DIMENSION THEORY IN POWER SERIES RINGS

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Let \( V \) be a valuation ring of finite rank \( n \). If \( V \) is discrete, then \( V[[X]] \) has dimension \( n + 1 \). If \( V \) is not discrete, then the dimension of \( V[[X]] \) is at least \( n + k + 1 \), where \( k \) is the number of idempotent proper prime ideals of \( V \).

Let \( R \) be a commutative ring with identity. If there exists a chain \( P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_n \) of \( n + 1 \) prime ideals of \( R \), where \( P_n \subset R \), but no such chain of \( n + 2 \) prime ideals, then we say that \( R \) has dimension \( n \) and we write \( \dim R = n \) [3]. In [3] and [4], Seidenberg has investigated the dimension theory of \( R[X_1, X_2, \ldots, X_m] \) where \( R \) has finite dimension and \( X_1, X_2, \ldots, X_m \) are indeterminates over \( R \). We investigate the dimension theory of \( V[[X]] \) where \( V \) is a valuation ring.

Throughout this paper, \( R \) denotes a commutative ring with identity; \( \omega \) is the set of natural numbers; \( \omega_0 \) is the set of non-negative integers; and \( Z \) is the set of integers. If
\[
f(X) = \sum_{i=0}^{\infty} f_i X^i \in R[[X]],
\]
we denote by \( A_f \) the ideal of \( R \) generated by the coefficients of \( f(X) \): \( A_f = \{ f_0, f_1, \ldots, f_k, \ldots \} R \). If \( A \) is an ideal of \( R \), we let
\[
A[[X]] = \{ f(X) = \sum_{i=0}^{\infty} f_i X^i : f_i \in A \text{ for each } i \in \omega \}
\]
and we define \( A \cdot R[[X]] \) to be the ideal of \( R[[X]] \) which is generated by \( A \). Then \( A \cdot R[[X]] = \{ f(X) : A_f \subseteq B \text{ for some finitely generated ideal } B \text{ of } R \text{ with } B \subseteq A \} \). It is clear that \( A \cdot R[[X]] \subseteq A[[X]] \); equality holds if and only if each countably generated ideal of \( R \) contained in \( A \) is contained in a finitely generated ideal of \( R \) contained in \( A \). In particular, if \( V \) is a valuation ring containing an ideal \( A \) which is countably generated but not finitely generated, then \( A \cdot V[[X]] \subset A[[X]] \). Finally, we note that if \( A \) is an ideal of \( R \), then \( R[[X]]/A[[X]] \cong (R/A)[[X]] \); hence \( A[[X]] \) is a prime ideal of \( R[[X]] \) if and only if \( A \) is a prime ideal of \( R \).

2. Discrete valuation rings. Let \( V \) be a valuation ring of rank \( k \) with associated valuation \( v \) and value group \( G \); let \( \{0\} = G_0 \subset G_1 \subset \cdots \subset G_k = G \) be the chain of isolated subgroups of \( G \) together with \( G \). In [2], Iwasawa proves that for \( 1 \leq i \leq k \),
$G_i/G_{i-1} \cong H_i$ where $H_i$ is a subgroup of the additive group of real numbers, this being an order-preserving isomorphism of groups. If for $1 \leq i \leq k$, $H_i \cong \mathbb{Z}$, we shall say that $V$ is a discrete valuation ring of rank $k$. This is equivalent to the condition that $V$ contains no idempotent proper prime ideal.

**Lemma 2.1.** Let $V$ be a valuation ring and let $P$ be a proper prime ideal of $V$. If $P$ is not idempotent, then in $V[[X]]$,\n\n$$\sqrt{(P \cdot V[[X]])} = P[[X]] \text{ and } (P[[X]])^2 \subseteq P \cdot V[[X]].$$

**Proof.** Let $\alpha \in P$, $\alpha \notin P^2$. Then \n\n$$(P[[X]])^2 \subseteq P^2[[X]] \subseteq (\alpha)V[[X]] \subseteq P \cdot V[[X]].$$

Hence $P[[X]] \subseteq \sqrt{(P \cdot V[[X]])}$ and the reverse containment is clear.

**Lemma 2.2.** Let $V$ be a valuation ring with quotient field $K$ and let $P$ be a proper prime ideal of $V$. Let \n\n$$D = V[[X]][K] = (V[[X]])_{V \setminus \{0\}}.$$ \n\nThen $D = (V_P[[X]])_{V_P \setminus \{0\}}$.

**Proof.** We first show that $V_P[[X]] \subseteq D$. Let \n\n$$f(X) = \sum_{i=0}^{\infty} f_i X^i \in V_P[[X]].$$

For each $i \in \omega_0$, there exists $r_i \in V \setminus P$ such that $r_i f_i \in V$. Let $a \in P \setminus \{0\}$; then for each $i \in \omega_0$, $a/r_i \in PV_P = P \subseteq V$, implying that $af_i = (a/r_i) (r_i f_i) \in V$; that is, $af(X) \in V[[X]]$. This implies that $f(X) \in (V[[X]])_{V \setminus \{0\}} = D$, showing that $V_P[[X]] \subseteq D$.

Since $D \supseteq K$, each nonzero element of $V_P$ is a unit in $D$. Thus $D \supseteq (V_P[[X]])_{V_P \setminus \{0\}}$ and the reverse containment is obvious.

**Corollary 2.3.** Let $V$ be a valuation ring and let $P$ be a proper prime ideal of $V$. There is a one-to-one correspondence between prime ideals of $V[[X]]$ which contract to $(0)$ in $V$ and prime ideals of $V_P[[X]]$ which contract to $(0)$ in $V_P$; this correspondence preserves containment.

**Proof.** Lemma 2.2 assures that there is a one-to-one, containment preserving correspondence between each of these classes of prime ideals and the class of prime ideals of $D$.

**Lemma 2.4.** Let $R$ be a quasi-local ring having maximal ideal
M. Let \( f(X) \in R[[X]] \), \( f(X) \in M[[X]] \) — say \( f_k \in R \setminus M \), \( k \) minimal. There exists \( g(X) \), a unit of \( R[[X]] \), such that \( f(X)g(X) \) has exactly one unit coefficient, namely \((fg)_k\).

Proof. For \( u(X) \in R[[X]] \), denote by \( \bar{u}(X) \) the canonical image of \( u(X) \) in \((R/M)[[X]]\). By choice of \( k \),
\[
\bar{f}(X) = \bar{f}_k X^k + \bar{f}_{k+1}X^{k+1} + \cdots = X^k(\bar{f}_k + \bar{f}_{k+1}X + \cdots),
\]
where \( \bar{f}_k \neq 0 \). Then \( \bar{f}_k + \bar{f}_{k+1}X + \cdots \) is a unit of \((R/M)[[X]]\), and we can choose \( g(X) \in R[[X]] \) such that \( \bar{g}(X) \cdot (\bar{f}_k + \bar{f}_{k+1}X + \cdots) = 1 \). Thus \( \bar{f}(X) \cdot \bar{g}(X) = X^k \), and \( f(X)g(X) - X^k \in M[[X]] \). This implies that only the coefficient of \( X^k \) in \( f(X)g(X) \) is not in \( M \).

COROLLARY 2.5. Let \( V \) be a valuation ring and let \( P \) be a proper prime ideal of \( V \). If \( Q \) is an ideal of \( V_P[[X]] \) and if \( Q \not\subseteq (PV_P) [[X]] \), then \( Q \cap V[[X]] \not\subseteq P[[X]] \).

Proof. Lemma 2.4 assures that there is a power series \( g(X) \) in \( Q \) with \( g(X) \) having exactly one unit coefficient, \( g_k \). Since \( g_k \) is a unit of \( V_P \), there in no loss of generality in assuming that, in fact, \( g_k = 1 \). Then for \( i \neq k \), \( g_i \in PV_P = P \subseteq V \), implying that \( g(X) \in Q \cap V[[X]] \) while \( g(X) \notin P[[X]] \).

LEMMA 2.6.¹ Let \( R \) be a Noetherian ring having dimension \( n \). Then \( R[[X_1, X_2, \cdots, X_m]] \) is Noetherian and has dimension \( n + m \).

Proof. It is well known that if \( R \) is Noetherian, then \( R[[X_1, X_2, \cdots, X_m]] \) is Noetherian. We shall show that the dimension of \( R[[X]] \) is \( n + 1 \); the lemma follows immediately by induction on \( m \).

Let \( M \) be a maximal ideal of \( R[[X]] \). Then \( M = M_i + (X) \) for some maximal ideal \( M_i \) of \( R \). Since \( \dim R = n \), the height of \( M_i \) is \( k \) where \( k \leq n \). There exists an ideal \( A = (a_1, a_2, \cdots, a_k) \) of \( R \) which admits \( M_i \) as an isolated prime ideal \([5; 242]\). It is straightforward to verify that \( M = M_i + (X) \) is an isolated prime ideal of \( A + (X) = (a_1, a_2, \cdots, a_k, X) \) \( R[[X]] \). This implies that the height of \( M \) is at most \( k + 1 \) \([5; 240]\); since \( k \leq n \), the height of \( M \) is at most \( n + 1 \). Since \( M \) was an arbitrary maximal ideal of \( R[[X]] \), we conclude that \( \dim R[[X]] \leq n + 1 \); the reverse inequality is clear.

Theorem 2.7. Let \( V \) be a discrete valuation ring of rank \( n \) and let \((0) = P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots \subseteq P_n \) be the nonunit prime ideals of

¹ The proof of Lemma 2.6 was pointed out to the author by William Heinzer.
Then \( \dim V[[X]] = n + 1 \).

**Proof.** We use induction on \( n \), the case \( n = 1 \) following from Lemma 2.6 since a rank one discrete valuation ring is Noetherian.

Assuming the result for discrete valuation rings of rank less than \( n \), let \( V \) be a discrete valuation ring of rank \( n \) and let \( (0) \subset Q_1 \subset Q_2 \subset \cdots \subset Q_t \) be a chain of prime ideals of \( V[[X]] \). We consider two cases.

**Case 1.** \( Q_i \cap V \neq (0) \). Here \( Q_i \cap V \supseteq P_i \), so that \( Q_i \supseteq P_i \cdot V[[X]] \), implying that \( Q_i \supseteq \sqrt{(P_i \cdot V[[X]])} = P_i[[X]] \), the latter equality being a consequence of Lemma 2.1. But the depth of \( P_i[[X]] \) cannot exceed \( \dim (V/P_i) [[X]] = n \); we conclude that \( t \leq n + 1 \).

**Case 2.** \( Q_i \cap V = (0) \). Corollary 2.3 asserts that \( Q_i = Q^* \cap V[[X]] \), where \( Q^* \) is a prime ideal of \( V_{P_1}[[X]] \) and \( Q^* \cap V_{P_1} = (0) \). Since \( \dim V_{P_1}[[X]] = 2 \), \( Q^* \nsubseteq (P_i V_{P_i})[[X]] \). By Corollary 2.5, \( Q_i \nsubseteq P_i[[X]] \). Since \( V_{P_1}[[X]] \) is two-dimensional and local, each proper prime ideal of \( V_{P_1}[[X]] \) which contracts to \( (0) \) in \( V_{P_1} \) is a minimal prime ideal of \( V_{P_1}[[X]] \). Corollary 2.3 now assures that each proper prime ideal of \( V[[X]] \) which contracts to \( (0) \) in \( V \) is a minimal prime ideal of \( V[[X]] \). It follows that \( Q_2 \cap V \neq (0) \), implying that \( Q_2 \supseteq P_1[[X]] \). Since also \( Q_2 \supseteq Q_1 \) and \( Q_1 \nsubseteq P_1[[X]] \), we conclude that \( Q_2 \supseteq P_i[[X]] \). Thus we have a chain \( (0) \subset P_1[[X]] \subset Q_2 \subset Q_3 \subset \cdots \subset Q_t \). It follows, as in Case 1, that \( t \leq n + 1 \).

Thus \( \dim V[[X]] \leq n + 1 \) and the reverse inequality is clear.

**3. Rank one nondiscrete valuation rings.** We note that if \( V \) is a rank one valuation ring, then the value group of \( v \) is Archimedian.

**Lemma 3.1.** Let \( V \) be a valuation ring and let \( B \) be an ideal of \( V \). If \( B \) is not finitely generated, then the following conditions are equivalent:

(a) \( f(X) \in B \cdot V[[X]] \).

(b) \( A_f \subseteq (b) \) for some \( b \in B \).

(c) \( f(X) = b g(X) \) for some \( b \in B \), \( g(X) \in V[[X]] \).

(d) \( A_f \subseteq B \).

**Proof.** We establish that (a) \( \rightarrow \) (b) \( \rightarrow \) (c) \( \rightarrow \) (a) and that (b) \( \rightarrow \) (d).

(a) \( \rightarrow \) (b): Let \( f(X) \in B \cdot V[[X]] \); then we can write

\[ f(X) = b_1[g^{(1)}(X)] + b_2[g^{(2)}(X)] + \cdots + b_r[g^{(r)}(X)] \]
where for $1 \leq i \leq t$, $b_i \in B$ and $g^{(i)}(X) = \sum_{j=0}^{\infty} g_{ij} X^j \in V[[X]]$. Thus $f(X) = \sum_{i=0}^{\infty} f_i X^i$ where $f_i = \sum_{k=1}^{t} b_k g_{ki}$. In $V$, $(b_1, b_2, \cdots, b_t) = (b_s)$ for some $s$, $1 \leq s \leq t$. Now for $i \in \omega$, $f_i = \sum_{k=1}^{s} b_k g_{ki} \in (b_s)$, implying that $A_f \subseteq (b_s)$ where $b_s \in B$.

(b) $\rightarrow$ (c): We assume that $A_f \subseteq (b)$; then for $i \in \omega$, $f_i = b g_i$ where $g_i \in V$. Let $g(X) = \sum_{i=0}^{\infty} g_i X^i$; it then is clear that $f(X) = b g(X)$.

(c) $\rightarrow$ (a): This is obvious.

(b) $\rightarrow$ (d): This is immediate from the assumption that $B$ is not finitely generated.

(d) $\rightarrow$ (b): Assuming that $A_f \subset B$, let $b \in B$, $b \not\in A_f$. Then $(b) \not\subseteq A_f$ so $A_f \subseteq (b)$ since $V$ is a valuation ring.

**Theorem 3.2.** Let $V$ be a rank one nondiscrete valuation ring having maximal ideal $M$. Then $M \cdot V[[X]] = \sqrt{(M \cdot V[[X]])}$.

**Proof.** Let $f(X) \in \sqrt{(M \cdot V[[X]])}$ — say $[f(X)]^k \in M \cdot V[[X]]$; we then can write $[f(X)]^k = r g(X)$ where $r \in M$ and $g(X) \in V[[X]]$. There exists an element $s$ of $M$ with $0 < v(s) \leq v(r)/k$; then $r = s^t t$ where $t \in V$, implying that $[f(X)]^k = r g(X) = s^t t g(X)$, so that

$$[f(X)]^k/s^k = [f(X)/s]^k = t g(X) \in V[[X]].$$

Therefore $f(X)/s$ is a root of $Z^k - t g(X) \in V[[X]][Z]$, whereby $f(X)/s$ is integral over $V[[X]]$. Also $f(X)/s$ clearly is in the quotient field of $V[[X]]$. But $V$ is completely integrally closed, implying that $V[[X]]$ is completely integrally closed, hence is integrally closed [1; 150]. Thus $f(X)/s = h(X) \in V[[X]]$ and $f(X) = s h(X) \in M \cdot V[[X]]$ since $s \in M$. Hence $\sqrt{(M \cdot V[[X]])} \subseteq M \cdot V[[X]]$, so that equality holds.

**Theorem 3.3.** Let $R$ be a quasi-local ring having maximal ideal $M$ and let $Q$ be a prime ideal of $R[[X]]$. If $Q \supseteq M \cdot R[[X]]$, then either $Q \supseteq M[[X]]$ or $Q \subseteq M[[X]]$.

**Proof.** We assume that $Q \not\subseteq M[[X]]$ and show that $Q \supseteq M[[X]]$. Let $f(X) = \sum_{i=0}^{\infty} f_i X^i \in Q$, $f(X) \in M[[X]]$. Let $t$ be the smallest integer $k$ for which $f_k$ is a unit of $R$. Let $g(X) = \sum_{i=0}^{t} f_i X^i$ if $t > 0$; let $g(X) = 0$ if $t = 0$. Then $g(X) \in M \cdot R[[X]] \subseteq Q$, implying that $f(X) - g(X) \in Q$. If $f(X) - g(X)$ has order zero, then $g(X) = 0$, so that $f_k$ is a unit of $R$, implying that $f(X)$ is a unit of $R[[X]]$, whence $Q = R[[X]] \supseteq M[[X]]$. If $f(X) - g(X)$ has positive order $n$, then $[f(X) - g(X)]_n$ is a unit of $R$ and $f(X) - g(X) = X^n h(X)$ where $h_0 = [f(X) - g(X)]_n$ is a unit of $R$, implying that $h(X)$ is a unit of $R[[X]]$. 


Since \( f(X) - g(X) = X^n h(X) \in Q \) and \( Q \) is a prime ideal of \( R[[X]] \), either \( X^n \in Q \) or \( h(X) \in Q \). If \( X^n \in Q \), then \( X \in Q \), implying that \( Q \supseteq M \cdot R[[X]] + (X) \supseteq M[[X]] \). If \( h(X) \in Q \), then \( Q = R[[X]] \supseteq M[[X]] \). Hence if \( Q \not\supseteq M[[X]] \), then \( Q \supseteq M[[X]] \).

**THEOREM 3.4.** Let \( V \) be a rank one nondiscrete valuation ring having maximal ideal \( M \).

(a) There is a prime ideal \( P \) of \( V[[X]] \) satisfying \( M \cdot V[[X]] \subseteq P \subseteq M[[X]] \).

(b) \( \dim V[[X]] \geq 3 \).

*Proof.* Theorem 3.2 asserts that

\[
\sqrt{(M \cdot V[[X]])} = M \cdot V[[X]] \subset M[[X]].
\]

Hence there is a prime ideal \( P \) of \( V[[X]] \) satisfying \( P \supseteq M \cdot V[[X]] \), \( P \not\supseteq M[[X]] \). Theorem 3.3 then asserts that \( P \subset M[[X]] \); hence (a) holds.

We now have a chain \((0) \subset P \subset M[[X]] \subset M \cdot V[[X]] + (X)\) of prime ideals of \( V[[X]] \), implying (b).


**LEMMA 4.1.** Let \( V \) be a valuation ring and let \( P \) be a proper prime ideal of \( V \). Then \( PV_r = P \); hence \( P \) is idempotent if and only if \( PV_r \) is idempotent.

The proof of Lemma 4.1 is straightforward and will therefore be omitted.

**LEMMA 4.2.** Let \( V \) be a valuation ring and let \( P \) be an idempotent proper prime ideal of \( V \). Then \( P \cdot V[[X]] = (PV_r) \cdot V_r[[X]] \).

*Proof.* Let \( f(X) \in (PV_r) \cdot V_r[[X]] \) — say \( f(X) = rh(X) \) where \( r \in PV_r \) and \( h(X) \in V_r[[X]] \). Since \( P = PV_r \) is idempotent, we can write \( r = st \) where \( s, t \in P \). Then for \( i \in \omega_b \), there exists \( a_i \in V \) such that \( a_i h_i \in V \). Since \( a_i \in V \) and \( t \in P \), we have that \( (t) \subseteq (a_i) \) so that \( t/a_i \in V \) for each \( i \in \omega_b \), implying that \( th_i = (t/a_i) (a_i h_i) \in V \) for each \( i \in \omega_b \) — that is, \( th(X) \in V[[X]] \). Since \( s \in P \), we conclude that \( f(X) = rh(X) = s(th(X)) \in P \cdot V[[X]] \), establishing that

\[
(PV_r) \cdot V_r[[X]] \subseteq P \cdot V[[X]].
\]

The reverse containment is obvious.
Theorem 4.3. Let $V$ be a valuation ring and let $P$ be a proper prime ideal of $V$. If $Q$ is a prime ideal of $V[[X]]$ and if $Q \supseteq P \cdot V[[X]]$, then either $Q \supseteq P[[X]]$ or $Q \subseteq P[[X]]$.

Proof. Assuming that $Q \not\subseteq P[[X]]$, we first establish that either $X \in Q$ or $Q$ contains $h(X)$, where $h(X) \in V[[X]]$ and $h \in P$. Let $f(X) = \sum_{i=0}^{\infty} f_i X^i \in Q$, $f(X) \in P[[X]]$. Let $t$ be the smallest integer $k$ for which $f_k \not\in P$. If $t = 0$, then we let $h(X) = f(X)$. If $t > 0$, then we let $g(X) = \sum_{i=0}^{t-1} f_i X^i$. Then $g(X) \in P \cdot V[[X]] \subseteq Q$, implying that $f(X) - g(X) \in Q$. Further, $f(X) - g(X) = X^t h(X)$ where $h = f_t \not\in P$. Since $Q$ is prime, either $X \in Q$ or $h(X) \in Q$. Hence if $Q \not\subseteq P[[X]]$, then either $X \in Q$ or $Q$ contains $h(X)$ where $h(X) \in V[[X]]$ and $h \in P$.

If $X \in Q$, then $Q \supseteq P[[X]]$; hence we consider the case where $h(X) \in Q$ with $h \not\in P$. Observe now that $h(X) \in V_p[[X]]$ and that $h$ is a unit of $V_p$, implying that $h(X)$ is a unit of $V_p[[X]]$ — that is $1/h(X) \in V_p[[X]]$. Now let $r(X) \in P[[X]]$; then

$$r(X)[1/h(X)] \in P[[X]] \cdot V_p[[X]] \subseteq P[[X]]$$

— in particular, $r(X)[1/h(X)] \in V[[X]]$. Since $h(X) \in Q$, we see that $r(X) = h(X)[r(X)/h(X)] \in Q$. Hence $Q \supseteq P[[X]]$.

Lemma 4.4. Let $V$ be a valuation ring having a minimal prime ideal $P$. If $P$ is idempotent, then $P \cdot V[[X]] = \sqrt{(P \cdot V[[X]])}$.

Proof. Let $f(X) \in \sqrt{(P \cdot V[[X]])}$. Then in

$$V_p[[X]], f(X) \in \sqrt{(P V_p) \cdot V_p[[X]]}$$

by Lemma 4.2. Since $V_p$ is a rank one nondiscrete valuation ring, Theorem 3.2 asserts that $\sqrt{(P V_p) \cdot V_p[[X]]} = (P V_p) \cdot V_p[[X]]$. Hence $f(X) \in (P V_p) \cdot V_p[[X]] = P \cdot V[[X]]$, the latter equality following from Lemma 4.2.

Theorem 4.5. Let $V$ be a valuation ring and let $P$ be a proper prime ideal of $V$. If $P$ is idempotent, then

$$P \cdot V[[X]] = \sqrt{(P \cdot V[[X]])}.$$ 

Proof. We shall say that $P$ is branched provided there exists a $P$-primary ideal distinct from $P[1; 173]$. We consider two cases.

Case 1. $P$ is branched. Then there is a prime ideal $Q$ of $V$ with $Q \subseteq P$ and such that there are no prime ideals of $V$ properly
between $Q$ and $P$ \([1; 173]\). Then $P/Q$ is a minimal prime ideal of $V/Q$ and $P/Q$ is idempotent. Lemma 4.4 assures that

$$(P/Q) \cdot (V/Q)[[X]] = \sqrt{(P/Q) \cdot (V/Q)[[X]]}.$$  

By considering the natural homomorphism from $V[[X]]$ to $(V/Q)[[X]]$, we conclude that $P \cdot V[[X]] = \sqrt{(P \cdot V[[X]])}$.

**Case 2.** $P$ is not branched. Then $P = \bigcup_{i} M_{i}$ where $\{M_{i}\}_{i \in \Lambda}$ is the collection of prime ideals of $V$ properly contained in $P[1; 173]$. Let $f(X) \in \sqrt{(P \cdot V[[X]])}$ — say $f(X)^{k} \in P \cdot V[[X]]$. Then $f(X)^{k} = rg(X)$ where $g(X) \in V[[X]]$ and $r \in P$, implying that $r \in M_{i}$ for some $\lambda_{i} \in \Lambda$. Thus $f(X)^{k} = rg(X) \in M_{i_{1}}[[X]]$, implying that $f(X) \in M_{i_{1}}[[X]]$. There exists $\lambda_{2} \in \Lambda$ such that $M_{i_{1}} \supset M_{i_{2}}$. Let $s \in M_{i_{2}}$, $s \notin M_{i_{1}}$; then $(s) \supseteq M_{i_{1}} \supseteq A_{r}$, so that $f(X) = sh(X)$ where $h(X) \in V[[X]]$. Since $s \in M_{i_{2}}$, $s \notin P$; hence $f(X) = sh(X) \in P \cdot V[[X]]$.

**Corollary 4.6.** Let $V$ be a valuation ring having a proper prime ideal $P$. If $P$ is idempotent, then there is a prime ideal $Q$ of $V[[X]]$ satisfying $P \cdot V[[X]] \subseteq Q \subset P[[X]]$.

**Proof.** Theorem 4.5 assures that

$$\sqrt{(P \cdot V[[X]])} = P \cdot V[[X]] \subset P[[X]].$$

Hence there is a prime ideal $Q$ of $V[[X]]$ satisfying $Q \supseteq P \cdot V[[X]]$, $Q \not\subseteq P[[X]]$. Theorem 4.3 then asserts that $Q \subset P[[X]]$.

**Theorem 4.7.** Let $V$ be a valuation ring of rank $n$ having $k$ distinct idempotent proper prime ideals. Then $\dim V[[X]] \geq n + k + 1$.

**Proof.** We use induction on $n$, the case $n = 1$ following from Theorem 2.7 and Theorem 3.4.

Assuming the result for valuation rings of rank $t$, let $V$ be a valuation ring of rank $t + 1$ having $k$ distinct idempotent proper prime ideals and let $(0) \subset P_{1} \subset P_{2} \subset \cdots \subset P_{t+1}$ be the chain of nonunit prime ideals of $V$. We consider two cases.

**Case 1.** $P_{i}$ is not idempotent. Here $V/P_{i}$ is a valuation ring of rank $t$ which has $k$ distinct idempotent proper prime ideals. By the induction hypothesis, $\dim (V/P_{i})[[X]] \geq t + k + 1$. Since $(V/P_{i})[[X]] \cong V[[X]]/P_{i}[[X]]$, this implies that the depth of $P_{i}[[X]]$ is at least $t + k + 1$. Since $P_{i}[[X]] \neq (0)$, $\dim V[[X]] \geq t + k + 2$.

**Case 2.** $P_{i}$ is idempotent. Here $V/P_{i}$ is a valuation ring of rank
t which has \(k - 1\) distinct idempotent proper prime ideals. By the induction hypothesis, \(\dim(V/P_t)[[X]] \geq t + (k - 1) + 1 = t + k\); hence the depth of \(P_t[[X]]\) is at least \(t + k\). Since \(P_t\) is idempotent, Corollary 4.6 asserts that there is a prime ideal \(Q\) of \(V[[X]]\) satisfying \(P_t \cdot V[[X]] \subseteq Q \subseteq P_t[[X]]\) — in particular, \((0) \subseteq Q \subseteq P_t[[X]]\). Since the depth of \(P_t[[X]]\) is at least \(t + k\), we see that \(\dim V[[X]] \geq t + k + 2\).

**Lemma 4.8.** Let \(V\) be valuation ring and let \(P\) be a proper prime ideal of \(V\).

(a) If \(Q'\) is a prime ideal of \(V_r[[X]]\) which satisfies \((PV_r) \cdot \) as \(V_r[[X]] \subseteq Q' \subseteq (PV_r)\), then \(Q'\) is a prime ideal of \(V[[X]]\) which satisfies \(P \cdot V[[X]] \subseteq Q' \subseteq P[[X]]\).

(b) Conversely, if \(Q\) is a prime ideal of \(V[[X]]\) which satisfies \(P \cdot V[[X]] \subseteq Q \subseteq P[[X]]\), then \(Q\) is a prime ideal of \(V_r[[X]]\) which satisfies \((PV_r) \cdot V_r[[X]] \subseteq Q \subseteq (PV_r)\).

**Proof.** To establish (a), we observe that \(Q' \subseteq (PV_r)\), whereby \(Q' \cap V[[X]] = Q'\).

We now establish (b); we begin by proving that \(Q\) is an ideal of \(V_r[[X]]\). Let \(f(X) \in Q\) and \(g(X) \in V_r[[X]]\); we show that \(f(X) \cdot g(X) \in Q\). Choose \(h(X) \in P[[X]], h(X) \in Q\). For each \(i, j \in \omega_0, g_i \in V_r\) and \(h_j \in P\), implying that \(g_i h_j \in PV_r = P\). Hence \(g(X) h(X) \in P[[X]] \subseteq V[[X]]\), implying that \(f(X) g(X) h(X) \in Q\). Since \(f(X) \in Q \subseteq P[[X]]\), each \(f_i \in P\); hence \(f(X) g(X) \in P[[X]] \subseteq V[[X]]\). Since \([f(X) g(X)] \cdot h(X) \in Q\) where \(f(X) g(X) \in V[[X]]\), \(h(X) \in V[[X]]\), and \(h(X) \in Q\), we conclude that \(f(X) g(X) \in Q\). Hence \(Q\) is an ideal of \(V_r[[X]]\).

We now prove that \(Q\) is a prime ideal of \(V_r[[X]]\). Let \(S = V[[X]] \cap Q\); then \(S\) is a multiplicative system in \(V[[X]]\), hence also in \(V_r[[X]]\), and \(S\) clearly does not meet the ideal \(Q\) of \(V_r[[X]]\). Hence there is a prime ideal \(Q^*\) of \(V_r[[X]]\) which satisfies \(Q \subseteq Q^*, Q^* \cap S = \emptyset\). Since \(Q \subseteq Q^*, Q \subseteq Q^* \cap V[[X]]\); since \(Q^* \cap S = \emptyset, Q^* \cap V[[X]] \subseteq Q\). Thus \(Q^* \cap V[[X]] = Q\). Observe now that \(Q^* \supseteq Q \supseteq P \cdot V[[X]] = (PV_r)\). By Theorem 4.3, \(Q^*\) compares with \((PV_r)\) = \(P[[X]]\). Since \(Q^*\) lies over \(Q\) we must have that \(Q^* \subseteq P[[X]] \subseteq V[[X]]\), implying that \(Q^* = Q\). Hence \(Q\) is a prime ideal of \(V_r[[X]]\).

That \((PV_r) \cdot V_r[[X]] \subseteq Q \subseteq (PV_r)\) is clear.

**Theorem 4.9.** The following conditions are equivalent:

(a) If \(V\) is a rank one nondiscrete valuation ring, then \(V[[X]]\) has finite dimension.

(b) If \(V\) is a valuation ring having finite rank \(n\), then \(V[[X]]\) has finite dimension.
Proof. It is clear that (b) $\rightarrow$ (a). We prove that (a) $\rightarrow$ (b) using induction on $n$, the case $n = 1$ being a consequence of (a) and Theorem 2.7.

We now assume that if $W$ is a valuation ring of rank $k$, then $W[[X]]$ has finite dimension. Let $V$ be a valuation ring of rank $k + 1$ which has minimal prime $P_1$. Let $(0) \subset Q_1 \subset Q_2 \subset \cdots \subset Q_t$ be a chain of prime ideals of $V[[X]]$. Let $d = \dim V_{P_1}[[X]]$. Corollary 2.3 assures that there are at most $d$ proper prime ideals in this chain which contract to $(0)$ in $V$. Choose $m$ so that $Q_m \cap V = (0)$ and $Q_{m+1} \cap V \neq (0)$; then $m \leq d$. For $r \geq m + 1$, $Q_r \cap V \supseteq P_1$; Theorem 4.3 assures that for $r \geq m + 1$, $Q_r$ compares with $P_1[[X]]$. Lemma 4.8 assures that at most $d$ of the ideals $Q_{m+1}, Q_{m+2}, \cdots, Q_t$ are contained in $P_1[[X]]$, whereby $Q_{m+d+1} \supset P_1[[X]]$. Since $m \leq d$, we have that $Q_{2d+1} \supseteq Q_{m+d+1} \supset P_1[[X]].$

By the induction hypothesis, $(V/P_1)[[X]]$ has finite dimension. The depth of $P_1[[X]]$ is at most $(\dim (V/P_1)[[X]] - 1)$. It follows that the depth of $Q_{2d+1}$ is at most $(\dim (V/P_1)[[X]] - 1)$, whereby

$$t \leq (2d + 1) + (\dim (V/P_1)[[X]] - 1) = 2d + \dim (V/P_1)[[X]].$$

We conclude that $\dim V[[X]] \leq 2d + \dim (V/P_1)[[X]]$, whereby $V[[X]]$ has finite dimension.

Theorem 4.10. The following conditions are equivalent:

(a) If $V$ is a rank one nondiscrete valuation ring, then the ascending chain condition for prime ideals holds in $V[[X]]$.

(b) If $V$ is a valuation ring having finite rank $n$, then the ascending chain condition for prime ideals holds in $V[[X]]$.

The proof of Theorem 4.10 is analogous to the proof of Theorem 4.9 and will therefore be omitted.

Added in proof. Jimmy T. Arnold has recently conveyed to me a paper of his, On Krull Dimension in Power Series Rings, in which he has established the following result.

Let $R$ be a commutative ring with identity. If there exists a prime ideal $P$ of $R$ such that $\sqrt{(P \cdot R[[X]])} \neq P[[X]]$, then $R[[X]]$ has infinite dimension.

It follows immediately that if $V$ is a valuation ring which is not discrete, then $V[[X]]$ has infinite dimension.
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