

Pacific Journal of Mathematics

**SOME TOPOLOGICAL PROPERTIES WEAKER THAN
COMPACTNESS**

G. GOSS AND GIOVANNI VIGLINO

SOME TOPOLOGICAL PROPERTIES WEAKER THAN COMPACTNESS

G. GOSS AND G. VIGLINO

Many topological properties may be described by covering relations which may also generally be easily described in terms of filter relations. For example, a space is compact if and only if each open cover of the space contains a finite subcover, or equivalently, if each filter has an adherent point. In this paper, characterizations are given of some topological properties weaker than compactness, both in terms of filters and coverings. In the final section a question posed by Viglino and by Dickman and Zame is answered.

2. Definitions and notations. (a) A space for which distinct points may be separated by disjoint closed neighborhoods (i.e., a Urysohn space) will be labeled a $T_{2(1/2)}$ -space. Let $\nu = 2, 2\frac{1}{2}$, or 3. A T_ν -space is said to be T_ν -closed if it is closed in each T_ν -extension. A T_ν -space (X, τ) is said to be T_ν -minimal if there exists no T_ν topology on X strictly weaker than τ .

(b) A Hausdorff space (X, τ) is C -compact if given a closed set Q of X and a τ -open cover \mathcal{O} of Q , then there exists a finite number of elements of \mathcal{O} , say $0_i, 1 \leq i \leq n$, with $Q \subset \text{cl}_X \bigcup_{i=1}^n 0_i$.

(c) A Hausdorff space X is *functionally compact* if for every open filter \mathcal{U} in X such that the intersection A of the elements of \mathcal{U} equals the intersection of the closures of the elements of \mathcal{U} , then \mathcal{U} is the neighborhood filter of A .

(d) A filter is *open* (closed) if it has a base of open (closed) sets. A *regular filter* is a filter which is both open and closed.

(e) Let A be a subset of a space X . An open cover, \mathcal{U} , of A will be said to be a *Urysohn cover* if for each $x \in A$ there exist elements $0_1, 0_2$ of \mathcal{U} with $x \in 0_1 \subset \text{cl } 0_1 \subset 0_2$.

(f) Let A be a subset of a space X . An open cover, \mathcal{S} , of A will be said to be a *strong cover* if for each $x \in A$ there exist $\{0_n\}_{n=1}^\infty \subset \mathcal{S}$ with $x \in 0_1$ and $\text{cl } 0_i \subset 0_{i+1}$ for each i .

(g) A closed subset Y of a space X is *regular closed* if given $x \in X \setminus Y$, then there exists an open set 0 with $x \in 0 \subset \text{cl } 0 \subset Y^c$.

3. Covering theorems. Filter characterizations for T_ν -closed and T_ν -minimal spaces are listed below. The proof of (a) may be found in [2]; (b) in [1]; (c), (d), and (e) in [5]; (f) in [4].

THEOREM I. (a) A T_2 -space is T_2 -closed if and only if every

open filter has an adherent point.

(b) A T_2 -space is T_2 -minimal if and only if every open filter with unique adherent point converges.

(c) A $T_{2(1/2)}$ -space is $T_{2(1/2)}$ -closed if and only if for each two open filters $\mathcal{F}_1, \mathcal{F}_3$ and each closed filter \mathcal{F}_2 such that $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3$, then \mathcal{F}_1 has an adherent point.

(d) A $T_{2(1/2)}$ -space is $T_{2(1/2)}$ -minimal if and only if for each two open filters $\mathcal{F}_1, \mathcal{F}_3$ and each closed filter \mathcal{F}_2 such that $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3$ with \mathcal{F}_1 having a unique adherent point, then \mathcal{F}_3 converges.

(e) A T_3 -space is T_3 -closed if and only if every regular filter has an adherent point.

(f) A T_3 -space is T_3 -minimal if and only if every regular filter with unique adherent point converges.

Of the six properties listed in the above theorem, only the first has a well known covering characterization. (See Theorem II (a'), below). Herrlick has listed in [5] covering characterizations for $T_{2(1/2)}$ -closed and T_3 -closed spaces. We list in the following theorem covering characterizations for each of the six properties. These characterizations emphasize the relationship between T_ν -closed and T_ν -minimal spaces. The proof of each part of Theorem II follows from its counterpart in Theorem I. We offer proofs for parts (d') and (f').

THEOREM II. (a') A T_2 -space X is T_2 -closed if and only if given an open cover, \mathcal{O} , of X , then there exists $0_i \in \mathcal{O}$, $1 \leq i \leq n$, such that $X = \text{cl } \bigcup_{i=1}^n 0_i$.

(b') A T_2 -space X is T_2 -minimal if and only if given $p \in X$, an open cover, \mathcal{O} , of $X \setminus \{p\}$, and an open neighborhood U of p , then there exist $0_i \in \mathcal{O}$, $1 \leq i \leq n$, such that $X = U \cup \text{cl } \bigcup_{i=1}^n 0_i$.

(c') A $T_{2(1/2)}$ -space X is $T_{2(1/2)}$ -closed if and only if given a Urysohn cover, \mathcal{U} of X , then there exist $0_i \in \mathcal{U}$, $1 \leq i \leq n$, such that $X = \text{cl } \bigcup_{i=1}^n 0_i$.

(d') A $T_{2(1/2)}$ -space X is $T_{2(1/2)}$ -minimal if and only if given $p \in X$, a Urysohn cover, \mathcal{U} of $X \setminus \{p\}$ and an open neighborhood U of p , then there exist $0_i \in \mathcal{U}$, $1 \leq i \leq n$, such that $X = U \cup \text{cl } \bigcup_{i=1}^n 0_i$.

(e') A T_3 -space X is T_3 -closed if and only if given a strong cover, \mathcal{S} , of X , then there exist $0_i \in \mathcal{S}$, $1 \leq i \leq n$, such that $X = \bigcup_{i=1}^n 0_i$.

(f') A T_3 -space X is T_3 -minimal if and only if given $p \in X$, a strong cover, \mathcal{S} , of $X \setminus \{p\}$, and an open neighborhood U of p , then there exist $0_i \in \mathcal{S}$, $1 \leq i \leq n$, such that $X = U \cup \bigcup_{i=1}^n 0_i$.

Proof of (d'). Let $p \in X$, U an open neighborhood of p , and \mathcal{U} a Urysohn cover of $X \setminus \{p\}$ such that the union of the closure of any

finite number of elements of \mathcal{U} fails to cover U^c . Since X is $T_{2(1/2)}$, we may assume that for each $x \in X \setminus \{p\}$ there exist $0_x^1, 0_x^3$ in \mathcal{U} with $x \in 0_x^1 \subset \text{cl } 0_x^1 \subset 0_x^3$, and $p \notin 0_x^3$. Let \mathcal{F}_3 denote the filter generated by $\{0_x^3\}_{x \in X \setminus \{p\}}$, \mathcal{F}_1 the filter generated by $\{\overline{0_x^1}\}_{x \in X \setminus \{p\}}$, and \mathcal{F}_2 the filter generated by $\{\overline{0_x^3}\}_{x \in X \setminus \{p\}}$. Then $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3$ with p the only adherent point of \mathcal{F}_1 and \mathcal{F}_3 not converging.

Conversely, let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3$ with \mathcal{F}_1 having a unique adherent point, p , and \mathcal{F}_3 not converging. Let U be an open neighborhood of p which contains no element of \mathcal{F}_3 . Then, $\mathcal{U} = \{\overline{F}_1^c/F_1 \in \mathcal{F}_1\} \cup \{\overline{F}_3^c/F_3 \in \mathcal{F}_3\}$ is a Urysohn cover of $X \setminus \{p\}$ such that union of the closure of any number of elements of \mathcal{U} fails to cover U^c .

Proof of (f'). Let $p \in X$, U an open neighborhood of p , and \mathcal{S} a strong cover of $X \setminus \{p\}$ such that the union of any finite number of elements of \mathcal{S} fails to cover U^c . Since X is T_3 , we may assume that for each $x \in X \setminus \{p\}$ there exist a sequence $\{0_x^n\}_{n=1}^\infty$ of elements of \mathcal{S} with $x \in 0_x^1, \text{cl } 0_x^n \subset 0_x^{n+1}$ and $p \notin 0_x^n$ for any n . Now, the filter generated by $\{0_x^n/x \in X \setminus \{p\}; n = 1, 2, 3, \dots\}$ is regular with unique adherent point, P , and does not converge.

Conversely, let \mathcal{F} be a regular filter with unique adherent point, p . Let U be an open neighborhood of p which contains no element of \mathcal{F} . Then $\{\overline{F}^c/F \in \mathcal{F}\}$ is a strong cover of $X \setminus \{p\}$ and no finite union of elements of this cover contains U^c .

By replacing in Theorem II (b') the point, p , by a closed (regular closed) set, we obtain the covering characterization for C -compact (functionally compact) spaces listed below. The proof follows easily from definition b (c).

THEOREM III. *A T_2 -space X is C -compact (functionally compact) if and only if given a closed (regular closed) subset C of X , an open cover \mathcal{O} , of $X \setminus C$, and an open neighborhood U of C , then there exist $0_i \in \mathcal{O}, 1 \leq i \leq n$, such that $X = U \cup \text{cl } \bigcup_{i=1}^n 0_i$.*

4. **A counterexample.** One can easily show that every continuous function from a C -compact space into a Hausdorff space is closed [6]. The question as to whether or not the converse is valid was posed in [6]. Dickman and Zame [4] have since then shown that a necessary and sufficient condition that a Hausdorff space be functionally compact is that each continuous function of the space into a Hausdorff space be closed. We resolve the question posed in [6] by constructing, in the following example, a space which is functionally compact but not C -compact. The example is a modification of that given in [4] showing that a functionally compact space need not be compact.

EXAMPLE. Let $I = [0, 1]$. For each integer $n \geq 2$, let $\{a_n^j\}_{j=1}^\infty$ be a strictly decreasing sequence in $(1/n, 1/n - 1)$ converging to $1/n$. Let $X = I \setminus \bigcup_{n \geq 2} \{a_n^j\}$. Topologize X as follows: Let $X \setminus (\{1/n\}_{n=1}^\infty \cup \{0\})$ retain the usual topology. Let a neighborhood system of the point 0 be composed of all sets of the form $\{x \in X \mid |x| < 1/m\} \setminus \{1/n\}_{n=1}^\infty$, m an integer. Let a neighborhood system of the point $1/n$ be composed of all sets of the form $0 \cap X$ where 0 is an open set in I with $\{1/n, a_{n-1}^1, a_{n-3}^2, \dots, a_2^{n-1/2}\} \subset 0$ in the case that n is odd, and with $\{1/n, a_{n-1}^1, a_{n-3}^2, \dots, a_3^{n/2-1}\} \subset 0$ in the case that n is even (where for $n = 2$ we simply have $\{1/2\}$). Clearly X is Hausdorff. Let $0_{2n} = \{x \in X \mid |x - 1/2n| < 1/3n\} \cup \bigcup_{i=1}^{n-1} \{x \in X \mid |x - a_{2n-2i+1}^i| < 1/3n\}$ for each $n > 1$. Then $\{0_{2n}\}_{n>1}$ is an open cover of the closed set $\{1/2n\}_{n>1}$ and the closure of any finite union of elements in $\{0_{2n}\}_{n>1}$ fails to contain $\{1/2n\}_{n>1}$. Hence, X is not C -compact. We apply Theorem III and show that X is functionally compact.

Let C be a regular closed subset of X , \mathcal{O} an open cover of $X \setminus C$, U an open neighborhood of C . Suppose first that C contains infinitely many elements of $\{1/n\}_{n=1}^\infty$. Then by the regularity of C , $\{1/n\}_{n=1}^\infty \cup \{0\} \subset C$ so that $X \setminus U$ is compact. Suppose that $0 \in C$ and that only finitely many elements of $\{1/n\}_{n=1}^\infty$ are contained in C . Choose $1/2n$ and $1/2n + 1$ such that neither is in C . Let $0_{2n}, 0_{2n+1}$ be elements of \mathcal{O} containing $1/2n$ and $1/2n + 1$ respectively. Then, $\{1/n \mid n \geq k\} \subset \text{cl}(0_{2n} \cup 0_{2n+1})$ for some k . It is easy to see that the closure of a finite number of elements of \mathcal{O} contains $X \setminus (\text{cl}(0_{2n} \cup 0_{2n+1}) \cup U)$. Suppose that $0 \notin C$ and that only finitely many elements of $\{1/n\}_{n=1}^\infty$ are contained in C . Once more let $0_{2n}, 0_{2n+1}$ be elements of \mathcal{O} containing $1/2n$ and $1/2n + 1$ respectively, with neither $1/2n$ nor $1/2n + 1$ in C . Let 0_i be an element of \mathcal{O} containing 0. It is easy to see that the closure of a finite number of elements of \mathcal{O} contains $X \setminus (\text{cl}(0_i \cup 0_{2n} \cup 0_{2n+1}) \cup U)$. Hence, X is functionally compact.

REFERENCES

1. M. P. Berri, *Minimal topological spaces*, Trans. Amer. Math. Soc. **108** 97-105.
2. M. P. Berri and R. Sorgenfrey, *Minimal regular spaces*, Proc. Amer. Math. Soc. **14** (1963), 454-458.
3. N. Bourbaki, *Topologie Generale*, 3^e ed, Ch. I, II, Hermann, Paris, 1961.
4. R. F. Dickman, Jr. and A. Zame, *Functionally compact spaces*, Pacific J. Math. **31** (1969), 303-312.
5. H. Herrlich, *T_v -Abgeschlossenheit und T_v -Minimalität*, Math. Zeit. **88** (1965), 285-294.
6. G. Viglino, *C -compact spaces*, Duke J. Math. **36** (1969), 761-764.

Received February 12, 1970.

WESLEYAN UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON
Stanford University
Stanford, California 94305

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

RICHARD PIERCE
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLE

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
TRW SYSTEMS
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Pacific Journal of Mathematics

Vol. 35, No. 3

November, 1970

John D. Arrison and Michael Rich, <i>On nearly commutative degree one algebras</i>	533
Bruce Alan Barnes, <i>Algebras with minimal left ideals which are Hilbert spaces</i>	537
Robert F. Brown, <i>An elementary proof of the uniqueness of the fixed point index</i>	549
Ronn L. Carpenter, <i>Principal ideals in F-algebras</i>	559
Chen Chung Chang and Yiannis (John) Nicolas Moschovakis, <i>The Suslin-Kleene theorem for V_κ with cofinality $(\kappa) = \omega$</i>	565
Theodore Seio Chihara, <i>The derived set of the spectrum of a distribution function</i>	571
Tae Geun Cho, <i>On the Choquet boundary for a nonclosed subspace of $C(S)$</i>	575
Richard Brian Darst, <i>The Lebesgue decomposition, Radon-Nikodym derivative, conditional expectation, and martingale convergence for lattices of sets</i>	581
David E. Fields, <i>Dimension theory in power series rings</i>	601
Michael Lawrence Fredman, <i>Congruence formulas obtained by counting irreducibles</i>	613
John Eric Gilbert, <i>On the ideal structure of some algebras of analytic functions</i>	625
G. Goss and Giovanni Viglino, <i>Some topological properties weaker than compactness</i>	635
George Grätzer and J. Sichler, <i>On the endomorphism semigroup (and category) of bounded lattices</i>	639
R. C. Lacher, <i>Cell-like mappings. II</i>	649
Shiva Narain Lal, <i>On a theorem of M. Izumi and S. Izumi</i>	661
Howard Barrow Lambert, <i>Differential mappings on a vector space</i>	669
Richard G. Levin and Takayuki Tamura, <i>Notes on commutative power joined semigroups</i>	673
Robert Edward Lewand and Kevin Mor McCrimmon, <i>Macdonald's theorem for quadratic Jordan algebras</i>	681
J. A. Marti, <i>On some types of completeness in topological vector spaces</i>	707
Walter J. Meyer, <i>Characterization of the Steiner point</i>	717
Saad H. Mohamed, <i>Rings whose homomorphic images are q-rings</i>	727
Thomas V. O'Brien and William Lawrence Reddy, <i>Each compact orientable surface of positive genus admits an expansive homeomorphism</i>	737
Robert James Plemmons and M. T. West, <i>On the semigroup of binary relations</i>	743
Calvin R. Putnam, <i>Unbounded inverses of hyponormal operators</i>	755
William T. Reid, <i>Some remarks on special disconjugacy criteria for differential systems</i>	763
C. Ambrose Rogers, <i>The convex generation of convex Borel sets in euclidean space</i>	773
S. Saran, <i>A general theorem for bilinear generating functions</i>	783
S. W. Smith, <i>Cone relationships of biorthogonal systems</i>	787
Wolmer Vasconcelos, <i>On commutative endomorphism rings</i>	795
Vernon Emil Zander, <i>Products of finitely additive set functions from Orlicz spaces</i>	799
G. Sankaranarayanan and C. Suyambulingom, <i>Correction to: "Some renewal theorems concerning a sequence of correlated random variables"</i>	805
Joseph Zaks, <i>Correction to: "Trivially extending decompositions of E^n"</i>	805
Dong Hoon Lee, <i>Correction to: "The adjoint group of Lie groups"</i>	805
James Edward Ward, <i>Correction to: "Two-groups and Jordan algebras"</i>	806