

# Pacific Journal of Mathematics

**ON THE CHOQUET BOUNDARY FOR A NONCLOSED  
SUBSPACE OF  $C(S)$**

TAE GEUN CHO

# ON THE CHOQUET BOUNDARY FOR A NONCLOSED SUBSPACE OF $C(S)$

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**In this paper, it is proved that if a separating (not necessarily closed) subspace  $X$  of  $C(S)$  which contains all the constant functions is generated by a weakly compact convex subset, then the peak points for  $X$  are dense in the Choquet boundary for  $X$ . In order to prove the theorem the extremal structure of convex subsets of the conjugate space of a normed linear space is studied.**

Let  $S$  be a compact Hausdorff space,  $C(S)$  the Banach space of all continuous complex functions on  $S$  with the sup norm and let  $X$  denote a separating subspace of  $C(S)$  which contains all the constant functions.  $X$  need not be closed under the sup norm. If  $X$  is a closed sub-algebra of  $C(S)$  and  $S$  is metrizable, then the Choquet boundary for  $X$  is exactly the set of peak points for  $X$ , [cf. 2]. If  $X$  is not an algebra, this conclusion may fail to hold. However, if  $X$  is closed and separable, then the peak points for  $X$  are dense in the Choquet boundary for  $X$  (cf. [5]). In this paper the latter will be generalized for certain nonclosed subspaces of  $C(S)$ . In § 2, it will be shown that if a subspace  $X$  is generated by a weakly compact convex subset than the set  $M = \{x^* \in X^*; x^*(1) = 1 = \|x^*\|\}$  is the weak\* closed convex hull of its weak\* absolute exposed points (see Definition 2.3 in § 2 for absolute exposed points). In § 3 it will be proved that a functional  $x^*$  in  $M$  is a weak\* absolute exposed point of  $M$  if and only if there is a peak point  $s \in S$  for  $X$  such that  $x^* = \phi(s)$  where  $\phi$  is the natural embedding of  $S$  into  $X^*$ . The main theorem is a simple consequence of the above two theorems.

2. Normed linear spaces generated by weakly compact convex subsets. Let  $K$  be a weakly compact subset of a normed linear space  $Y$ . If the linear span of  $K$  is norm dense in  $Y$ , then  $Y$  is said to be generated by a weakly compact subset  $K$ . The set  $K$  is called a fundamental subset of  $Y$ . In a Banach space, the closed convex hull of a weakly compact subset is weakly compact, and hence a Banach space is generated by a weakly compact convex subset if it is generated by a weakly compact subset. But there is an incomplete normed linear space generated by a weakly compact subset which does not contain a weakly compact convex fundamental subset (see Example 3 in § 3). It is clear that every separable normed linear space is generated by a weakly compact subset. Therefore, every

norm bounded linear image of a separable Banach space is generated by a weakly compact convex subset.

Let  $F$  be a subspace of the conjugate space  $Y^*$  of a normed linear space  $Y$ .

**DEFINITION 2.1.** A point  $x$  of a convex subset  $C$  of  $Y$  is an  $F$ -exposed point of  $C$  if there exists a functional  $f$  in  $F$  such that  $\operatorname{Re} f(x) > \operatorname{Re} f(y)$  for all  $y \in C$ ,  $y \neq x$ .

If  $F$  coincides with the conjugate space  $Y^*$ , then an  $F$ -exposed point is called an exposed point. If  $Y$  is a conjugate space of a normed linear space and  $F$  is the set of all weak\* continuous functionals on  $Y$ , then an  $F$ -exposed point is called a weak\* exposed point. General information about exposed points can be found in either [3] or [4].

Our first theorem is an easy consequence of methods used by Amir and Lindenstrauss in proving a related result, Theorem 4 of [1].

**THEOREM 2.2.** *Let  $Y$  be a normed linear space generated by a weakly compact convex subset. Then every weak\* compact convex subset  $C$  of the conjugate space  $Y^*$  is the weak\* closed convex hull of its weak\* exposed points.*

*Proof.* It is clear from the proof of Proposition 2 of [1] that the latter is valid for an incomplete space if it is generated by a weakly compact convex set. The reasoning of Theorem 4 of [1] applies to yield the desired conclusion.

**DEFINITION 2.3.** A point  $x$  of a convex subset  $C$  of a normed linear space  $Y$  is an (weak\*) absolute exposed point of  $C$  if there is a (weak\*) continuous linear functional  $f$  such that

$$f(x) = \sup \{ |f(y)| : y \in C \} \text{ and } f(x) \neq \operatorname{Re} f(y) \text{ for all } y \in C, y \neq x.$$

If  $x$  is an absolute exposed point of a convex set  $C$  and if  $f$  is a functional which realizes its maximum modulus over  $C$  at  $x$  then the affine functional  $f + 1$  peaks at  $x$ . An absolute exposed point is an exposed point but the converse does not hold, (see Example 1 in § 3). However, it is clear from the definition that every exposed point of a circled convex set is an absolute exposed point of the set.

**LEMMA 2.4.** *Suppose that  $z = \sum_{j=1}^n t_j \alpha_j$ , where  $|a_j| \leq 1$  and  $t_j > 0$  for each  $j$  and  $\sum_{j=1}^n t_j = 1$ . If  $\operatorname{Re} z > \sqrt{1 - \delta^2}$  for a given  $0 < \delta < 1$ , then  $\sum_{j=1}^n t_j |\operatorname{Im} \alpha_j| < \delta$ .*

*Proof.* Let  $z_1 = \sum_{j=1}^n t_j(\operatorname{Re} \alpha_j + i |\operatorname{Im} \alpha_j|)$ . Then  $\operatorname{Re} z = \operatorname{Re} z_1$  and  $|z_1| \leq 1$ . Now

$$\left(\sum_{j=1}^n t_j |\operatorname{Im} \alpha_j|\right)^2 = (\operatorname{Im} z_1)^2 = |z_1|^2 - (\operatorname{Re} z_1)^2 < 1 - (1 - \delta^2) = \delta^2.$$

**THEOREM 2.5.** *Let  $X$  be a separating subspace of  $C(S)$  with  $1 \in X$ . If  $X$  is generated by a weakly compact convex subset, then  $M = \{x^* \in X^*; x^*(1) = 1 = \|x^*\|\}$  is the weak\* closed convex hull of its weak\* absolute exposed points.*

*Proof.* Let  $M_1$  be the weak\* closed convex hull of

$$M_0 = \{\alpha x^*; \alpha = a + ib \text{ with } |\alpha| \leq 1 \text{ and } x^* \in M\}.$$

Since  $M_1$  is a circled weak\* compact convex set, it is the weak\* closed convex hull of its weak\* absolute exposed points by Theorem 2.2. Let  $C$  be the weak\* closed convex hull of all the weak\* absolute exposed points of  $M_1$  which are in  $M$ . It suffices to show that  $C = M$ . Suppose that  $C \neq M$  and let  $z^*$  be a functional in  $M - C$ . By the separation theorem, we may choose a function  $z$  in  $X$  with  $\|z\| = 1$  and a number  $\delta$ ,  $0 < \delta < 1$ , such that

$$\operatorname{Re} z^*(z) > 2\delta + \sup\{\operatorname{Re} x^*(z); x^* \in C\}.$$

Since  $x^*(1) = 1$  for all  $x^*$  in  $M$  we may assume that  $\operatorname{Re} x^*(z) \geq 0$  for all  $x^*$  in  $M$ . On the other hand, since the functional  $z^*$  is in  $M_1$ , the weak\* closed convex hull of weak\* absolute exposed points of itself, for the number  $\delta$  we may choose a functional

$$y^* = \sum_{i=1}^n t_i y_i^*$$

where  $\sum_{i=1}^n t_i = 1$ ,  $0 < t_i < 1$  and  $y_i^*$  is a weak\* absolute exposed point of  $M_1$ ,  $i = 1, 2, \dots, n$ , such that

$$(1) \quad |z^*(z) - y^*(z)| < \delta$$

and

$$(2) \quad |z^*(1) - y^*(1)| < 1 - \sqrt{1 - \delta^2}.$$

Note that  $y_i^* = \alpha_i z_i^*$ , where  $\alpha_i$  is a complex number with  $|\alpha_i| \leq 1$  and  $z_i^*$  is a function in  $M$  which is a weak\* absolute exposed point of  $M_1$ , since every exposed point of  $M_1$  belongs to  $M_0$  by Milman's theorem. Therefore,

$$y^* = \sum_{i=1}^n (t_i \alpha_i) z_i^*.$$

Since  $z^*, z_i^* \in M$ ,  $z^*(1) = 1$  and  $z_i^*(1) = 1$ , hence, taking the real part of  $z^*(1) - y^*(1)$  of (2) we see that  $\operatorname{Re} y^*(1) > \sqrt{1 - \delta^2}$ . Therefore,  $\sum_{i=1}^n t_i |\operatorname{Im} \alpha_i| < \delta$  by the lemma.

Now,

$$\begin{aligned} |z^*(z) - y^*(z)| &\geq |\operatorname{Re} z^*(z) - \operatorname{Re} y^*(z)| \\ &= \left| \sum_{i=1}^n t_i [\operatorname{Re} z^*(z) - (\operatorname{Re} \alpha_i) (\operatorname{Re} z_i^*(z))] \right. \\ &\quad \left. + \sum_{i=1}^n t_i (\operatorname{Im} \alpha_i) (\operatorname{Im} z_i^*(z)) \right| \\ &\geq 2\delta - \sum_{i=1}^n t_i |\operatorname{Im} \alpha_i| \\ &> \delta. \end{aligned}$$

This contradicts (1). Therefore  $M = C$ .

**3. Function spaces generated by weakly compact convex subsets.** Throughout this section,  $S$  will denote a compact Hausdorff space and  $X$  a (not necessarily closed) subspace of  $C(S)$  with the sup norm. The mapping  $\phi: S \rightarrow X^*$ , defined by  $\phi(s)x = x(s)$  for all  $x \in X$  and for each  $s \in S$ , is a homeomorphism between  $S$  and  $\phi(S)$  with respect to the weak\* topology of  $X^*$ . The convex set

$$M = \{x^* \in X^*; x^*(1) = 1 = \|x^*\|\}$$

is the weak\* closed convex hull of  $\phi(S)$  and if  $x^*$  is an extreme point of  $M$ , there is a point  $s \in S$  such that  $\phi(s) = x^*$ . The set of extreme points of  $M$  is called the Choquet boundary for  $X$  (cf. [2] and [5]). By a peak point for  $X$  we mean a point  $s$  of  $S$  such that there exists a function  $x$  in  $X$  with the property that  $|x(s)| > |x(t)|$  for all  $t \in S$ ,  $t \neq s$ .

**THEOREM 3.1.** *Let  $X$  be a separating subspace of  $C(S)$  with  $1 \in X$  and let  $M = \{x^* \in X^*; x^*(1) = 1 = \|x^*\|\}$ . Then a linear functional  $x^* \in M$  is a weak\* absolute exposed point of  $M$  if and only if there exists a peak point  $s \in S$  for  $X$  such that  $x^* = \phi(s)$ .*

*Proof.* ( $\Rightarrow$ ) If  $x \in X$  exposes  $x^* = \phi(s)$  absolutely, it follows easily that  $x + 1$  peaks at  $s$ .

( $\Leftarrow$ ) Suppose that  $s \in S$  is a peak point for  $X$  and let  $x$  be a function in  $X$  which peaks at  $s$ . Then  $\phi(s)$  is the only functional in  $\phi(S)$  such that  $\phi(s)x = 1$ . Let

$$M_x = \{x^* \in M; x^*(x) = 1\}.$$

Since every extreme point of the weak\* compact convex set  $M_x$  is an extreme point of  $M$ , hence in  $\phi(S)$ , we see that  $M_x = \{\phi(s)\}$  and therefore  $\phi(s)$  is a weak\* absolute exposed point of  $M$ .

The following example shows a weak\* exposed point which is not a weak\* absolute exposed point.

EXAMPLE 1. Let  $S = \{\zeta = \xi + i\eta; \xi^4 + \eta^4 \leq 1\}$  and let  $X \subset C(S)$  be the linear span of  $x$  and 1, where  $x(\zeta) = \zeta$  for each  $\zeta \in S$ . Then the boundary of  $S$  is the Choquet boundary for  $X$  since  $M$  is affinely homeomorphic to  $S$ . The points  $\pm 1$ ,  $\pm i$  are not weak\* absolute exposed points of  $M$  (i.e., they are not peak points for  $X$ ), although they are weak\* exposed points of  $M$ .

Our main theorem is an immediate consequence of Theorem 2.5 and Theorem 3.1.

THEOREM 3.2. *Let  $X$  be a separating subspace (not necessarily closed) of  $C(S)$  such that  $1 \in X$ . If  $X$  is generated by a weakly compact convex subset, then the peak points for  $X$  are dense in the Choquet boundary for  $X$ .*

*Proof.* The set  $M = \{x^* \in X^*; x^*(1) = 1 = \|x^*\|\}$  is the weak\* closed convex hull of its weak\* absolute exposed points. Since weak\* absolute exposed points of  $M$  are peak points for  $X$  the theorem holds by Mil'man's theorem.

REMARK. The real case of Theorem 3.2 can be proved without the need of Theorem 2.5.

COROLLARY 3.3. *Let  $X$  be a separating subspace of  $C(S)$  such that  $1 \in X$ . If there is a Banach space  $Y$  generated by a weakly compact subset and a bounded linear operator from  $Y$  onto  $X$ , then the peak points for  $X$  are dense in the Choquet boundary for  $X$ .*

*Proof.* Let  $K$  be a weakly compact fundamental subset of  $Y$ . Then the continuous linear image of the closed convex hull of  $K$  is a weakly compact convex fundamental subset of  $X$ .

EXAMPLE 2. Let  $X$  be a separable, commutative, semi-simple Banach algebra with identity.  $X$  is isomorphic to a subspace of  $C(\mathcal{M})$  where  $\mathcal{M}$  is the maximal ideal space of  $X$ . By the Corollary 3.3 peak points for  $X$  are dense in the Choquet boundary for  $X$ .

EXAMPLE 3. Let  $S$  be the Cantor set in  $[0, 1]$ . Let

$$X = \{f \in C(S); f \text{ is a simple function}\}.$$

$X$  is clearly a separating subalgebra of  $C(S)$  with  $1 \in X$  but  $X$  contains no peaking function and hence there is no peak point for  $X$  in  $S$ . Since  $X$  is separable, it contains a weakly compact fundamental subset, however it contains no weakly compact convex fundamental subset by Theorem 3.2.

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