Pacific Journal of Mathematics

DIMENSION THEORY IN POWER SERIES RINGS

DAVID E. FIELDS

Vol. 35, No. 3 November 1970

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Let V be a valuation ring of finite rank n. If V is discrete, then V [[X]] has dimension n+1. If V is not discrete, then the dimension of V [[X]] is at least n+k+1, where k is the number of idempotent proper prime ideals of V.

Let R be a commutative ring with identity. If there exists a chain $P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_n$ of n+1 prime ideals of R, where $P_n \subset R$, but no such chain of n+2 prime ideals, then we say that R has dimension n and we write dim R=n [3]. In [3] and [4], Seidenberg has investigated the dimension theory of $R[X_1, X_2, \cdots, X_m]$ where R has finite dimension and X_1, X_2, \cdots, X_m are indeterminates over R. We investigate the dimension theory of V[X] where V is a valuation ring.

Throughout this paper, R denotes a commutative ring with identity; ω is the set of natural numbers; ω_0 is the set of nonnegative integers; and Z is the set of integers. If

$$f(X) = \sum\limits_{i=0}^{\infty} f_i X^i \! \in R \left[\left[X
ight]
ight]$$
 ,

we denote by A_f the ideal of R generated by the coefficients of f(X): $A_f = \{f_0, f_1, \dots, f_k, \dots\} R$. If A is an ideal of R, we let

$$A \hspace{.1cm} \hbox{\it [[X]]} = \{ f(X) = \sum\limits_{i=0}^{\infty} f_i X^i \hbox{\it :} \hspace{.1cm} f_i \hspace{.05cm} \in A \hspace{.1cm} ext{for each} \hspace{.1cm} i \hspace{.05cm} \in \omega_0 \}$$

and we define $A \cdot R[[X]]$ to be the ideal of R[[X]] which is generated by A. Then $A \cdot R[[X]] = \{f(X): A_f \subseteq B \text{ for some finitely generated ideal } B \text{ of } R \text{ with } B \subseteq A\}$. It is clear that $A \cdot R[[X]] \subseteq A[[X]]$; equality holds if and only if each countably generated ideal of R contained in A is contained in a finitely generated ideal of R contained in A. In particular, if V is a valuation ring containing an ideal A which is countably generated but not finitely generated, then $A \cdot V[[X]] \subset A[[X]]$. Finally, we note that if A is an ideal of R, then $R[[X]]/A[[X]] \cong (R/A)[[X]]$; hence A[[X]] is a prime ideal of R[[X]] if and only if A is a prime ideal of R.

2. Discrete valuation rings. Let V be a valuation ring of rank k with associated valuation v and value group G; let $\{0\} = G_0 \subset G_1 \subset \cdots \subset G_k = G$ be the chain of isolated subgroups of G together with G. In [2], Iwasawa proves that for $1 \le i \le k$,

 $G_i/G_{i-1} \cong H_i$ where H_i is a subgroup of the additive group of real numbers, this being an order-preserving isomorphism of groups. If for $1 \leq i \leq k$, $H_i \cong Z$, we shall say that V is a discrete valuation ring of rank k. This is equivalent to the condition that V contains no idempotent proper prime ideal.

LEMMA 2.1. Let V be a valuation ring and let P be a proper prime ideal of V. If P is not idempotent, then in V[[X]], $\sqrt{(P \cdot V[[X]])} = P[[X]]$ and $(P[[X]])^2 \subseteq P \cdot V[[X]]$.

Proof. Let $\alpha \in P$, $\alpha \notin P^2$. Then

$$(P[[X]])^2 \subseteq P^2[[X]] \subseteq (\alpha) V[[X]] \subseteq P \cdot V[[X]].$$

Hence $P[[X]] \subseteq \sqrt{P \cdot V[[X]]}$ and the reverse containment is clear.

LEMMA 2.2. Let V be a valuation ring with quotient field K and let P be a proper prime ideal of V. Let

$$D = V[[X]][K] = (V[[X]])_{V\setminus\{0\}}$$
.

Then $D = (V_P[[X]])_{V_P \setminus \{0\}}$.

Proof. We first show that $V_P[[X]] \subseteq D$. Let

$$f(X) = \sum\limits_{i=0}^{\infty} f_i X^i \in V_P[[X]]$$
 .

For each $i \in \omega_0$, there exists $r_i \in V \setminus P$ such that $r_i f_i \in V$. Let $a \in P \setminus \{0\}$; then for each $i \in \omega_0$, $a/r_i \in PV_P = P \subseteq V$, implying that $af_i = (a/r_i) \ (r_i f_i) \in V$; that is, $af(X) \in V[[X]]$. This implies that $f(X) \in (V[[X]])_{V \setminus \{0\}} = D$, showing that $V_P[[X]] \subseteq D$.

Since $D \supseteq K$, each nonzero element of V_P is a unit in D. Thus $D \supseteq (V_P[[X]])_{V_P \setminus \{0\}}$ and the reverse containment is obvious.

COROLLARY 2.3. Let V be a valuation ring and let P be a proper prime ideal of V. There is a one-to-one correspondence between prime ideals of V[X] which contract to (0) in V and prime ideals of $V_P[X]$ which contract to (0) in V_P ; this correspondence preserves containment.

Proof. Lemma 2.2 assures that there is a one-to-one, containment preserving correspondence between each of these classes of prime ideals and the class of prime ideals of D.

LEMMA 2.4. Let R be a quasi-local ring having maximal ideal

M. Let $f(X) \in R[[X]]$, $f(X) \notin M[[X]] - say \ f_k \in R \setminus M$, k minimal. There exists g(X), a unit of R[[X]], such that f(X)g(X) has exactly one unit coefficient, namely $(fg)_k$.

Proof. For $u(X) \in R[[X]]$, denote by $\overline{u}(X)$ the canonical image of u(X) in (R/M)[[X]]. By choice of k,

$$\bar{f}(X) = \bar{f}_k X^k + \bar{f}_{k+1} X^{k+1} + \cdots = X^k (\bar{f}_k + \bar{f}_{k+1} X + \cdots)$$
,

where $\bar{f}_k \neq 0$. Then $\bar{f}_k + \bar{f}_{k+1}X + \cdots$ is a unit of (R/M)[[X]], and we can choose $g(X) \in R[[X]]$ such that $\bar{g}(X) \cdot (\bar{f}_k + \bar{f}_{k+1}X + \cdots) = 1$. Thus $\bar{f}(X) \cdot \bar{g}(X) = X^k$, and $f(X)g(X) - X^k \in M[[X]]$. This implies that only the coefficient of X^k in f(X)g(X) is not in M.

COROLLARY 2.5. Let V be a valuation ring and let P be a proper prime ideal of V. If Q is an ideal of $V_P[[X]]$ and if $Q \nsubseteq (PV_P)[[X]]$, then $Q \cap V[[X]] \nsubseteq P[[X]]$.

Proof. Lemma 2.4 assures that there is a power series g(X) in Q with g(X) having exactly one unit coefficient, g_k . Since g_k is a unit of V_P , there in no loss of generality in assuming that, in fact, $g_k = 1$. Then for $i \neq k$, $g_i \in PV_P = P \subseteq V$, implying that $g(X) \in Q \cap V[[X]]$ while $g(X) \notin P[[X]]$.

LEMMA 2.6. Let R be a Noetherian ring having dimension n. Then $R[[X_1, X_2, \dots, X_m]]$ is Noetherian and has dimension n + m.

Proof. It is well known that if R is Noetherian, then $R[[X_1, X_2, \dots, X_m]]$ is Noetherian. We shall show that the dimension of R[[X]] is n+1; the lemma follows immediately by induction on m.

Let M be a maximal ideal of R[[X]]. Then $M=M_1+(X)$ for some maximal ideal M_1 of R. Since dim R=n, the height of M_1 is k where $k \leq n$. There exists an ideal $A=(a_1,a_2,\cdots,a_k)$ of R which admits M_1 as an isolated prime ideal [5; 242]. It is straightforward to verify that $M=M_1+(X)$ is an isolated prime ideal of $A+(X)=(a_1,a_2,\cdots,a_k,X)$ R[[X]]. This implies that the height of M is at most k+1 [5; 240]; since $k \leq n$, the height of M is at most n+1. Since M was an arbitrary maximal ideal of R[[X]], we conclude that dim $R[[X]] \leq n+1$; the reverse inequality is clear.

Theorem 2.7. Let V be a discrete valuation ring of rank n and let $(0) = P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_n$ be the nonunit prime ideals of

¹ The proof of Lemma 2.6 was pointed out to the author by William Heinzer.

V. Then dim V[[X]] = n + 1.

Proof. We use induction on n, the case n = 1 following from Lemma 2.6 since a rank one discrete valuation ring is Noetherian.

Assuming the result for discrete valuation rings of rank less than n, let V be a discrete valuation ring of rank n and let $(0) \subset Q_1 \subset Q_2 \subset \cdots \subset Q_t$ be a chain of prime ideals of V[[X]]. We consider two cases.

- Case 1. $Q_1 \cap V \neq (0)$. Here $Q_1 \cap V \supseteq P_1$, so that $Q_1 \supseteq P_1 \cdot V[[X]]$, implying that $Q_1 \supseteq V(P_1 \cdot V[[X]]) = P_1[[X]]$, the latter equality being a consequence of Lemma 2.1. But the depth of $P_1[[X]]$ cannot exceed dim $(V/P_1)[[X]] = n$; we conclude that $t \leq n + 1$.
- Case 2. $Q_1 \cap V = (0)$. Corollary 2.3 asserts that $Q_1 = Q^* \cap V[[X]]$, where Q^* is a prime ideal of $V_{P_1}[[X]]$ and $Q^* \cap V_{P_1} = (0)$. Since dim $V_{P_1}[[X]] = 2$, $Q^* \nsubseteq (P_1 V_{P_1})[[X]]$. By Corollary 2.5, $Q_1 \nsubseteq P_1[[X]]$. Since $V_{P_1}[[X]]$ is two-dimensional and local, each proper prime ideal of $V_{P_1}[[X]]$ which contracts to (0) in V_{P_1} is a minimal prime ideal of $V_{P_1}[[X]]$. Corollary 2.3 now assures that each proper prime ideal of V[[X]] which contracts to (0) in V is a minimal prime ideal of V[[X]]. It follows that $Q_2 \cap V \neq (0)$, implying that $Q_2 \supseteq P_1[[X]]$. Since also $Q_2 \supseteq Q_1$ and $Q_1 \nsubseteq P_1[[X]]$, we conclude that $Q_2 \supset P_1[[X]]$. Thus we have a chain $(0) \subset P_1[[X]] \subset Q_2 \subset Q_3 \subset \cdots \subset Q_t$. It follows, as in Case 1, that $t \leq n+1$.

Thus dim $V[[X]] \leq n+1$ and the reverse inequality is clear.

- 3. Rank one nondiscrete valuation rings. We note that if V is a rank one valuation ring, then the value group of v is Archimedian.
- Lemma 3.1. Let V be a valuation ring and let B be an ideal of V. If B is not finitely generated, then the following conditions are equivalent:
 - (a) $f(X) \in B \cdot V[[X]]$.
 - (b) $A_f \subseteq (b)$ for some $b \in B$.
 - (c) f(X) = bg(X) for some $b \in B$, $g(X) \in V[[X]]$.
 - (d) $A_f \subset B$.

Proof. We establish that $(a) \to (b) \to (c) \to (a)$ and that $(b) \leftrightarrow (d)$. $(a) \to (b)$: Let $f(X) \in B \cdot V[[X]]$; then we can write

$$f(X) = b_1[g^{(1)}(X)] + b_2[g^{(2)}(X)] + \cdots + b_t[g^{(t)}(X)]$$

where for $1 \leq i \leq t$, $b_i \in B$ and $g^{(i)}(X) = \sum_{j=0}^{\infty} g_{ij} \ X^j \in V[[X]]$. Thus $f(X) = \sum_{i=0}^{\infty} f_i X^i$ where $f_i = \sum_{k=1}^t b_k g_{ki}$. In V, $(b_1, b_2, \dots, b_t) = (b_s)$ for some s, $1 \leq s \leq t$. Now for $i \in \omega_0$, $f_i = \sum_{k=1}^t b_k g_{ki} \in (b_s)$, implying that $A_f \subseteq (b_s)$ where $b_s \in B$.

- (b) \rightarrow (c): We assume that $A_f \subseteq (b)$; then for $i \in \omega_0$, $f_i = bg_i$ where $g_i \in V$. Let $g(X) = \sum_{i=0}^{\infty} g_i X^i$; it then is clear that f(X) = bg(X).
 - $(c) \rightarrow (a)$: This is obvious.
- (b) \rightarrow (d): This is immediate from the assumption that B is not finitely generated.
- (d) \rightarrow (b): Assuming that $A_f \subset B$, let $b \in B$, $b \notin A_f$. Then (b) $\nsubseteq A_f$ so $A_f \subseteq (b)$ since V is a valuation ring.

THEOREM 3.2. Let V be a rank one nondiscrete valuation ring having maximal ideal M. Then $M \cdot V[[X]] = \sqrt{(M \cdot V[[X]])}$.

Proof. Let $f(X) \in \sqrt{(M \cdot V[[X]])} - \text{say} [f(X)]^k \in M \cdot V[[X]];$ we then can write $[f(X)]^k = rg(X)$ where $r \in M$ and $g(X) \in V[[X]].$ There exists an element s of M with $0 < v(s) \le v(r)/k$; then $r = s^k t$ where $t \in V$, implying that $[f(X)]^k = rg(X) = s^k tg(X)$, so that

$$[f(X)]^k/s^k = [f(X)/s]^k = tg(X) \in V[[X]]$$
.

Therefore f(X)/s is a root of $Z^k - tg(X) \in V[[X]][Z]$, whereby f(X)/s is integral over V[[X]]. Also f(X)/s clearly is in the quotient field of V[[X]]. But V is completely integrally closed, implying that V[[X]] is completely integrally closed, hence is integrally closed [1; 150]. Thus $f(X)/s = h(X) \in V[[X]]$ and $f(X) = sh(X) \in M \cdot V[[X]]$ since $s \in M$. Hence $\sqrt{M \cdot V[[X]]} \subseteq M \cdot V[[X]]$, so that equality holds.

THEOREM 3.3. Let R be a quasi-local ring having maximal ideal M and let Q be a prime ideal of R [[X]]. If $Q \supseteq M \cdot R$ [[X]], then either $Q \supseteq M[[X]]$ or $Q \subseteq M[[X]]$.

Proof. We assume that $Q \nsubseteq M[[X]]$ and show that $Q \supseteq M[[X]]$. Let $f(X) = \sum_{i=0}^{\infty} f_i X^i \in Q$, $f(X) \notin M[[X]]$. Let $f(X) = \sum_{i=0}^{t-1} f_i X^i \in Q$ as unit of $f(X) = \sum_{i=0}^{t-1} f_i X^i$ if f(X) = 0; let f(X) = 0 if f(X) = 0. Then $f(X) \in M \cdot R[[X]] \subseteq Q$, implying that $f(X) = f(X) \in Q$. If f(X) = f(X) has order zero, then f(X) = f(X) so that f(X) = f(X) = f(X) is a unit of f(X) = f(X), whence f(X) = f(X) = f(X) has positive order f(X) = f(X) = f(X) = f(X) is a unit of f(X) = f(X) = f(X) where f(X) = f(X) = f(X) is a unit of f(X) = f(X) is a unit of f(X) = f(X).

Since $f(X) - g(X) = X^n h(X) \in Q$ and Q is a prime ideal of R[[X]], either $X^n \in Q$ or $h(X) \in Q$. If $X^n \in Q$, then $X \in Q$, implying that $Q \supseteq M \cdot R[[X]] + (X) \supseteq M[[X]]$. If $h(X) \in Q$, then $Q = R[[X]] \supseteq M[[X]]$. Hence if $Q \nsubseteq M[[X]]$, then $Q \supseteq M[[X]]$.

Theorem 3.4. Let V be a rank one nondiscrete valuation ring having maximal ideal M.

- (a) There is a prime ideal P of V[[X]] satisfying $M \cdot V[[X]] \subseteq P \subset M[[X]]$.
 - (b) dim $V[[X]] \ge 3$.

Proof. Theorem 3.2 asserts that

$$\sqrt{(M \cdot V[[X]])} = M \cdot V[[X]] \subset M[[X]]$$
.

Hence there is a prime ideal P of V[[X]] satisfying $P \supseteq M \cdot V[[X]]$, $P \not\supseteq M[[X]]$. Theorem 3.3 then asserts that $P \subset M[[X]]$; hence (a) holds.

We now have a chain $(0) \subset P \subset M[[X]] \subset M \cdot V[[X]] + (X)$ of prime ideals of V[[X]], implying (b).

4. Valuation rings of finite rank.

LEMMA 4.1. Let V be a valuation ring and let P be a proper prime ideal of V. Then $PV_P = P$; hence P is idempotent if and only if PV_P is idempotent.

The proof of Lemma 4.1 is straightforward and will therefore be omitted.

LEMMA 4.2. Let V be a valution ring and let P be an idempotent proper prime ideal of V. Then $P \cdot V[X] = (PV_P) \cdot V_P[X]$.

Proof. Let $f(X) \in (PV_P) \cdot V_P[[X]] - \text{say} \quad f(X) = rh(X)$ where $r \in PV_P$ and $h(X) \in V_P[[X]]$. Since $P = PV_P$ is idempotent, we can write r = st where $s, t \in P = PV_P$; then for $i \in \omega_0$, there exists $a_i \in V \setminus P$ such that $a_ih_i \in V$. Since $a_i \in V \setminus P$ and $t \in P$, we have that $(t) \subseteq (a_i)$ so that $t/a_i \in V$ for each $i \in \omega_0$, implying that $th_i = (t/a_i) \quad (a_ih_i) \in V$ for each $i \in \omega_0$ — that is, $th(X) \in V[[X]]$. Since $s \in P$, we conclude that $f(X) = rh(X) = s(th(X)) \in P \cdot V[[X]]$, establishing that

$$(PV_P) \cdot V_P[[X]] \subseteq P \cdot V[[X]]$$
.

The reverse containment is obvious.

THEOREM 4.3. Let V be a valuation ring and let P be a proper prime ideal of V. If Q is a prime ideal of V[[X]] and if $Q \supseteq P \cdot V[[X]]$, then either $Q \supseteq P[[X]]$ or $Q \subseteq P[[X]]$.

Proof. Assuming that $Q \nsubseteq P[[X]]$, we first establish that either $X \in Q$ or Q contains h(X), where $h(X) \in V[[X]]$ and $h_0 \notin P$. Let $f(X) = \sum_{i=0}^{\infty} f_i X^i \in Q$, $f(X) \notin P[[X]]$. Let t be the smallest integer k for which $f_k \notin P$. If t = 0, then we let h(X) = f(X). If t > 0, then we let $g(X) = \sum_{i=0}^{t-1} f_i X^i$. Then $g(X) \in P \cdot V[[X]] \subseteq Q$, implying that $f(X) - g(X) \in Q$. Further, $f(X) - g(X) = X^t h(X)$ where $h_0 = f_t \notin P$. Since Q is prime, either $X \in Q$ or $h(X) \in Q$. Hence if $Q \nsubseteq P[[X]]$, then either $X \in Q$ or Q contains h(X) where $h(X) \in V[[X]]$ and $h_0 \notin P$.

If $X \in Q$, then $Q \supseteq P[[X]]$; hence we consider the case where $h(X) \in Q$ with $h_0 \notin P$. Observe now that $h(X) \in V_P[[X]]$ and that h_0 is a unit of V_P , implying that h(X) is a unit of $V_P[[X]]$ — that is $1/h(X) \in V_P[[X]]$. Now let $r(X) \in P[[X]]$; then

$$r(X)[1/h(X)] \in P[[X]] \cdot V_P[[X]] \subseteq P[[X]]$$

- in particular, $r(X)[1/h(X)] \in V[[X]]$. Since $h(X) \in Q$, we see that $r(X) = h(X)[r(X)/h(X)] \in Q$. Hence $Q \supseteq P[[X]]$.

LEMMA 4.4. Let V be a valuation ring having a minimal prime ideal P. If P is idempotent, then $P \cdot V[[X]] = \sqrt{P \cdot V[[X]]}$.

Proof. Let
$$f(X) \in \sqrt{(P \cdot V[[X]])}$$
. Then in

$$V_P[[X]], f(X) \in \sqrt{(PV_P) \cdot V_P[[X]]}$$

by Lemma 4.2. Since V_P is a rank one nondiscrete valuation ring, Theorem 3.2 asserts that $\sqrt{((PV_P)\cdot V_P[[X]])}=(PV_P)\cdot V_P[[X]]$. Hence $f(X)\in (PV_P)\cdot V_P[[X]]=P\cdot V[[X]]$, the latter equality following from Lemma 4.2.

Theorem 4.5. Let V be a valuation ring and let P be a proper prime ideal of V. If P is idempotent, then

$$P \cdot V[[X]] = \sqrt{(P \cdot V[[X]])}$$
.

Proof. We shall say that P is branched provided there exists a P-primary ideal distinct from P[1; 173]. We consider two cases.

Case 1. P is branched. Then there is a prime ideal Q of V with $Q \subset P$ and such that there are no prime ideals of V properly

between Q and P [1; 173]. Then P/Q is a minimal prime ideal of V/Q and P/Q is idempotent. Lemma 4.4 assures that

$$(P/Q) \cdot (V/Q)[[X]] = \sqrt{((P/Q) \cdot (V/Q)[[X]])}$$
.

By considering the natural homomorphism from V[[X]] to (V/Q)[[X]], we conclude that $P \cdot V[[X]] = \sqrt{P \cdot V[[X]]}$.

Case 2. P is not branched. Then $P = \bigcup_{\lambda} M_{\lambda}$ where $\{M_{\lambda}\}_{\lambda \in \Lambda}$ is the collection of prime ideals of V properly contained in P[1; 173]. Let $f(X) \in \sqrt{(P \cdot V[[X]])} - \operatorname{say} f(X)^k \in P \cdot V[[X]]$. Then $f(X)^k = rg(X)$ where $g(X) \in V[[X]]$ and $r \in P$, implying that $r \in M_{\lambda_1}$ for some $\lambda_1 \in \Lambda$. Thus $f(X)^k = rg(X) \in M_{\lambda_1}[[X]]$, implying that $f(X) \in M_{\lambda_1}[[X]]$. There exists $\lambda_2 \in \Lambda$ such that $M_{\lambda_1} \subset M_{\lambda_2}$. Let $s \in M_{\lambda_2}$, $s \notin M_{\lambda_1}$; then $(s) \supseteq M_{\lambda_1} \supseteq A_f$, so that f(X) = sh(X) where $h(X) \in V[[X]]$. Since $s \in M_{\lambda_2}$, $s \in P$; hence $f(X) = sh(X) \in P \cdot V[[X]]$.

COROLLARY 4.6. Let V be a valuation ring having a proper prime ideal P. If P is idempotent, then there is a prime ideal Q of V[[X]] satisfying $P \cdot V[[X]] \subseteq Q \subset P[[X]]$.

Proof. Theorem 4.5 assures that

$$\sqrt{(P \cdot V[[X]])} = P \cdot V[[X]] \subset P[[X]]$$
.

Hence there is a prime ideal Q of V[[X]] satisfying $Q \supseteq P \cdot V[[X]]$, $Q \not\supseteq P[[X]]$. Theorem 4.3 then asserts that $Q \subset P[[X]]$.

THEOREM 4.7. Let V be a valuation ring of rank n having k distinct idempotent proper prime ideals. Then dim $V[[X]] \ge n + k + 1$.

Proof. We use induction on n, the case n = 1 following from Theorem 2.7 and Theorem 3.4.

Assuming the result for valuation rings of rank t, let V be a valuation ring of rank t+1 having k distinct idempotent proper prime ideals and let $(0) \subset P_1 \subset P_2 \subset \cdots \subset P_{t+1}$ be the chain of nonunit prime ideals of V. We consider two cases.

Case 1. P_1 is not idempotent. Here V/P_1 is a valuation ring of rank t which has k distinct idempotent proper prime ideals. By the induction hypothesis, $\dim (V/P_1)[[X]] \ge t + k + 1$. Since $(V/P_1)[[X]] \cong V[[X]]/P_1[[X]]$, this implies that the depth of $P_1[[X]]$ is at least t + k + 1. Since $P_1[[X]] \ne (0)$, $\dim V[[X]] \ge t + k + 2$.

Case 2. P_1 is idempotent. Here V/P_1 is a valuation ring of rank

t which has k-1 distinct idempotent proper prime ideals. By the induction hypothesis. $\dim (V/P_1)[[X]] \ge t + (k-1) + 1 = t + k$; hence the depth of $P_1[[X]]$ is at least t+k. Since P_1 is idempotent, Corollary 4.6 asserts that there is a prime ideal Q of V[[X]] satisfying $P_1 \cdot V[[X]] \subseteq Q \subset P_1[[X]]$ in particular, $(0) \subset Q \subset P_1[[X]]$. Since the depth of $P_1[[X]]$ is at least t+k, we see that $\dim V[[X]] \ge t+k+2$.

Lemma 4.8. Let V be valuation ring and let P be a proper prime ideal of V.

- (a) If Q' is a prime ideal of $V_P[[X]]$ which satisfies $(PV_P) \cdot as$ $V_P[[X]] \subseteq Q' \subset (PV_P)[[X]]$, then Q' is a prime ideal of V[[X]] which satisfies $P \cdot V[[X]] \subseteq Q' \subset P[[X]]$.
- (b) Conversely, if Q is a prime ideal of V[[X]] which satisfies $P \cdot V[[X]] \subseteq Q \subset P[[X]]$, then Q is a prime ideal of $V_P[[X]]$ which satisfies $(PV_P) \cdot V_P[[X]] \subseteq Q \subset (PV_P)[[X]]$.

Proof. To establish (a), we observe that $Q' \subseteq (PV_P)[[X]] = P[[X]] \subseteq V[[X]]$, whereby $Q' \cap V[[X]] = Q'$.

We now establish (b); we begin by proving that Q is an ideal of $V_P[[X]]$. Let $f(X) \in Q$ and $g(X) \in V_P[[X]]$; we show that $f(X) \cdot$ as $g(X) \in Q$. Choose $h(X) \in P[[X]]$, $h(X) \notin Q$. For each $i, j \in \omega_0$, $g_i \in V_P$ and $h_j \in P$, implying that $g_i h_j \in PV_P = P$. Hence $g(X)h(X) \in P[[X]] \subseteq V[[X]]$, implying that $f(X)[g(X)h(X)] \in Q$. Since $f(X) \in Q \subseteq P[[X]]$, each $f_i \in P$; hence $f(X)g(X) \in P[[X]] \subseteq V[[X]]$. Since $[f(X)g(X)] \cdot h(X) \in Q$ where $f(X)g(X) \in V[[X]]$, $h(X) \in V[[X]]$, and $h(X) \notin Q$, we conclude that $f(X)g(X) \in Q$. Hence Q is an ideal of $V_P[[X]]$.

We now prove that Q is a prime ideal of $V_P[[X]]$. Let $S=V[[X]]\backslash Q$; then S is a multiplicative system in V[[X]], hence also in $V_P[[X]]$, and S clearly does not meet the ideal Q of $V_P[[X]]$. Hence there is a prime ideal Q^* of $V_P[[X]]$ which satisfies $Q \subseteq Q^*$, $Q^* \cap S = \varnothing$. Since $Q \subseteq Q^*$, $Q \subseteq Q^* \cap V[[X]]$; since $Q^* \cap S = \varnothing$, $Q^* \cap V[[X]] \subseteq Q$. Thus $Q^* \cap V[[X]] = Q$. Observe now that $Q^* \supseteq Q \supseteq P \cdot V[[X]] = (PV_P) \cdot V_P[[X]]$. By Theorem 4.3, Q^* compares with $(PV_P)[[X]] = P[[X]]$. Since Q^* lies over Q we must have that $Q^* \subset P[[X]] \subseteq V[[X]]$, implying that $Q^* = Q$. Hence Q is a prime ideal of $V_P[[X]]$.

That $(PV_P) \cdot V_P[[X]] \subseteq Q \subset (PV_P)[[X]]$ is clear.

Theorem 4.9. The following conditions are equivalent:

- (a) If V is a rank one nondiscrete valuation ring, then V[[X]] has finite dimension.
- (b) If V is a valuation ring having finite rank n, then V[[X]] has finite dimension.

Proof. It is clear that $(b) \rightarrow (a)$. We prove that $(a) \rightarrow (b)$ using induction on n, the case n = 1 being a consequence of (a) and Theorem 2.7.

We now assume that if W is a valuation ring of rank k, then W[[X]] has finite dimension. Let V be a valuation ring of rank k+1 which has minimal prime P_1 . Let $(0) \subset Q_1 \subset Q_2 \subset \cdots \subset Q_t$ be a chain of prime ideals of V[[X]]. Let $d=\dim V_{P_1}[[X]]$. Corollary 2.3 assures that there are at most d proper prime ideals in this chain which contract to (0) in V. Choose m so that $Q_m \cap V = (0)$ and $Q_{m+1} \cap V \neq (0)$; then $m \leq d$. For $r \geq m+1$, $Q_r \cap V \supseteq P_1$; Theorem 4.3 assures that for $r \geq m+1$, Q_r compares with $P_1[[X]]$. Lemma 4.8 assures that at most d of the ideals $Q_{m+1}, Q_{m+2}, \cdots, Q_t$ are contained in $P_1[[X]]$, whereby $Q_{m+d+1} \supset P_1[[X]]$. Since $m \leq d$, we have that $Q_{2d+1} \supseteq Q_{m+d+1} \supset P_1[[X]]$.

By the induction hypothesis, $(V/P_1)[[X]]$ has finite dimension. The depth of $P_1[[X]]$ is at most $(\dim(V/P_1)[[X]] - 1)$. It follows that the depth of Q_{2d+1} is at most $(\dim(V/P_1)[[X]] - 1)$, whereby

$$t \le (2d+1) + (\dim(V/P_1)[[X]] - 1) = 2d + \dim(V/P_1)[[X]]$$
.

We conclude that dim $V[[X]] \leq 2d + \dim(V/P_i)[[X]]$, whereby V[[X]] has finite dimension.

Theorem 4.10. The following conditions are equivalent:

- (a) If V is a rank one nondiscrete valuation ring, then the ascending chain condition for prime ideals holds in V[[X]].
- (b) If V is a valuation ring having finite rank n, then the ascending chain condition for prime ideals holds in V[[X]].

The proof of Theorem 4.10 is analogous to the proof of Theorem 4.9 and will therefore be omitted.

Added in proof. Jimmy T. Arnold has recently conveyed to me a paper of his, On Krull Dimension in Power Series Rings, in which he has established the following result.

Let R be a commutative ring with identity. If there exists a prime ideal P of R such that $\sqrt{(P \cdot R[[X]])} \neq P[[X]]$, then R[[X]] has infinite dimension.

It follows immediately that if V is a valuation ring which is not discrete, then V[[X]] has infinite dimension.

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Received May 12, 1970. This paper is part of the author's dissertation, written under the direction of Professor Robert Gilmer of Florida State University.

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PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

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