DIMENSION THEORY IN POWER SERIES RINGS

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Let \( V \) be a valuation ring of finite rank \( n \). If \( V \) is discrete, then \( V[[X]] \) has dimension \( n + 1 \). If \( V \) is not discrete, then the dimension of \( V[[X]] \) is at least \( n + k + 1 \), where \( k \) is the number of idempotent proper prime ideals of \( V \).

Let \( R \) be a commutative ring with identity. If there exists a chain \( P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots \subseteq P_n \) of \( n + 1 \) prime ideals of \( R \), where \( P_n \subseteq R \), but no such chain of \( n + 2 \) prime ideals, then we say that \( R \) has dimension \( n \) and we write \( \dim R = n \) \cite{3}. In \cite{3} and \cite{4}, Seidenberg has investigated the dimension theory of \( R[X_1, X_2, \cdots, X_m] \) where \( R \) has finite dimension and \( X_1, X_2, \cdots, X_m \) are indeterminates over \( R \). We investigate the dimension theory of \( V[[X]] \) where \( V \) is a valuation ring.

Throughout this paper, \( R \) denotes a commutative ring with identity; \( \omega \) is the set of natural numbers; \( \omega_0 \) is the set of non-negative integers; and \( Z \) is the set of integers. If

\[
f(X) = \sum_{i=0}^{\infty} f_i X^i \in R[[X]],
\]

we denote by \( A_f \) the ideal of \( R \) generated by the coefficients of \( f(X) \):

\[
A_f = \{ f_0, f_1, \cdots, f_k, \cdots \} R.
\]

If \( A \) is an ideal of \( R \), we let

\[
A[[X]] = \{ f(X) = \sum_{i=0}^{\infty} f_i X^i : f_i \in A \text{ for each } i \in \omega_0 \}
\]

and we define \( A \cdot R[[X]] \) to be the ideal of \( R[[X]] \) which is generated by \( A \). Then \( A \cdot R[[X]] = \{ f(X) : A_f \subseteq B \text{ for some finitely generated ideal } B \text{ of } R \text{ with } B \subseteq A \} \). It is clear that \( A \cdot R[[X]] \subseteq A[[X]] \); equality holds if and only if each countably generated ideal of \( R \) contained in \( A \) is contained in a finitely generated ideal of \( R \) contained in \( A \). In particular, if \( V \) is a valuation ring containing an ideal \( A \) which is countably generated but not finitely generated, then \( A \cdot V[[X]] \subseteq A[[X]] \). Finally, we note that if \( A \) is an ideal of \( R \), then \( R[[X]]/A[[X]] \cong (R/A)[[X]] \); hence \( A[[X]] \) is a prime ideal of \( R[[X]] \) if and only if \( A \) is a prime ideal of \( R \).

2. Discrete valuation rings. Let \( V \) be a valuation ring of rank \( k \) with associated valuation \( v \) and value group \( G \); let \( \{0\} = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_k = G \) be the chain of isolated subgroups of \( G \) together with \( G \). In \cite{2}, Iwasawa proves that for \( 1 \leq i \leq k \),
$G_i/G_{i-1} \cong H_i$ where $H_i$ is a subgroup of the additive group of real numbers, this being an order-preserving isomorphism of groups. If for $1 \leq i \leq k$, $H_i \cong \mathbb{Z}$, we shall say that $V$ is a discrete valuation ring of rank $k$. This is equivalent to the condition that $V$ contains no idempotent proper prime ideal.

**Lemma 2.1.** Let $V$ be a valuation ring and let $P$ be a proper prime ideal of $V$. If $P$ is not idempotent, then in $V[[X]]$,

$$\sqrt{(P \cdot V[[X]])} = P[[X]]$$

and $(P[[X]])^2 \subseteq P \cdot V[[X]]$.

**Proof.** Let $\alpha \in P$, $\alpha \notin P^2$. Then

$$(P[[X]])^2 \subseteq P^2[[X]] \subseteq (\alpha) V[[X]] \subseteq P \cdot V[[X]].$$

Hence $P[[X]] \subseteq \sqrt{(P \cdot V[[X]])}$ and the reverse containment is clear.

**Lemma 2.2.** Let $V$ be a valuation ring with quotient field $K$ and let $P$ be a proper prime ideal of $V$. Let

$$D = V[[X]] [K] = (V[[X]])_{\mathfrak{m} \setminus \{0\}}.$$ 

Then $D = (V_P[[X]])_{\mathfrak{m} \setminus \{0\}}$.

**Proof.** We first show that $V_P[[X]] \subseteq D$. Let

$$f(X) = \sum_{i=0}^{n} f_i X^i \in V_P[[X]].$$

For each $i \in \omega$, there exists $r_i \in V \setminus P$ such that $r_i f_i \in V$. Let $a \in P \setminus \{0\}$; then for each $i \in \omega$, $a/r_i \in PV_P = P \subseteq V$, implying that $af_i = (a/r_i) (r_i f_i) \in V$; that is, $af(X) \in V[[X]]$. This implies that $f(X) \in (V[[X]])_{\mathfrak{m} \setminus \{0\}} = D$, showing that $V_P[[X]] \subseteq D$.

Since $D \supseteq K$, each nonzero element of $V_P$ is a unit in $D$. Thus $D \supseteq (V_P[[X]])_{\mathfrak{m} \setminus \{0\}}$ and the reverse containment is obvious.

**Corollary 2.3.** Let $V$ be a valuation ring and let $P$ be a proper prime ideal of $V$. There is a one-to-one correspondence between prime ideals of $V[[X]]$ which contract to $(0)$ in $V$ and prime ideals of $V_P[[X]]$ which contract to $(0)$ in $V_P$; this correspondence preserves containment.

**Proof.** Lemma 2.2 assures that there is a one-to-one, containment preserving correspondence between each of these classes of prime ideals and the class of prime ideals of $D$.

**Lemma 2.4.** Let $R$ be a quasi-local ring having maximal ideal
M. Let \( f(X) \in R[[X]] \), \( f(X) \in M[[X]] \) — say \( f_k \in R \setminus M \), \( k \) minimal. There exists \( g(X) \), a unit of \( R[[X]] \), such that \( f(X)g(X) \) has exactly one unit coefficient, namely \((fg)_k\).

Proof. For \( u(X) \in R[[X]] \), denote by \( \bar{u}(X) \) the canonical image of \( u(X) \) in \( (R/M)[[X]] \). By choice of \( k \),

\[
\bar{f}(X) = \bar{f}_k X^k + \bar{f}_{k+1} X^{k+1} + \cdots = X^k(\bar{f}_k + \bar{f}_{k+1} X + \cdots),
\]

where \( \bar{f}_k \neq 0 \). Then \( \bar{f}_k + \bar{f}_{k+1} X + \cdots \) is a unit of \((R/M)[[X]]\), and we can choose \( g(X) \in R[[X]] \) such that \( \bar{g}(X) \cdot (\bar{f}_k + \bar{f}_{k+1} X + \cdots) = 1 \). Thus \( \bar{f}(X) \cdot \bar{g}(X) = X^k \), and \( f(X)g(X) - X^k \in M[[X]] \). This implies that only the coefficient of \( X^k \) in \( f(X)g(X) \) is not in \( M \).

Corollary 2.5. Let \( V \) be a valuation ring and let \( P \) be a proper prime ideal of \( V \). If \( Q \) is an ideal of \( V_P[[X]] \) and if \( Q \not\subseteq (PV_P)[[X]] \), then \( Q \cap V[[X]] \not\subseteq P[[X]] \).

Proof. Lemma 2.4 assures that there is a power series \( g(X) \) in \( Q \) with \( g(X) \) having exactly one unit coefficient, \( g_k \). Since \( g_k \) is a unit of \( V_P \), there is no loss of generality in assuming that, in fact, \( g_k = 1 \). Then for \( i \neq k \), \( g_i \in PV_P = P \subseteq V \), implying that \( g(X) \in Q \cap V[[X]] \) while \( g(X) \notin P[[X]] \).

Lemma 2.6.¹ Let \( R \) be a Noetherian ring having dimension \( n \). Then \( R[[X_1, X_2, \cdots, X_n]] \) is Noetherian and has dimension \( n + m \).

Proof. It is well known that if \( R \) is Noetherian, then \( R[[X_1, X_2, \cdots, X_m]] \) is Noetherian. We shall show that the dimension of \( R[[X]] \) is \( n + 1 \); the lemma follows immediately by induction on \( m \).

Let \( M \) be a maximal ideal of \( R[[X]] \). Then \( M = M_i + (X) \) for some maximal ideal \( M_i \) of \( R \). Since \( \dim R = n \), the height of \( M_i \) is \( k \) where \( k \leq n \). There exists an ideal \( A = (a_1, a_2, \cdots, a_m) \) of \( R \) which admits \( M_i \) as an isolated prime ideal [5; 242]. It is straightforward to verify that \( M = M_i + (X) \) is an isolated prime ideal of \( A + (X) = (a_1, a_2, \cdots, a_m, X) R[[X]] \). This implies that the height of \( M \) is at most \( k + 1 \) [5; 240]: since \( k \leq n \), the height of \( M \) is at most \( n + 1 \). Since \( M \) was an arbitrary maximal ideal of \( R[[X]] \), we conclude that \( \dim R[[X]] \leq n + 1 \); the reverse inequality is clear.

Theorem 2.7. Let \( V \) be a discrete valuation ring of rank \( n \) and let \( (0) = P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_n \) be the nonunit prime ideals of

¹ The proof of Lemma 2.6 was pointed out to the author by William Heinzer.
V. Then \( \dim V[[X]] = n + 1 \).

Proof. We use induction on \( n \), the case \( n = 1 \) following from Lemma 2.6 since a rank one discrete valuation ring is Noetherian.

Assuming the result for discrete valuation rings of rank less than \( n \), let \( V \) be a discrete valuation ring of rank \( n \) and let \( (0) \subseteq Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_t \) be a chain of prime ideals of \( V[[X]] \). We consider two cases.

Case 1. \( Q_t \cap V \neq (0) \). Here \( Q_t \cap V \supseteq P_i \), so that \( Q_t \supseteq P_i \cdot V[[X]] \), implying that \( Q_t \supseteq \sqrt{(P_i \cdot V[[X]])} = P_i[[X]] \), the latter equality being a consequence of Lemma 2.1. But the depth of \( P_i[[X]] \) cannot exceed \( \dim (V/P_i) [[X]] = n \); we conclude that \( t \leq n + 1 \).

Case 2. \( Q_t \cap V = (0) \). Corollary 2.3 asserts that \( Q_t = Q^* \cap V[[X]] \), where \( Q^* \) is a prime ideal of \( V_{P_i}[[X]] \) and \( Q^* \cap V_{P_i} = (0) \). Since \( \dim V_{P_i}[[X]] = 2 \), \( Q^* \not\subseteq (P_i V_{P_i})[[X]] \). By Corollary 2.5, \( Q_t \not\subseteq P_i[[X]] \). Since \( V_{P_i}[[X]] \) is two-dimensional and local, each proper prime ideal of \( V_{P_i}[[X]] \) which contracts to \( (0) \) in \( V_{P_i} \) is a minimal prime ideal of \( V_{P_i}[[X]] \). Corollary 2.3 now assures that each proper prime ideal of \( V[[X]] \) which contracts to \( (0) \) in \( V \) is a minimal prime ideal of \( V[[X]] \). It follows that \( Q_t \cap V \neq (0) \), implying that \( Q_t \supseteq P_i[[X]] \). Since also \( Q_t \supseteq Q_i \) and \( Q_i \not\subseteq P_i[[X]] \), we conclude that \( Q_t \supseteq P_i[[X]] \). Thus we have a chain \( (0) \subset P_i[[X]] \subset Q_2 \subset Q_3 \subset \cdots \subset Q_t \). It follows, as in Case 1, that \( t \leq n + 1 \).

Thus \( \dim V[[X]] \leq n + 1 \) and the reverse inequality is clear.

3. Rank one nondiscrete valuation rings. We note that if \( V \) is a rank one valuation ring, then the value group of \( v \) is Archimedean.

Lemma 3.1. Let \( V \) be a valuation ring and let \( B \) be an ideal of \( V \). If \( B \) is not finitely generated, then the following conditions are equivalent:

(a) \( f(X) \in B \cdot V[[X]] \).
(b) \( A_f \subseteq (b) \) for some \( b \in B \).
(c) \( f(X) = bg(X) \) for some \( b \in B \), \( g(X) \in V[[X]] \).
(d) \( A_f \subseteq B \).

Proof. We establish that \( (a) \rightarrow (b) \rightarrow (c) \rightarrow (a) \) and that \( (b) \leftrightarrow (d) \).

(a) \rightarrow (b): Let \( f(X) \in B \cdot V[[X]] \); then we can write

\[
f(X) = b_1[g^{(1)}(X)] + b_2[g^{(2)}(X)] + \cdots + b_l[g^{(l)}(X)]
\]
where for $1 \leq i \leq t$, $b_i \in B$ and $g^{(i)}(X) = \sum_{j=0}^{\infty} g_{i,j} X^j \in V[[X]]$. Thus $f(X) = \sum_{i=0}^{\infty} f_i X^i$ where $f_i = \sum_{j=1}^{s} b_j g_{i,j}$. In $V$, $(b_1, b_2, \ldots, b_s) = (b_s)$ for some $s$, $1 \leq s \leq t$. Now for $i \in \omega$, $f_i = \sum_{k=1}^{t} b_k g_{i,k} \in (b_i)$, implying that $A_f \subseteq (b_i)$ where $b_i \in B$.

(b)$\Rightarrow$(c): We assume that $A_f \subseteq (b)$; then for $i \in \omega$, $f_i = bg_i$ where $g_i \in V$. Let $g(X) = \sum_{i=0}^{\infty} g_i X^i$; it then is clear that $f(X) = bg(X)$.

(c)$\Rightarrow$(a): This is obvious.

(b)$\Rightarrow$(d): This is immediate from the assumption that $B$ is not finitely generated.

(d)$\nsubseteq$(b): Assuming that $A_f \subseteq B$, let $b \in B$, $b \not\in A_f$. Then $b \nsubseteq A_f$ so $A_f \nsubseteq (b)$ since $V$ is a valuation ring.

**Theorem 3.2.** Let $V$ be a rank one nondiscrete valuation ring having maximal ideal $M$. Then $M \cdot V[[X]] = \sqrt{(M \cdot V[[X]])}$.

**Proof.** Let $f(X) \in \sqrt{(M \cdot V[[X]])}$—say $[f(X)]^k \in M \cdot V[[X]]$; we then can write $[f(X)]^k = rg(X)$ where $r \in M$ and $g(X) \in V[[X]]$. There exists an element $s$ of $M$ with $0 < v(s) \leq v(r)/k$; then $r = s^kt$ where $t \in V$, implying that $[f(X)]^k = rg(X) = s^ktg(X)$, so that

$$[f(X)]^k/s^k = [f(X)/s]^k = tg(X) \in V[[X]].$$

Therefore $f(X)/s$ is a root of $Z^k - tg(X) \in V[[X]][Z]$, whereby $f(X)/s$ is integral over $V[[X]]$. Also $f(X)/s$ clearly is in the quotient field of $V[[X]]$. But $V$ is completely integrally closed, implying that $V[[X]]$ is completely integrally closed, hence is integrally closed [1; 150]. Thus $f(X)/s = h(X) \in V[[X]]$ and $f(X) = sh(X) \in M \cdot V[[X]]$ since $s \in M$. Hence $\sqrt{(M \cdot V[[X]])} \subseteq M \cdot V[[X]]$, so that equality holds.

**Theorem 3.3.** Let $R$ be a quasi-local ring having maximal ideal $M$ and let $Q$ be a prime ideal of $R[[X]]$. If $Q \nsubseteq M \cdot R[[X]]$, then either $Q \nsubseteq M[[X]]$ or $Q \subseteq M[[X]]$.

**Proof.** We assume that $Q \nsubseteq M[[X]]$ and show that $Q \nsubseteq M[[X]]$. Let $f(X) = \sum_{i=0}^{\infty} f_i X^i \in Q$, $f(X) \in M[[X]]$. Let $t$ be the smallest integer $k$ for which $f_k$ is a unit of $R$. Let $g(X) = \sum_{i=0}^{t-1} f_i X^i$ if $t > 0$; let $g(X) = 0$ if $t = 0$. Then $g(X) \in M \cdot R[[X]] \subseteq Q$, implying that $f(X) - g(X) \in Q$. If $f(X) - g(X)$ has order zero, then $g(X) = 0$, so that $f_t$ is a unit of $R$, implying that $f(X)$ is a unit of $R[[X]]$, whence $Q = R[[X]] \nsubseteq M[[X]]$. If $f(X) - g(X)$ has positive order $n$, then $[f(X) - g(X)]_n$ is a unit of $R$ and $f(X) - g(X) = X^nh(X)$ where $h_0 = [f(X) - g(X)]_n$ is a unit of $R$, implying that $h(X)$ is a unit of $R[[X]]$. 


Since \( f(X) - g(X) = X^n h(X) \in Q \) and \( Q \) is a prime ideal of \( R[[X]] \), either \( X^n \in Q \) or \( h(X) \in Q \). If \( X^n \in Q \), then \( X \in Q \), implying that \( Q \supseteq M \cdot R[[X]] + (X) \supseteq M[[X]] \). If \( h(X) \in Q \), then \( Q = R[[X]] \supseteq M[[X]] \). Hence if \( Q \not\subseteq M[[X]] \), then \( Q \supseteq M[[X]] \).

**Theorem 3.4.** Let \( V \) be a rank one nondiscrete valuation ring having maximal ideal \( M \).

(a) There is a prime ideal \( P \) of \( V[[X]] \) satisfying \( M \cdot V[[X]] \subseteq P \subseteq M[[X]] \).

(b) \( \dim V[[X]] \geq 3 \).

**Proof.** Theorem 3.2 asserts that

\[
\sqrt{(M \cdot V[[X]])} = M \cdot V[[X]] \subseteq M[[X]].
\]

Hence there is a prime ideal \( P \) of \( V[[X]] \) satisfying \( P \supseteq M \cdot V[[X]] \), \( P \not\subseteq M[[X]] \). Theorem 3.3 then asserts that \( P \subseteq M[[X]] \); hence (a) holds.

We now have a chain \((0) \subseteq P \subseteq M[[X]] \subseteq M \cdot V[[X]] + (X)\) of prime ideals of \( V[[X]] \), implying (b).

**4. Valuation rings of finite rank.**

**Lemma 4.1.** Let \( V \) be a valuation ring and let \( P \) be a proper prime ideal of \( V \). Then \( PV_P = P \); hence \( P \) is idempotent if and only if \( PV_P \) is idempotent.

The proof of Lemma 4.1 is straightforward and will therefore be omitted.

**Lemma 4.2.** Let \( V \) be a valuation ring and let \( P \) be an idempotent proper prime ideal of \( V \). Then \( P \cdot V[[X]] = (PV_P) \cdot V[[X]] \).

**Proof.** Let \( f(X) \in (PV_P) \cdot V[[X]] \) — say \( f(X) = rh(X) \) where \( r \in PV_P \) and \( h(X) \in V_P[[X]] \). Since \( P = PV_P \) is idempotent, we can write \( r = st \) where \( s, t \in P = PV_P \); then for \( i \in \omega \), there exists \( a_i \in V \setminus P \) such that \( a_i h_i \in V \). Since \( a_i \in V \setminus P \) and \( t \in P \), we have that \( (t) \subseteq \langle a_i \rangle \) so that \( t/a_i \in V \) for each \( i \in \omega \), implying that \( th_i = (t/a_i) (a_i h_i) \in V \) for each \( i \in \omega \) — that is, \( th(X) \in V[[X]] \). Since \( s \in P \), we conclude that \( f(X) = rh(X) = s(th(X)) \in P \cdot V[[X]] \), establishing that

\[
(PV_P) \cdot V[[X]] \subseteq P \cdot V[[X]].
\]

The reverse containment is obvious.
**Theorem 4.3.** Let \( V \) be a valuation ring and let \( P \) be a proper prime ideal of \( V \). If \( Q \) is a prime ideal of \( V[[X]] \) and if \( Q \supseteq P \cdot V[[X]] \), then either \( Q \supseteq P[[X]] \) or \( Q \subseteq P[[X]] \).

**Proof.** Assuming that \( Q \not\subseteq P[[X]] \), we first establish that either \( X \in Q \) or \( Q \) contains \( h(X) \), where \( h(X) \in V[[X]] \) and \( h_0 \notin P \). Let \( f(X) = \sum_{i=0}^{\infty} f_i X^i \in Q \), \( f(X) \in P[[X]] \). Let \( t \) be the smallest integer \( k \) for which \( f_k \notin P \). If \( t = 0 \), then we let \( h(X) = f(X) \). If \( t > 0 \), then we let \( g(X) = \sum_{i=0}^{t-1} f_i X^i \). Then \( g(X) \in P \cdot V[[X]] \subseteq Q \), implying that \( f(X) - g(X) \in Q \). Further, \( f(X) - g(X) = X^t h(X) \) where \( h_0 = f_t \in P \). Since \( Q \) is prime, either \( X \in Q \) or \( h(X) \in Q \). Hence if \( Q \not\subseteq P[[X]] \), then either \( X \in Q \) or \( Q \) contains \( h(X) \) where \( h(X) \in V[[X]] \) and \( h_0 \notin P \).

If \( X \in Q \), then \( Q \supseteq P[[X]] \); hence we consider the case where \( h(X) \in Q \) with \( h_0 \notin P \). Observe now that \( h(X) \in V_r[[X]] \) and that \( h_0 \) is a unit of \( V_r \), implying that \( h(X) \) is a unit of \( V_r[[X]] \) — that is \( 1/h(X) \in V_r[[X]] \). Now let \( r(X) \in P[[X]] \); then
\[
r(X)[1/h(X)] \in P[[X]] \cdot V_r[[X]] \subseteq P[[X]]
\]
— in particular, \( r(X)[1/h(X)] \in V[[X]] \). Since \( h(X) \in Q \), we see that \( r(X) = h(X)[r(X)/h(X)] \in Q \). Hence \( Q \supseteq P[[X]] \).

**Lemma 4.4.** Let \( V \) be a valuation ring having a minimal prime ideal \( P \). If \( P \) is idempotent, then \( P \cdot V[[X]] = \sqrt{(P \cdot V[[X]])} \).

**Proof.** Let \( f(X) \in \sqrt{(P \cdot V[[X]])} \). Then in
\[
V_r[[X]], \ f(X) \in \sqrt{(PV_r \cdot V_r[[X]])}
\]
by Lemma 4.2. Since \( V_r \) is a rank one nondiscrete valuation ring, Theorem 3.2 asserts that \( \sqrt{(PV_r \cdot V_r[[X]])} = (PV_r) \cdot V_r[[X]] \). Hence \( f(X) \in (PV_r) \cdot V_r[[X]] = P \cdot V[[X]] \), the latter equality following from Lemma 4.2.

**Theorem 4.5.** Let \( V \) be a valuation ring and let \( P \) be a proper prime ideal of \( V \). If \( P \) is idempotent, then
\[
P \cdot V[[X]] = \sqrt{(P \cdot V[[X]])}.
\]

**Proof.** We shall say that \( P \) is branched provided there exists a \( P \)-primary ideal distinct from \( P[1; 173] \). We consider two cases.

**Case 1.** \( P \) is branched. Then there is a prime ideal \( Q \) of \( V \) with \( Q \supseteq P \) and such that there are no prime ideals of \( V \) properly
between Q and P [1; 173]. Then P/Q is a minimal prime ideal of V/Q and P/Q is idempotent. Lemma 4.4 assures that 

$$(P/Q) \cdot (V/Q)[[X]] = \sqrt{(P/Q) \cdot (V/Q)[[X]]}.$$ 

By considering the natural homomorphism from V[[X]] to (V/Q)[[X]], we conclude that $P \cdot V[[X]] = \sqrt{P \cdot V[[X]]}$.

**Case 2.** P is not branched. Then $P = \bigcup M_{\lambda}$ where $\{M_{\lambda}\}_{\lambda \in \Lambda}$ is the collection of prime ideals of V properly contained in P [1; 173]. Let $f(X) \in \sqrt{(P \cdot V[[X]])}$ — say $f(X)^k \in P \cdot V[[X]]$. Then $f(X)^k = rg(X)$ where $g(X) \in V[[X]]$ and $r \in P$, implying that $r \in M_{\lambda_i}$ for some $\lambda_i \in \Lambda$. Thus $f(X)^k = rg(X) \in M_{\lambda_i}[[X]]$, implying that $f(X) \in M_{\lambda_i}[[X]]$. There exists $\lambda \in \Lambda$ such that $M_{\lambda_i} \subset M_{\lambda}$. Let $s \in M_{\lambda_i}$, $s \notin M_{\lambda_i}$; then $(s) \supseteq M_{\lambda_i} \supseteq A_i$, so that $f(X) = sh(X)$ where $h(X) \in V[[X]]$. Since $s \in M_{\lambda_i}$, $s \in P$; hence $f(X) = sh(X) \in P \cdot V[[X]]$.

**Corollary 4.6.** Let V be a valuation ring having a proper prime ideal P. If P is idempotent, then there is a prime ideal Q of V[[X]] satisfying $P \cdot V[[X]] \subseteq Q \subset P[[X]]$.

**Proof.** Theorem 4.5 assures that 

$$\sqrt{(P \cdot V[[X]])} = P \cdot V[[X]] \subset V[[X]].$$ 

Hence there is a prime ideal Q of V[[X]] satisfying $Q \supseteq P \cdot V[[X]]$, $Q \nsubseteq P[[X]]$. Theorem 4.3 then asserts that $Q \subset P[[X]]$.

**Theorem 4.7.** Let V be a valuation ring of rank n having k distinct idempotent proper prime ideals. Then $\dim V[[X]] \geq n + k + 1$.

**Proof.** We use induction on n, the case $n = 1$ following from Theorem 2.7 and Theorem 3.4.

Assuming the result for valuation rings of rank t, let V be a valuation ring of rank $t + 1$ having k distinct idempotent proper prime ideals and let $(0) \subset P_1 \subset P_2 \subset \cdots \subset P_{t+1}$ be the chain of nonunit prime ideals of V. We consider two cases.

**Case 1.** $P_1$ is not idempotent. Here $V/P_1$ is a valuation ring of rank t which has k distinct idempotent proper prime ideals. By the induction hypothesis, $\dim (V/P_1)[[X]] \geq t + k + 1$. Since $(V/P_1)[[X]] \cong V[[X]]/P_1[[X]]$, this implies that the depth of $P_1[[X]]$ is at least $t + k + 1$. Since $P_1[[X]] \neq (0)$, $\dim V[[X]] \geq t + k + 2$.

**Case 2.** $P_1$ is idempotent. Here $V/P_1$ is a valuation ring of rank
which has \(k - 1\) distinct idempotent proper prime ideals. By the induction hypothesis, \(\dim (V/P_i)[X] \geq t + (k - 1) + 1 = t + k\); hence the depth of \(P_i[[X]]\) is at least \(t + k\). Since \(P_i\) is idempotent, Corollary 4.6 asserts that there is a prime ideal \(Q\) of \(V[[X]]\) satisfying \(P_i \cdot V[[X]] \subseteq Q \subseteq P_i[[X]]\) — in particular, \((0) \subseteq Q \subseteq P_i[[X]]\). Since the depth of \(P_i[[X]]\) is at least \(t + k\), we see that \(\dim V[[X]] \geq t + k + 2\).

**Lemma 4.8.** Let \(V\) be valuation ring and let \(P\) be a proper prime ideal of \(V\).

(a) If \(Q'\) is a prime ideal of \(V_p[[X]]\) which satisfies \((P V_p) \cdot V_p[[X]] \subseteq Q' \subseteq (P V_p)[[X]]\), then \(Q'\) is a prime ideal of \(V[[X]]\) which satisfies \(P \cdot V[[X]] \subseteq Q' \subseteq P[[X]]\).

(b) Conversely, if \(Q\) is a prime ideal of \(V[[X]]\) which satisfies \(P \cdot V[[X]] \subseteq Q \subseteq P[[X]]\), then \(Q\) is a prime ideal of \(V_p[[X]]\) which satisfies \((P V_p) \cdot V_p[[X]] \subseteq Q \subseteq (P V_p)[[X]]\).

**Proof.** To establish (a), we observe that \(Q' \subseteq (P V_p)[[X]] = P[[X]] \subseteq V[[X]]\), whereby \(Q' \cap V[[X]] = Q'\).

We now establish (b); we begin by proving that \(Q\) is an ideal of \(V_p[[X]]\). Let \(f(X) \in Q\) and \(g(X) \in V_p[[X]]\); we show that \(f(X) \cdot g(X) \in Q\). Choose \(h(X) \in P[[X]]\), \(h(X) \in Q\). For each \(i, j \in \omega_0\), \(g_i \in V_p\) and \(h_j \in P_i\), implying that \(g_i h_j \in P V_p = P\). Hence \(g(X)h(X) \in P[[X]] \subseteq V[[X]]\), implying that \(f(X)[g(X)h(X)] \in Q\). Since \(f(X) \in Q \subseteq P[[X]]\), each \(f_i \in P\); hence \(f(X)g(X) \in P[[X]] \subseteq V[[X]]\). Since \([f(X)g(X)] \cdot h(X) \in Q\) where \(f(X)g(X) \in V[[X]]\), \(h(X) \in V[[X]]\), and \(h(X) \in Q\), we conclude that \(f(X)g(X) \in Q\). Hence \(Q\) is an ideal of \(V_p[[X]]\).

We now prove that \(Q\) is a prime ideal of \(V_p[[X]]\). Let \(S = V[[X]] \cap Q\); then \(S\) is a multiplicative system in \(V[[X]]\), hence also in \(V_p[[X]]\), and \(S\) clearly does not meet the ideal \(Q\) of \(V_p[[X]]\). Hence there is a prime ideal \(Q^*\) of \(V_p[[X]]\) which satisfies \(Q = Q^* \cap S = Q\). Since \(Q \subseteq Q^*, Q \subseteq Q^* \cap V[[X]]\); since \(Q^* \cap S = \emptyset\), \(Q^* \cap V[[X]] \subseteq Q\). Thus \(Q^* \cap V[[X]] = Q\). Observe now that \(Q^* \supseteq Q \supseteq P \cdot V[[X]] = (P V_p) \cdot V_p[[X]]\). By Theorem 4.3, \(Q^*\) compares with \((P V_p)[[X]] = P[[X]]\). Since \(Q^*\) lies over \(Q\) we must have that \(Q^* \subseteq P[[X]] \subseteq V[[X]]\), implying that \(Q^* = Q\). Hence \(Q\) is a prime ideal of \(V_p[[X]]\).

That \((P V_p) \cdot V_p[[X]] \subseteq Q \subseteq (P V_p)[[X]]\) is clear.

**Theorem 4.9.** The following conditions are equivalent:

(a) If \(V\) is a rank one nondiscrete valuation ring, then \(V[[X]]\) has finite dimension.

(b) If \(V\) is a valuation ring having finite rank \(n\), then \(V[[X]]\) has finite dimension.
Proof. It is clear that \((b) \rightarrow (a)\). We prove that \((a) \rightarrow (b)\) using induction on \(n\), the case \(n = 1\) being a consequence of \((a)\) and Theorem 2.7.

We now assume that if \(W\) is a valuation ring of rank \(k\), then \(W[[X]]\) has finite dimension. Let \(V\) be a valuation ring of rank \(k + 1\) which has minimal prime \(P_i\). Let \((0) \subset Q_1 \subset Q_2 \subset \cdots \subset Q_i\) be a chain of prime ideals of \(V[[X]]\). Let \(d = \dim V_{P_i}[[X]]\). Corollary 2.3 assures that there are at most \(d\) proper prime ideals in this chain which contract to \((0)\) in \(V\). Choose \(m\) so that \(Q_m \cap V = (0)\) and \(Q_{m+1} \cap V \neq (0)\); then \(m \leq d\). For \(r \geq m + 1\), \(Q_r \cap V \supseteq P_i\); Theorem 4.3 assures that for \(r \geq m + 1\), \(Q_r\) compares with \(P_i[[X]]\). Lemma 4.8 assures that at most \(d\) of the ideals \(Q_{m+1}, Q_{m+2}, \ldots, Q_d\) are contained in \(P_i[[X]]\), whereby \(Q_{m+d+1} \supset P_i[[X]]\). Since \(m \leq d\), we have that \(Q_{2d+1} \supseteq Q_{m+d+1} \supset P_i[[X]]\).

By the induction hypothesis, \((V/P_i)[[X]]\) has finite dimension. The depth of \(P_i[[X]]\) is at most \((\dim (V/P_i)[[X]] - 1)\). It follows that the depth of \(Q_{2d+1}\) is at most \((\dim (V/P_i)[[X]] - 1)\), whereby

\[
t \leq (2d + 1) + (\dim (V/P_i)[[X]] - 1) = 2d + \dim (V/P_i)[[X]].
\]

We conclude that \(\dim V[[X]] \leq 2d + \dim (V/P_i)[[X]]\), whereby \(V[[X]]\) has finite dimension.

**Theorem 4.10.** The following conditions are equivalent:

(a) If \(V\) is a rank one nondiscrete valuation ring, then the ascending chain condition for prime ideals holds in \(V[[X]]\).

(b) If \(V\) is a valuation ring having finite rank \(n\), then the ascending chain condition for prime ideals holds in \(V[[X]]\).

The proof of Theorem 4.10 is analogous to the proof of Theorem 4.9 and will therefore be omitted.

*Added in proof.* Jimmy T. Arnold has recently conveyed to me a paper of his, *On Krull Dimension in Power Series Rings*, in which he has established the following result.

Let \(R\) be a commutative ring with identity. If there exists a prime ideal \(P\) of \(R\) such that \(\sqrt{(P \cdot R[[X]])} \neq P[[X]]\), then \(R[[X]]\) has infinite dimension.

It follows immediately that if \(V\) is a valuation ring which is not discrete, then \(V[[X]]\) has infinite dimension.
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