), we have

$$F(z) = \frac{A_N(z) - A_{N-1}(z)F_N(z)}{B_N(z) - B_{N-1}(z)F_N(z)}$$

for $z \in D$, z not a zero of $B_N - B_{N-1}F_N$.

Since F_N cannot be a rational function, $B_N - B_{N-1}F_N$ can have at most finitely many zeros in D . That is, F has at most finitely many singularities in D , which in turn implies that ψ has at most

finitely many points of increase in $[\lambda - \delta, \lambda + \delta]$. Thus we again reach a contradiction.

4. REMARKS. 1. W. R. Allaway has shown (private communication) that when $\{a_n\}$ is bounded, then a theorem of Krein [1, pp. 230-231] can be used to prove Maki's conjecture in the case $S(\psi)'$ is finite.

2. Maki's proof shows that (3.1) holds if for some $\{n_k\}$,

$$(4.1) \quad b_{n_{k-1}} \rightarrow 0, b_{n_k} \rightarrow 0, a_{n_k} \rightarrow \lambda \text{ (finite) } (n \rightarrow \infty)$$

so that (4.1) is sufficient for $\lambda \in S(\psi)'$ (assuming a determined moment problem).

3. Maki showed that if A denotes the self-adjoint operator, $Af = xf$, defined on a dense subset of $L^2(\psi)$, then $\sigma(A) \subset S(\psi)$, where $\sigma(A)$ denotes the spectrum of A .

Consideration of the characteristic function for the singleton set $\{\lambda\}$ shows that λ is an eigenvalue of A if and only if λ is a point of discontinuity of ψ . That is, $P\sigma(A) = D(\psi)$, $C\sigma(A) \subset S(\psi) \setminus D(\psi)$, where $P\sigma(A)$, $C\sigma(A)$ denote the point and continuous spectra of A and $D(\psi)$ is the set of jump points of ψ .

On the other hand, if $\lambda \in S(\psi) \setminus D(\psi)$ (λ finite), then there is a measurable function f_n with support in $[\lambda - 1/n, \lambda + 1/n]$ and with $\|f_n\| = 1$. Then

$$\|(A - \lambda)f_n\|^2 \leq n^{-2} \int_{\lambda-1/n}^{\lambda+1/n} f_n^2 d\psi = n^{-2}$$

so that $\lambda \in \sigma(A)$. It follows that $C\sigma(A) = S(\psi) \setminus D(\psi)$.

REFERENCES

1. N. I. Ahiezer and M. Krein, *Some Questions in the Theory of Moments*. Transl. Math. Monographs, Vol. 2, Amer. Math. Soc., 1962.
2. T. S. Chihara, *Chain sequences and orthogonal polynomials*, Trans. Amer. Math. Soc. **104** (1962), 1-16.
3. D. P. Maki, *A note on recursively defined orthogonal polynomials*, Pacific J. Math. **28** (1969), 611-613.
4. J. Shohat and J. Tamarkin, *The Problem of Moments*, Math. Surveys No. 1, Amer. Math. Soc., 1943, 1950.
5. H. S. Wall, *Analytic Theory of Continued Fractions*, van Nostrand, New York, 1948.

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John D. Arrison and Michael Rich, <i>On nearly commutative degree one algebras</i>	533
Bruce Alan Barnes, <i>Algebras with minimal left ideals which are Hilbert spaces</i>	537
Robert F. Brown, <i>An elementary proof of the uniqueness of the fixed point index</i>	549
Ronn L. Carpenter, <i>Principal ideals in F-algebras</i>	559
Chen Chung Chang and Yiannis (John) Nicolas Moschovakis, <i>The Suslin-Kleene theorem for V_κ with cofinality $(\kappa) = \omega$</i>	565
Theodore Seio Chihara, <i>The derived set of the spectrum of a distribution function</i>	571
Tae Geun Cho, <i>On the Choquet boundary for a nonclosed subspace of $C(S)$</i>	575
Richard Brian Darst, <i>The Lebesgue decomposition, Radon-Nikodym derivative, conditional expectation, and martingale convergence for lattices of sets</i>	581
David E. Fields, <i>Dimension theory in power series rings</i>	601
Michael Lawrence Fredman, <i>Congruence formulas obtained by counting irreducibles</i>	613
John Eric Gilbert, <i>On the ideal structure of some algebras of analytic functions</i>	625
G. Goss and Giovanni Viglino, <i>Some topological properties weaker than compactness</i>	635
George Grätzer and J. Sichler, <i>On the endomorphism semigroup (and category) of bounded lattices</i>	639
R. C. Lacher, <i>Cell-like mappings. II</i>	649
Shiva Narain Lal, <i>On a theorem of M. Izumi and S. Izumi</i>	661
Howard Barrow Lambert, <i>Differential mappings on a vector space</i>	669
Richard G. Levin and Takayuki Tamura, <i>Notes on commutative power joined semigroups</i>	673
Robert Edward Lewand and Kevin Mor McCrimmon, <i>Macdonald's theorem for quadratic Jordan algebras</i>	681
J. A. Marti, <i>On some types of completeness in topological vector spaces</i>	707
Walter J. Meyer, <i>Characterization of the Steiner point</i>	717
Saad H. Mohamed, <i>Rings whose homomorphic images are q-rings</i>	727
Thomas V. O'Brien and William Lawrence Reddy, <i>Each compact orientable surface of positive genus admits an expansive homeomorphism</i>	737
Robert James Plemmons and M. T. West, <i>On the semigroup of binary relations</i>	743
Calvin R. Putnam, <i>Unbounded inverses of hyponormal operators</i>	755
William T. Reid, <i>Some remarks on special disconjugacy criteria for differential systems</i>	763
C. Ambrose Rogers, <i>The convex generation of convex Borel sets in euclidean space</i>	773
S. Saran, <i>A general theorem for bilinear generating functions</i>	783
S. W. Smith, <i>Cone relationships of biorthogonal systems</i>	787
Wolmer Vasconcelos, <i>On commutative endomorphism rings</i>	795
Vernon Emil Zander, <i>Products of finitely additive set functions from Orlicz spaces</i>	799
G. Sankaranarayanan and C. Suyambulingom, <i>Correction to: "Some renewal theorems concerning a sequence of correlated random variables"</i>	805
Joseph Zaks, <i>Correction to: "Trivially extending decompositions of E^n"</i>	805
Dong Hoon Lee, <i>Correction to: "The adjoint group of Lie groups"</i>	805
James Edward Ward, <i>Correction to: "Two-groups and Jordan algebras"</i>	806