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Let E and F be two normed vector spaces with real scalars, and G an open subset of E. A mapping $f: E \to F$ is said to be differentiable at x in G if there is a bounded linear map $L: E \to F$ such that for every y in G,

$$f(y) - f(x) = L(y - x) + R(x, y)\ ,$$

where $R: E \to F$ and

$$\lim_{||y-x||\to 0} \frac{||R(x, y)||}{||y - x||} = 0\ .$$

L is called the differential of f at x. An extension of this definition is possible is such a way as to include a point x on the boundary of G. In such cases f is said to have a differential at x from side G. Some properties of side differentials and relationships between the differential of f at x and its side differentials at x are shown in this paper.

Theorems 1, 2, 3, and 4, listed below without proofs, are known theorems. The balance of the paper will be used to extend this theory.

THEOREM 1. *If f is differentiable at x, then for every y in G,*

$$L(y) = \lim_{a\to 0} \frac{f(x + ay) - f(x)}{a} \quad [1, \text{p. } 32].$$

THEOREM 2. *If f is differentiable at x, then the differential is unique* [1, p. 33].

THEOREM 3. *The differential of f is independent of the norms in E and F* [1, p. 35].

THEOREM 4. *If f is a real valued functional and differentiable, a necessary condition for f to have an extreme value at x is that $L(x - y)$ vanish for all y in G* [2, p. 13].

DEFINITION 1. Let h be a vector in E, then

$$|m| = \lim_{a\to 0} \frac{||f(x + ah) - f(x)||}{||ah||}$$

if it exists, is said to be the absolute-slope of f at x in the h direction.

THEOREM 5. *If f is differentiable at x then the absolute-slope of f at x in the h direction exists and furthermore*

$$|m| = \left\| L\left(\frac{h}{\|h\|}\right)\right\| .$$

Proof. Writing $f(y) - f(x) = L(ah) + R(x, x + ah)$ and dividing by $\|ah\|$ we have

$$\frac{f(x + ah) - f(x)}{\|ah\|} = \frac{L(ah)}{\|ah\|} + \frac{R(x, x + ah)}{\|ah\|} ,$$

and

$$\lim_{a\to 0} \frac{\|f(x + ah) - f(x)\|}{\|ah\|} = \lim_{a\to 0} \left\|\frac{a}{|a|} L\left(\frac{h}{\|h\|}\right)\right\| = \left\| L \frac{h}{\|h\|}\right\| .$$

COROLLARY. $|m| \leqq \|L\|$ *for all h in E.*

The proof is immediate since L is bounded and

$$\|L\| = \sup_{\|h\|=1} \|L(h)\| .$$

DEFINITION 2. A function S from E to F with the properties
(i) $S(x + y) = S(x) + S(y)$ and
(ii) $S(ax) = aS(x)$ if $a > 0$ will be called almost-linear.

It should be noted that if the domain of S is the whole space, then S is linear.

DEFINITION 3. Suppose that f is differentiable on an open set G and x is on the boundary of G. f is said to have a derivative from side G if there exists an almost-linear function S such that:

$$f(y) - f(x) = S(y - x) + R(x, y)$$

for all y such that $(1 - a)x + ay$ is in G whenever $0 < a \leqq 1$ and

$$\lim_{\|y-x\|\to 0} \frac{\|R(x, y)\|}{\|y - x\|} = 0 .$$

THEOREM 6. *If f is differentiable at x from side G then*

$$S(y - x) = \lim_{a\to 0^+} \frac{f((1 - a)x + ay) - f(x)}{a} .$$

The proof is similar to that of Theorem 1.

COROLLARY.

$$S(y) = \lim_{a \to 0^+} \frac{f(x + ay) - f(x)}{a} .$$

THEOREM 7. *If f has a differential at x from side G, then this differential is unique.*

The proof is similar to that of Theorem 2.

THEOREM 8. *Suppose that $x \in G$, an open set. Let $\{F_i\}$ be the collection of all open subsets of G such that x is on the boundary of F_i. Then f is differentiable at x if and only if f has a differential from side F_i for each i and all of these side differentials are equal.*

Proof. The only if proof is trivial. For the converse, we need only show that the almost-linear function S is a linear function. Pick F_1 and F_2 so that $F_1 \cup F_2 \cup \{x\}$ is balanced with respect to x. For each y in F_2, we can write $f(y) - f(x) = S(x - y) + R(x, y)$, where S is almost-linear and

$$\lim_{||y-x|| \to 0} \frac{||R(x, y)||}{||y - x||} = 0 .$$

Since the differential from side F_1 is the same as the differential from side F_2, we have for each z in F_2, $f(z) - f(x) = S(z - x) + P(x, z)$, where

$$\lim_{||z-x|| \to 0} \frac{||P(x, z)||}{||z - x||} = 0 .$$

If z is chosen so that $z - x = x - y$, then

$$\begin{aligned} S(x - y) = S(z - x) &= \lim_{a \to 0^+} \frac{f((1 - a)x + az) - f(x)}{a} \\ &= \lim_{a \to 0^+} \frac{f(x - ay + 2ax - ay) - f(x)}{a} \\ &= \lim_{a \to 0^+} \frac{f((1 + a)x - ay) - f(x)}{a} \\ &= \lim_{a \to 0^-} \frac{f((1 - a)x + ay) - f(x)}{a} \\ &= -\lim_{a \to 0^-} \frac{f((1 - a)x + ay) - f(x)}{a} \\ &= -S(y - x) . \end{aligned}$$

Hence S is linear.

THEOREM 9. *Suppose that G and H are open sets with a non-*

empty intersection and that x is a boundary point of G, H and $G \cap H$. If f has a differential at x from the G and H sides, then f has a differential from the $G \cup H$ side.

Proof. Since f has a differential from the G side we may write $f(y) - f(x) = S(y - x) + R(x, y)$ for all y such that $(1 - a)x + ay$ is in G whenever $0 < a \leqq 1$, where S is almost-linear and

$$\lim_{||x-y||\to 0} \frac{||R(x, y)||}{||x - y||} = 0 .$$

Also we may write $f(y) - f(x) = T(y - x) + P(x, y)$ for all y such that $(1 - a)x + ay$ is in H whenever $0 < a \leqq 1$, where T is almost-linear and

$$\lim_{||x-y||\to 0} \frac{||P(x, y)||}{||x - y||} = 0 .$$

By Theorem 6, $S(y - x)$ and $T(y - x)$ must agree for all y such that $(1 - a)x + ay$ is in $G \cap H$ whenever $0 < a \leqq 1$. Hence T and S are extensions of each other and are unique by Theorem 7. Let

$$V(y - x) = \begin{cases} S(y - x), y \text{ in } G \\ T(y - x), y \text{ in } H . \end{cases}$$

Then

$$f(y) - f(x) = V(y - x) + \begin{cases} R(x, y), y \text{ in } G \\ P(x, y), y \text{ in } H - G \end{cases}$$

for all y such that $(1 - a)x + ay$ is in $G \cup H$. Therefore f has a differential at x from the $G \cup H$ side.

THEOREM 10. *Let U be an open set containing x. Then f has a differential at x if there exist in U a finite number of open sets $\{G_i\}_{i=1}^n$ such that $G_i \cap G_{i+1} \neq \emptyset$ (Take $G_{n+1} = G_1$), x is on the boundary of each G_i and $G_i \cap G_{i+1}$, $U = \{x\} \cup \bigcup_{i=1}^n G_i$ and f has a differential from the G_i side for each i.*

This is a corollary to Theorems 7 and 9.

REFERENCES

1. Casper Goffman, *Calculus of Several Variables*, Harper & Row, Publisher, New York, 1965.
2. I. M. Gilfand and S. V. Fomin, *Calculus of Variations*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963.

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