

Pacific Journal of Mathematics

**ON THE CHOQUET BOUNDARY FOR A NONCLOSED
SUBSPACE OF $C(S)$**

TAE GEUN CHO

ON THE CHOQUET BOUNDARY FOR A NONCLOSED SUBSPACE OF $C(S)$

TAE-GUEN CHO

In this paper, it is proved that if a separating (not necessarily closed) subspace X of $C(S)$ which contains all the constant functions is generated by a weakly compact convex subset, then the peak points for X are dense in the Choquet boundary for X . In order to prove the theorem the extremal structure of convex subsets of the conjugate space of a normed linear space is studied.

Let S be a compact Hausdorff space, $C(S)$ the Banach space of all continuous complex functions on S with the sup norm and let X denote a separating subspace of $C(S)$ which contains all the constant functions. X need not be closed under the sup norm. If X is a closed sub-algebra of $C(S)$ and S is metrizable, then the Choquet boundary for X is exactly the set of peak points for X , [cf. 2]. If X is not an algebra, this conclusion may fail to hold. However, if X is closed and separable, then the peak points for X are dense in the Choquet boundary for X (cf. [5]). In this paper the latter will be generalized for certain nonclosed subspaces of $C(S)$. In § 2, it will be shown that if a subspace X is generated by a weakly compact convex subset then the set $M = \{x^* \in X^*; x^*(1) = 1 = \|x^*\|\}$ is the weak* closed convex hull of its weak* absolute exposed points (see Definition 2.3 in § 2 for absolute exposed points). In § 3 it will be proved that a functional x^* in M is a weak* absolute exposed point of M if and only if there is a peak point $s \in S$ for X such that $x^* = \phi(s)$ where ϕ is the natural embedding of S into X^* . The main theorem is a simple consequence of the above two theorems.

2. Normed linear spaces generated by weakly compact convex subsets. Let K be a weakly compact subset of a normed linear space Y . If the linear span of K is norm dense in Y , then Y is said to be generated by a weakly compact subset K . The set K is called a fundamental subset of Y . In a Banach space, the closed convex hull of a weakly compact subset is weakly compact, and hence a Banach space is generated by a weakly compact convex subset if it is generated by a weakly compact subset. But there is an incomplete normed linear space generated by a weakly compact subset which does not contain a weakly compact convex fundamental subset (see Example 3 in § 3). It is clear that every separable normed linear space is generated by a weakly compact subset. Therefore, every

norm bounded linear image of a separable Banach space is generated by a weakly compact convex subset.

Let F be a subspace of the conjugate space Y^* of a normed linear space Y .

DEFINITION 2.1. A point x of a convex subset C of Y is an F -exposed point of C if there exists a functional f in F such that $\operatorname{Re} f(x) > \operatorname{Re} f(y)$ for all $y \in C$, $y \neq x$.

If F coincides with the conjugate space Y^* , then an F -exposed point is called an exposed point. If Y is a conjugate space of a normed linear space and F is the set of all weak* continuous functionals on Y , then an F -exposed point is called a weak* exposed point. General information about exposed points can be found in either [3] or [4].

Our first theorem is an easy consequence of methods used by Amir and Lindenstrauss in proving a related result, Theorem 4 of [1].

THEOREM 2.2. *Let Y be a normed linear space generated by a weakly compact convex subset. Then every weak* compact convex subset C of the conjugate space Y^* is the weak* closed convex hull of its weak* exposed points.*

Proof. It is clear from the proof of Proposition 2 of [1] that the latter is valid for an incomplete space if it is generated by a weakly compact convex set. The reasoning of Theorem 4 of [1] applies to yield the desired conclusion.

DEFINITION 2.3. A point x of a convex subset C of a normed linear space Y is an (weak*) absolute exposed point of C if there is a (weak*) continuous linear functional f such that

$$f(x) = \sup \{ |f(y)| : y \in C \} \text{ and } f(x) \neq \operatorname{Re} f(y) \text{ for all } y \in C, y \neq x.$$

If x is an absolute exposed point of a convex set C and if f is a functional which realizes its maximum modulus over C at x then the affine functional $f + 1$ peaks at x . An absolute exposed point is an exposed point but the converse does not hold, (see Example 1 in § 3). However, it is clear from the definition that every exposed point of a circled convex set is an absolute exposed point of the set.

LEMMA 2.4. *Suppose that $z = \sum_{j=1}^n t_j \alpha_j$, where $|a_j| \leq 1$ and $t_j > 0$ for each j and $\sum_{j=1}^n t_j = 1$. If $\operatorname{Re} z > \sqrt{1 - \delta^2}$ for a given $0 < \delta < 1$, then $\sum_{j=1}^n t_j |\operatorname{Im} \alpha_j| < \delta$.*

Proof. Let $z_1 = \sum_{j=1}^n t_j(\operatorname{Re} \alpha_j + i |\operatorname{Im} \alpha_j|)$. Then $\operatorname{Re} z = \operatorname{Re} z_1$ and $|z_1| \leq 1$. Now

$$\left(\sum_{j=1}^n t_j |\operatorname{Im} \alpha_j|\right)^2 = (\operatorname{Im} z_1)^2 = |z_1|^2 - (\operatorname{Re} z_1)^2 < 1 - (1 - \delta^2) = \delta^2 .$$

THEOREM 2.5. *Let X be a separating subspace of $C(S)$ with $1 \in X$. If X is generated by a weakly compact convex subset, then $M = \{x^* \in X^*; x^*(1) = 1 = \|x^*\|\}$ is the weak* closed convex hull of its weak* absolute exposed points.*

Proof. Let M_1 be the weak* closed convex hull of

$$M_0 = \{\alpha x^*; \alpha = a + ib \text{ with } |\alpha| \leq 1 \text{ and } x^* \in M\} .$$

Since M_1 is a circled weak* compact convex set, it is the weak* closed convex hull of its weak* absolute exposed points by Theorem 2.2. Let C be the weak* closed convex hull of all the weak* absolute exposed points of M_1 which are in M . It suffices to show that $C = M$. Suppose that $C \neq M$ and let z^* be a functional in $M - C$. By the separation theorem, we may choose a function z in X with $\|z\| = 1$ and a number δ , $0 < \delta < 1$, such that

$$\operatorname{Re} z^*(z) > 2\delta + \sup \{\operatorname{Re} x^*(z); x^* \in C\} .$$

Since $x^*(1) = 1$ for all x^* in M we may assume that $\operatorname{Re} x^*(z) \geq 0$ for all x^* in M . On the other hand, since the functional z^* is in M_1 , the weak* closed convex hull of weak* absolute exposed points of itself, for the number δ we may choose a functional

$$y^* = \sum_{i=1}^n t_i y_i^*$$

where $\sum_{i=1}^n t_i = 1$, $0 < t_i < 1$ and y_i^* is a weak* absolute exposed point of M_1 , $i = 1, 2, \dots, n$, such that

$$(1) \quad |z^*(z) - y^*(z)| < \delta$$

and

$$(2) \quad |z^*(1) - y^*(1)| < 1 - \sqrt{1 - \delta^2} .$$

Note that $y_i^* = \alpha_i z_i^*$, where α_i is a complex number with $|\alpha_i| \leq 1$ and z_i^* is a function in M which is a weak* absolute exposed point of M_1 , since every exposed point of M_1 belongs to M_0 by Milman's theorem. Therefore,

$$y^* = \sum_{i=1}^n (t_i \alpha_i) z_i^* .$$

Since $z^*, z_i^* \in M$, $z^*(1) = 1$ and $z_i^*(1) = 1$, hence, taking the real part of $z^*(1) - y^*(1)$ of (2) we see that $\operatorname{Re} y^*(1) > \sqrt{1 - \delta^2}$. Therefore, $\sum_{i=1}^n t_i |\operatorname{Im} \alpha_i| < \delta$ by the lemma.

Now,

$$\begin{aligned} |z^*(z) - y^*(z)| &\geq |\operatorname{Re} z^*(z) - \operatorname{Re} y^*(z)| \\ &= \left| \sum_{i=1}^n t_i [\operatorname{Re} z^*(z) - (\operatorname{Re} \alpha_i) (\operatorname{Re} z_i^*(z))] \right. \\ &\quad \left. + \sum_{i=1}^n t_i (\operatorname{Im} \alpha_i) (\operatorname{Im} z_i^*(z)) \right| \\ &\geq 2\delta - \sum_{i=1}^n t_i |\operatorname{Im} \alpha_i| \\ &> \delta. \end{aligned}$$

This contradicts (1). Therefore $M = C$.

3. Function spaces generated by weakly compact convex subsets. Throughout this section, S will denote a compact Hausdorff space and X a (not necessarily closed) subspace of $C(S)$ with the sup norm. The mapping $\phi: S \rightarrow X^*$, defined by $\phi(s)x = x(s)$ for all $x \in X$ and for each $s \in S$, is a homeomorphism between S and $\phi(S)$ with respect to the weak* topology of X^* . The convex set

$$M = \{x^* \in X^*; x^*(1) = 1 = \|x^*\|\}$$

is the weak* closed convex hull of $\phi(S)$ and if x^* is an extreme point of M , there is a point $s \in S$ such that $\phi(s) = x^*$. The set of extreme points of M is called the Choquet boundary for X (cf. [2] and [5]). By a peak point for X we mean a point s of S such that there exists a function x in X with the property that $|x(s)| > |x(t)|$ for all $t \in S$, $t \neq s$.

THEOREM 3.1. *Let X be a separating subspace of $C(S)$ with $1 \in X$ and let $M = \{x^* \in X^*; x^*(1) = 1 = \|x^*\|\}$. Then a linear functional $x^* \in M$ is a weak* absolute exposed point of M if and only if there exists a peak point $s \in S$ for X such that $x^* = \phi(s)$.*

Proof. (\Rightarrow) If $x \in X$ exposes $x^* = \phi(s)$ absolutely, it follows easily that $x + 1$ peaks at s .

(\Leftarrow) Suppose that $s \in S$ is a peak point for X and let x be a function in X which peaks at s . Then $\phi(s)$ is the only functional in $\phi(S)$ such that $\phi(s)x = 1$. Let

$$M_x = \{x^* \in M; x^*(x) = 1\}.$$

Since every extreme point of the weak* compact convex set M_x is an extreme point of M , hence in $\phi(S)$, we see that $M_x = \{\phi(s)\}$ and therefore $\phi(s)$ is a weak* absolute exposed point of M .

The following example shows a weak* exposed point which is not a weak* absolute exposed point.

EXAMPLE 1. Let $S = \{\zeta = \xi + i\eta; \xi^4 + \eta^4 \leq 1\}$ and let $X \subset C(S)$ be the linear span of x and 1, where $x(\zeta) = \zeta$ for each $\zeta \in S$. Then the boundary of S is the Choquet boundary for X since M is affinely homeomorphic to S . The points $\pm 1, \pm i$ are not weak* absolute exposed points of M (i.e., they are not peak points for X), although they are weak* exposed points of M .

Our main theorem is an immediate consequence of Theorem 2.5 and Theorem 3.1.

THEOREM 3.2. *Let X be a separating subspace (not necessarily closed) of $C(S)$ such that $1 \in X$. If X is generated by a weakly compact convex subset, then the peak points for X are dense in the Choquet boundary for X .*

Proof. The set $M = \{x^* \in X^*; x^*(1) = 1 = \|x^*\|\}$ is the weak* closed convex hull of its weak* absolute exposed points. Since weak* absolute exposed points of M are peak points for X the theorem holds by Mil'man's theorem.

REMARK. The real case of Theorem 3.2 can be proved without the need of Theorem 2.5.

COROLLARY 3.3. *Let X be a separating subspace of $C(S)$ such that $1 \in X$. If there is a Banach space Y generated by a weakly compact subset and a bounded linear operator from Y onto X , then the peak points for X are dense in the Choquet boundary for X .*

Proof. Let K be a weakly compact fundamental subset of Y . Then the continuous linear image of the closed convex hull of K is a weakly compact convex fundamental subset of X .

EXAMPLE 2. Let X be a separable, commutative, semi-simple Banach algebra with identity. X is isomorphic to a subspace of $C(\mathcal{M})$ where \mathcal{M} is the maximal ideal space of X . By the Corollary 3.3 peak points for X are dense in the Choquet boundary for X .

EXAMPLE 3. Let S be the Cantor set in $[0, 1]$. Let

$$X = \{f \in C(S); f \text{ is a simple function}\}.$$

X is clearly a separating subalgebra of $C(S)$ with $1 \in X$ but X contains no peaking function and hence there is no peak point for X in S . Since X is separable, it contains a weakly compact fundamental subset, however it contains no weakly compact convex fundamental subset by Theorem 3.2.

The author would like to express his sincere gratitude to Professor P. C. Curtis, Jr., the author's thesis advisor, for valuable advice and consultation.

BIBLIOGRAPHY

1. D. Amir and J. Lindenstrauss, *The structure of weakly compact subsets in Banach spaces*, Ann. of Math. **88** (1968), 35-46.
2. E. Bishop and K. deLeeuw, *The representation of linear functionals by measures on sets of extreme points*, Ann. Inst. Fourier (Grenoble) **9** (1959), 305-331.
3. M. M. Day, *Normed Linear Spaces*, Springer-Verlag, 1962.
4. V. L. Klee, Jr., *Some new results on smoothness and rotundity in normed linear spaces*, Math. Ann. **139** (1959), 51-63.
5. R. Phelps, *Lectures on Choquet's Theorem*, Van Nostrand, 1966.

Received November 7, 1969. This is a part of the author's doctoral dissertation submitted to UCLA in 1969.

STATE UNIVERSITY OF NEW YORK, ALBANY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON
Stanford University
Stanford, California 94305

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

RICHARD PIERCE
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLE

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
TRW SYSTEMS
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Pacific Journal of Mathematics

Vol. 35, No. 3

November, 1970

John D. Arrison and Michael Rich, <i>On nearly commutative degree one algebras</i>	533
Bruce Alan Barnes, <i>Algebras with minimal left ideals which are Hilbert spaces</i>	537
Robert F. Brown, <i>An elementary proof of the uniqueness of the fixed point index</i>	549
Ronn L. Carpenter, <i>Principal ideals in F-algebras</i>	559
Chen Chung Chang and Yiannis (John) Nicolas Moschovakis, <i>The Suslin-Kleene theorem for V_κ with cofinality $(\kappa) = \omega$</i>	565
Theodore Seio Chihara, <i>The derived set of the spectrum of a distribution function</i>	571
Tae Geun Cho, <i>On the Choquet boundary for a nonclosed subspace of $C(S)$</i>	575
Richard Brian Darst, <i>The Lebesgue decomposition, Radon-Nikodym derivative, conditional expectation, and martingale convergence for lattices of sets</i>	581
David E. Fields, <i>Dimension theory in power series rings</i>	601
Michael Lawrence Fredman, <i>Congruence formulas obtained by counting irreducibles</i>	613
John Eric Gilbert, <i>On the ideal structure of some algebras of analytic functions</i>	625
G. Goss and Giovanni Viglino, <i>Some topological properties weaker than compactness</i>	635
George Grätzer and J. Sichler, <i>On the endomorphism semigroup (and category) of bounded lattices</i>	639
R. C. Lacher, <i>Cell-like mappings. II</i>	649
Shiva Narain Lal, <i>On a theorem of M. Izumi and S. Izumi</i>	661
Howard Barrow Lambert, <i>Differential mappings on a vector space</i>	669
Richard G. Levin and Takayuki Tamura, <i>Notes on commutative power joined semigroups</i>	673
Robert Edward Lewand and Kevin Mor McCrimmon, <i>Macdonald's theorem for quadratic Jordan algebras</i>	681
J. A. Marti, <i>On some types of completeness in topological vector spaces</i>	707
Walter J. Meyer, <i>Characterization of the Steiner point</i>	717
Saad H. Mohamed, <i>Rings whose homomorphic images are q-rings</i>	727
Thomas V. O'Brien and William Lawrence Reddy, <i>Each compact orientable surface of positive genus admits an expansive homeomorphism</i>	737
Robert James Plemmons and M. T. West, <i>On the semigroup of binary relations</i>	743
Calvin R. Putnam, <i>Unbounded inverses of hyponormal operators</i>	755
William T. Reid, <i>Some remarks on special disconjugacy criteria for differential systems</i>	763
C. Ambrose Rogers, <i>The convex generation of convex Borel sets in euclidean space</i>	773
S. Saran, <i>A general theorem for bilinear generating functions</i>	783
S. W. Smith, <i>Cone relationships of biorthogonal systems</i>	787
Wolmer Vasconcelos, <i>On commutative endomorphism rings</i>	795
Vernon Emil Zander, <i>Products of finitely additive set functions from Orlicz spaces</i>	799
G. Sankaranarayanan and C. Suyambulingom, <i>Correction to: "Some renewal theorems concerning a sequence of correlated random variables"</i>	805
Joseph Zaks, <i>Correction to: "Trivially extending decompositions of E^n"</i>	805
Dong Hoon Lee, <i>Correction to: "The adjoint group of Lie groups"</i>	805
James Edward Ward, <i>Correction to: "Two-groups and Jordan algebras"</i>	806