ON A THEOREM OF M. IZUMI AND S. IZUMI

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This paper establishes a theorem on the absolute Nörlund summability of Fourier series which generalizes and unifies generalizations by the author and by M. and S. Izumi of an earlier result by McFadden.

Let \( \sum a_n \) be a series with partial sums \( S_n \) and let \( p_n \) be a sequence of real constants with

\[
P_n = \sum_{v=0}^{n} p_v, \quad p_0 > 0, \quad P_{-1} = p_{-1} = 0.
\]

The series \( \sum a_n \) is said to be summable \( |N, p_n| \) if

\[
\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty,
\]

where

\[
t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} S_v.
\]

We write \( P(t) = P_{[t]} \) and in the sequel we assume that \( p_n \) is nonnegative, nonincreasing and \( \lim_{n \to \infty} p_n = 0 \).

2. Let \( f(t) \) be a periodic function with period \( 2\pi \) and integrable (L) in \( (-\pi, \pi) \). The Fourier series of \( f(t) \) is

\[
\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t),
\]

where \( a_n \) and \( b_n \) are given by the usual Euler-Fourier formulae. We write

\[
\phi(t) = f(x + t) + f(x - t) - 2f(x),
\]

\[
\alpha(t) = \sum_{v=0}^{\infty} p_v \cos vt, \quad \beta(t) = \sum_{v=0}^{\infty} p_v \sin vt,
\]

\[
\alpha_n = \int_{0}^{\pi} \phi(t) \alpha(t) \cos nt \, dt, \quad \beta_n = \int_{0}^{\pi} \phi(t) \beta(t) \sin nt \, dt,
\]

\[
w(\delta) = \sup_{0 \leq |t| \leq \delta} |f(x + t) - f(x)|.
\]

\( p \) and \( q \) are mutually conjugate indices in the sense that \( \frac{1}{p} + \frac{1}{q} = 1 \).

Recently M. Izumi and S. Izumi ([2, Th. 3]) proved the following
**THEOREM A.** Let \( \{p_n\} \) be a positive decreasing and convex sequence tending to zero and satisfying the condition

\[
\sum_{n=1}^{\infty} p^n_n n^{p-2} < \infty , \quad (1 < p \leq 2).
\]

If the modulus of continuity \( \omega(\delta) \) of \( f \) satisfies the conditions

\[
\sum_{n=1}^{\infty} \frac{\omega\left(\frac{1}{n}\right)}{n^{1/p} P_n} < \infty ,
\]

and

\[(2.1) \quad \sum_{m=n}^{\infty} \frac{1}{m^p \left(\frac{1}{m}\right)^{p-1}} \leq \frac{C}{\left(n \omega\left(\frac{1}{n}\right)\right)^{p-1}} \]

then the Fourier series of \( f \) is \( |N, p_n| \) summable.

In this note we prove that the condition (2.1) of the above theorem is redundant in that the assertion of the theorem holds without the condition (2.1) as well. The final result is then embodied in the following

**THEOREM.** Let \( \{p_n - p_{n+1}\} \) be a nonincreasing sequence and

\[(2.2) \quad \sum_{n=1}^{\infty} p^n_n n^{p-2} < C , \quad (1 < p \leq 2).
\]

If the modulus of continuity of the continuous function \( f(x) \) satisfies the condition

\[(2.3) \quad \sum_{n=1}^{\infty} \omega(n^{-1}) P_n^{-1} n^{-1/q} < C ,
\]

then the Fourier series of \( f \) is \( |N, p_n| \) summable.

It is known that (see [4, Chapter XII, proof of Lemma 6.6]) the condition (2.2) of the theorem implies that

\[
\sum_{n=1}^{\infty} P^n_n n^{-2} < C .
\]

Also it is easy to show that the above condition implies the condition (2.2) of the theorem. Since \( p_n \) is nonnegative and nonincreasing

---

1 Throughout the paper \( C \) denotes a positive constant, not necessarily the same at each occurrence.
we have \( np_n \leq P_n \) and therefore
\[
\sum_{n=1}^{\infty} P_n^p n^{p-2} \leq \sum_{n=1}^{\infty} P_n^p n^{-2} .
\]
Thus the conditions (2.2) and \( \sum_{n=1}^{\infty} P_n^p n^{-2} < C \) are equivalent. In view of this equivalence it follows that the theorem established here generalises an earlier result of the author [3] as well.

3. The following lemmas are required for the proof of the theorem.

**Lemma 1.** Under the condition (2.2) of the theorem
\[
\int_0^{\gamma/\sqrt{n}} \omega(t)P(t^{-1})dt \leq C \omega(n^{-1})n^{-1/\alpha} .
\]

*Proof.* Remembering that the condition (2.2) of the theorem implies that
\[
\sum_{n=1}^{\infty} P_n^p n^{-2} < C ,
\]
we have
\[
\int_0^{\gamma/\sqrt{n}} \omega(t)P(t^{-1})dt \leq \sum_{n=1}^{\infty} \omega(n^{-1})P_n n^{-2}
\]
\[
\leq \left[ \sum_{n=1}^{\infty} \{\omega(n^{-1})n^{-2+1/p}\}^p \right]^{1/p} \left[ \sum_{n=1}^{\infty} P_n^p n^{-2} \right]^{1/p}
\]
\[
\leq C \omega(n^{-1})n^{2/p-1} ,
\]
which is equivalent to the assertion of the lemma.

**Lemma 2.** ([4, Chapter XII, Lemma 6.6]). For the function \( \alpha(t) \) to belong to the class \( L^p(p > 1) \) it is necessary and sufficient that the condition (2.2) of the theorem is satisfied.

**Lemma 3.** ([1, Lemmas 5.11, 5.14 and 5.32]). If \( P_n \) is non-negative and nonincreasing, then for \( 0 \leq a < b \leq \infty \), \( 0 < \alpha \leq \pi \) and any \( n \)
\[
(3.1) \quad \left| \sum_{\gamma=a}^{b} P_{\gamma}e^{i(n-\gamma)t} \right| \leq C P(t^{-1}) ,
\]
\[
(3.2) \quad \sum_{\gamma=a}^{b} \frac{\nu(\gamma - P_{\gamma-1})}{P_\gamma P_{\gamma-1}} \leq C P_{n-1}^{-1} ,
\]
and
as \( \lambda \to \infty \).

**Lemma 4.** ([3, Lemma 5.20]). If \( p_n \) is nonnegative and non-increasing and if we take

\[
\gamma(t) = \sum_{n=0}^{\infty} p_ne^{it}
\]

then for \( t \) in \((h, \pi)\)

\[
|\gamma(t + 2h) - \gamma(t)| \leq Ch^{-1}P(h^{-1}) .
\]

**Lemma 5.** ([1, see proof of Lemma 5.16]). If \( p_n \) is nonnegative and nonincreasing, \( \lim_{n \to \infty} p_n = 0 \) and \( \{p_n - p_{n+1}\} \) is nonincreasing, then

\[
\frac{1}{P_{n-1}} \left| \int_{1/n}^{\varepsilon} \phi(t) \left\{ \sum_{\nu=0}^{n-1} p_{\nu} \cos (n - \nu)t + \sum_{\nu=0}^{n-1} \frac{p_{\nu}}{P_{n}} \cos (n - \nu)t \right\} dt \right|
\]

\[
\leq C \frac{p_n}{P_n P_{n-1}} + C \left[ \frac{n(p_n - p_{n-1})}{P_n P_{n-1}} + \frac{P_n}{P_n P_{n-1}} \right] \int_{1/n}^{\varepsilon} |\phi(t)| P(t^{-1})t^{-1} dt .
\]

**Lemma 6.** ([2]). Under the conditions (2.2) and (2.3) of the theorem

\[
\sum_{n=1}^{\infty} \omega \left( \frac{1}{n} \right) \leq C .
\]

4. **Proof of the theorem.** For the Fourier series

\[
S_n(x) - f(x) = \frac{1}{\pi} \int_{0}^{\pi} \phi(t) \left( \frac{1}{2} + \sum_{k=1}^{\infty} \cos kt \right) dt ,
\]

so that from (1.1) and Abel’s transformation we have

\[
\frac{\pi}{2} \left| t_n - t_{n-1} \right| \\
= \left| \int_{0}^{\pi} \phi(t) \left\{ \sum_{\nu=0}^{n-1} \left( \frac{P_{n-\nu-1}}{P_n} - \frac{P_{n-\nu-2}}{P_{n-1}} \right) \cos (n + 1)t \right\} dt \right| \\
= \frac{1}{P_n P_{n-1}} \left| \int_{0}^{\pi} \phi(t) \sum_{\nu=0}^{n-1} (p_{\nu} P_n - p_{\nu} P_{n-1}) \cos (n + 1)t \ dt \right| \\
= \left| \frac{1}{P_{n-1}} \int_{0}^{\pi} \phi(t) \sum_{\nu=0}^{n-1} p_{\nu} n \cos (n + 1)t \ dt \right| \\
- \frac{1}{P_n P_{n-1}} \left| \int_{0}^{\pi} \phi(t) \sum_{\nu=n-1}^{\infty} p_{\nu} \cos (n - \nu)t + \sum_{\nu=0}^{n-1} p_{\nu} P_n \cos (n - \nu)t \ dt \right| \\
(4.1)
\]
\[
\frac{1}{P_{n-1}} \left[ \int_0^\pi \phi(t) \alpha(t) \cos nt \, dt \right] + \frac{1}{P_{n-1}} \left[ \int_0^\pi \phi(t) \beta(t) \sin nt \, dt \right] \\
+ \frac{1}{P_{n-1}} \left[ \int_0^{1/n} \phi(t) \sum_{e=1}^{\infty} p_e \cos (n - v) t \, dt \right] \\
+ \frac{p_n}{P_n P_{n-1}} \left[ \int_0^{1/n} \phi(t) \sum_{e=1}^{n-1} p_e \cos (n - v) t \, dt \right] \\
+ \frac{1}{P_{n-1}} \left[ \int_0^{\pi} \phi(t) \left\{ \sum_{e=1}^{\infty} p_e \cos (n - v) t + \sum_{v=0}^{n-1} \frac{p_n}{P_n} p_e \cos (n - v) t \right\} \, dt \right].
\]

= \sum_{r=1}^{5} |x_n^{(r)}|, \text{ say.}

From (4.1) and the definition of the absolute Nörlund summability it is clear that for establishing the theorem we have to prove that

\[
(4.2) \quad \sum_{n=2}^{\infty} |x_n^{(r)}| < \infty, \quad (r = 1, 2, \cdots, 5).
\]

Now

\[
\sum_{n=2}^{\infty} |x_n^{(1)}| = \sum_{n=2}^{\infty} \sum_{k=1}^{2^l+1} |\alpha_n| P_{n-1}
\]

\[
\leq \sum_{n=2}^{\infty} \left( \sum_{k=1}^{2^l+1} |\alpha_n|^q \right)^{1/q} \left( \sum_{n=2^l+1}^{2^{l+1}} P_{n-1}^{-1/p} \right)^{1/p}
\]

\[
\leq C \sum_{l=1}^{\infty} 2^{l/2} P^{-1/2} \left( \sum_{n=1}^{\infty} |\alpha_n| \sin \frac{n\pi}{2^{l+1}} \right)^{1/q}
\]

making use of (3.3) of Lemma 3.

Since the function \( \phi(t) \) is bounded in \([0, \pi]\) and by Lemma 2, under the condition (2.2) of the theorem, \( \alpha(t) \in L^p \), it follows that \( \phi(t) \alpha(t) \in L^p \). Also, it is known [1] that the Fourier series of \( \phi(t + h) \alpha(t + h) - \phi(t - h) \alpha(t - h) \) is \(-4/\pi \sum \alpha_n \sin nt \sin nh\), and therefore by Hausdorff-Young inequality we get

\[
\left( \sum_{n=1}^{\infty} \left| \alpha_n \sin nh \right| \right)^{p/q}
\]

\[
\leq C \int_0^{\pi} \left| \phi(t + h) \alpha(t + h) - \phi(t - h) \alpha(t - h) \right|^p \, dt
\]

\[
\leq C \int_0^{\pi} \left| \phi(t + h) - \phi(t - h) \right|^p \left| \alpha(t + h) \right|^p \, dt
\]

\[
+ C \int_0^{\pi} \left| \alpha(t + h) - \alpha(t - h) \right|^p \left| \phi(t - h) \right|^p \, dt
\]

\[
(4.4) \quad \leq C \omega^p(h) \int_0^{\pi} \left| \alpha(t + h) \right|^p \, dt + C \int_{-h}^{\pi-h} \left| \alpha(t + 2h) - \alpha(t) \right|^p \left| \phi(t) \right|^p \, dt
\]

\[
\leq C \omega^p(h) + C \int_{-h}^{h} \omega^p(|t|) \left| \alpha(t + 2h) \right|^p \, dt
\]
\[ + C \int_{-h}^{h} \omega^p(t) | \alpha(t) |^p dt + C \int_{-h}^{h} | \alpha(t + 2h) - \alpha(t) |^p \omega^p(t) dt \]

\[ \leq C \omega^p(h) + Ch^p P^p(h^{-1}) \int_{-h}^{h} \omega^p(t) t^{-p} dt \]

using Lemma 4 and remembering that by virtue of Lemma 2, \( \alpha(t) \in L^p \).

Taking \( h = \pi/2 \) in the estimate (4.4) and then substituting it in (4.3) we have

\[
\sum_{n=2}^{\infty} | x_n^{(1)} | \leq C \sum_{n=1}^{\infty} 2^{\lambda/(1/p) - 1} \sum_{m=1}^{2^{\lambda/(1/p) - 1}} \left( \left( \int_{1/2}^{1} \omega(n^{-1}) n^{-1} \right) \right)^{1/p} \]

\[ \leq C + C \sum_{n=1}^{\infty} 2^{\lambda/(1/p) - 1} \sum_{m=1}^{2^{\lambda/(1/p) - 1}} \left( \sum_{n=m^{-1}+1}^{2m} \omega(n^{-1}) n^{-1} \right)^{1/p} \]

\[ \leq C + C \sum_{n=1}^{\infty} \omega(n^{-1}) n^{-1} \leq C , \]

by virtue of the condition (2.3) of the theorem and Lemma 6.

Similarly, we can prove that

\[ \sum_{n=2}^{\infty} | x_n^{(2)} | < \infty . \]

Also,

\[ \sum_{n=2}^{\infty} | x_n^{(3)} | \leq C \sum_{n=2}^{\infty} P_{n-1}^{-1} \int_{0}^{1/2} \omega(t) P(t^{-1}) dt \]

\[ \leq C \sum_{n=1}^{\infty} \omega(n^{-1}) P_{n-1}^{-1} n^{-1/2} \leq C , \]

by the application of (3.1) of Lemma 3, Lemma 1 and the condition (2.3) of the theorem.

For the proof of

\[ \sum_{n=2}^{\infty} | x_n^{(4)} | < C , \]

see the proof of \( \sum_{n=1}^{\infty} K_n < \infty \) in [2]. Finally, by Lemma 5 we have
\[
\sum_{n=2}^\infty |a_n^{(5)}| \leq C \sum_{n=1}^\infty \frac{p_n}{P_n P_{n-1}} + C \sum_{n=1}^\infty \left[ \frac{n(p_n - p_{n+1})}{P_n P_{n-1}} + \frac{p_n}{P_n P_{n-1}} \right] \int_1^T \omega(t) P(t^{-1}) t^{-1} dt
\]
\[
+ C \sum_{n=1}^\infty \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^n \omega(v^{-1}) P_v v^{-1}
\]
\[
+ C \sum_{n=1}^\infty \frac{n(p_n - p_{n+1})}{P_n P_{n-1}} \sum_{v=1}^n \omega(v-1) P_v v^{-1}
\]
\[
\leq C + C \sum_{v=1}^\infty \omega(v^{-1}) P_v v^{-1} \sum_{n=v}^\infty \frac{p_n}{P_n P_{n-1}}
\]
\[
+ C \sum_{v=1}^\infty \omega(v^{-1}) P_v v^{-1} \sum_{n=v}^\infty \frac{n(p_n - p_{n+1})}{P_n P_{n-1}}
\]
\[
\leq C + C \sum_{v=1}^\infty \omega(v^{-1}) v^{-1} \leq C ,
\]
by the application of the estimate (3.2) of Lemma 3 and Lemma 6.

Combining the estimates in (4.5) — (4.9) we find that (4.2) is established. This completes the proof of the theorem.

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REFERENCES


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