

Pacific Journal of Mathematics

DIFFERENTIAL MAPPINGS ON A VECTOR SPACE

HOWARD BARROW LAMBERT

DIFFERENTIAL MAPPINGS ON A VECTOR SPACE

HOWARD B. LAMBERT

Let E and F be two normed vector spaces with real scalars, and G an open subset of E . A mapping $f: E \rightarrow F$ is said to be differentiable at x in G if there is a bounded linear map $L: E \rightarrow F$ such that for every y in G ,

$$f(y) - f(x) = L(y - x) + R(x, y),$$

where $R: E \rightarrow F$ and

$$\lim_{\|y-x\| \rightarrow 0} \frac{\|R(x, y)\|}{\|y-x\|} = 0.$$

L is called the differential of f at x . An extension of this definition is possible in such a way as to include a point x on the boundary of G . In such cases f is said to have a differential at x from side G . Some properties of side differentials and relationships between the differential of f at x and its side differentials at x are shown in this paper.

Theorems 1, 2, 3, and 4, listed below without proofs, are known theorems. The balance of the paper will be used to extend this theory.

THEOREM 1. *If f is differentiable at x , then for every y in G ,*

$$L(y) = \lim_{a \rightarrow 0} \frac{f(x + ay) - f(x)}{a} \quad [1, \text{p. } 32].$$

THEOREM 2. *If f is differentiable at x , then the differential is unique [1, p. 33].*

THEOREM 3. *The differential of f is independent of the norms in E and F [1, p. 35].*

THEOREM 4. *If f is a real valued functional and differentiable, a necessary condition for f to have an extreme value at x is that $L(x - y)$ vanish for all y in G [2, p. 13].*

DEFINITION 1. Let h be a vector in E , then

$$|m| = \lim_{a \rightarrow 0} \frac{\|f(x + ah) - f(x)\|}{\|ah\|}$$

if it exists, is said to be the absolute-slope of f at x in the h direction.

THEOREM 5. *If f is differentiable at x then the absolute-slope of f at x in the h direction exists and furthermore*

$$|m| = \left\| L\left(\frac{h}{\|h\|}\right) \right\|.$$

Proof. Writing $f(y) - f(x) = L(ah) + R(x, x + ah)$ and dividing by $\|ah\|$ we have

$$\frac{f(x + ah) - f(x)}{\|ah\|} = \frac{L(ah)}{\|ah\|} + \frac{R(x, x + ah)}{\|ah\|},$$

and

$$\lim_{a \rightarrow 0} \frac{\|f(x + ah) - f(x)\|}{\|ah\|} = \lim_{a \rightarrow 0} \left\| \frac{a}{|a|} L\left(\frac{h}{\|h\|}\right) \right\| = \left\| L\frac{h}{\|h\|} \right\|.$$

COROLLARY. $|m| \leq \|L\|$ for all h in E .

The proof is immediate since L is bounded and

$$\|L\| = \sup_{\|h\|=1} \|L(h)\|.$$

DEFINITION 2. A function S from E to F with the properties

- (i) $S(x + y) = S(x) + S(y)$ and
- (ii) $S(ax) = aS(x)$ if $a > 0$ will be called almost-linear.

It should be noted that if the domain of S is the whole space, then S is linear.

DEFINITION 3. Suppose that f is differentiable on an open set G and x is on the boundary of G . f is said to have a derivative from side G if there exists an almost-linear function S such that:

$$f(y) - f(x) = S(y - x) + R(x, y)$$

for all y such that $(1 - a)x + ay$ is in G whenever $0 < a \leq 1$ and

$$\lim_{\|y-x\| \rightarrow 0} \frac{\|R(x, y)\|}{\|y - x\|} = 0.$$

THEOREM 6. *If f is differentiable at x from side G then*

$$S(y - x) = \lim_{a \rightarrow 0^+} \frac{f((1 - a)x + ay) - f(x)}{a}.$$

The proof is similar to that of Theorem 1.

COROLLARY.

$$S(y) = \lim_{a \rightarrow 0^+} \frac{f(x + ay) - f(x)}{a} .$$

THEOREM 7. *If f has a differential at x from side G , then this differential is unique.*

The proof is similar to that of Theorem 2.

THEOREM 8. *Suppose that $x \in G$, an open set. Let $\{F_i\}$ be the collection of all open subsets of G such that x is on the boundary of F_i . Then f is differentiable at x if and only if f has a differential from side F_i for each i and all of these side differentials are equal.*

Proof. The only if proof is trivial. For the converse, we need only show that the almost-linear function S is a linear function. Pick F_1 and F_2 so that $F_1 \cup F_2 \cup \{x\}$ is balanced with respect to x . For each y in F_2 , we can write $f(y) - f(x) = S(x - y) + R(x, y)$, where S is almost-linear and

$$\lim_{\|y-x\| \rightarrow 0} \frac{\|R(x, y)\|}{\|y-x\|} = 0 .$$

Since the differential from side F_1 is the same as the differential from side F_2 , we have for each z in F_2 , $f(z) - f(x) = S(z - x) + P(x, z)$, where

$$\lim_{\|z-x\| \rightarrow 0} \frac{\|P(x, z)\|}{\|z-x\|} = 0 .$$

If z is chosen so that $z - x = x - y$, then

$$\begin{aligned} S(x - y) &= S(z - x) = \lim_{a \rightarrow 0^+} \frac{f((1 - a)x + az) - f(x)}{a} \\ &= \lim_{a \rightarrow 0^+} \frac{f(x - ay + 2ax - ay) - f(x)}{a} \\ &= \lim_{a \rightarrow 0^+} \frac{f((1 + a)x - ay) - f(x)}{a} \\ &= \lim_{a \rightarrow 0^-} \frac{f((1 - a)x + ay) - f(x)}{a} \\ &= - \lim_{a \rightarrow 0^-} \frac{f((1 - a)x + ay) - f(x)}{a} \\ &= -S(y - x) . \end{aligned}$$

Hence S is linear.

THEOREM 9. *Suppose that G and H are open sets with a non-*

empty intersection and that x is a boundary point of G , H and $G \cap H$. If f has a differential at x from the G and H sides, then f has a differential from the $G \cup H$ side.

Proof. Since f has a differential from the G side we may write $f(y) - f(x) = S(y - x) + R(x, y)$ for all y such that $(1 - a)x + ay$ is in G whenever $0 < a \leq 1$, where S is almost-linear and

$$\lim_{\|x-y\| \rightarrow 0} \frac{\|R(x, y)\|}{\|x - y\|} = 0.$$

Also we may write $f(y) - f(x) = T(y - x) + P(x, y)$ for all y such that $(1 - a)x + ay$ is in H whenever $0 < a \leq 1$, where T is almost-linear and

$$\lim_{\|x-y\| \rightarrow 0} \frac{\|P(x, y)\|}{\|x - y\|} = 0.$$

By Theorem 6, $S(y - x)$ and $T(y - x)$ must agree for all y such that $(1 - a)x + ay$ is in $G \cap H$ whenever $0 < a \leq 1$. Hence T and S are extensions of each other and are unique by Theorem 7. Let

$$V(y - x) = \begin{cases} S(y - x), & y \text{ in } G \\ T(y - x), & y \text{ in } H. \end{cases}$$

Then

$$f(y) - f(x) = V(y - x) + \begin{cases} R(x, y), & y \text{ in } G \\ P(x, y), & y \text{ in } H - G \end{cases}$$

for all y such that $(1 - a)x + ay$ is in $G \cup H$. Therefore f has a differential at x from the $G \cup H$ side.

THEOREM 10. *Let U be an open set containing x . Then f has a differential at x if there exist in U a finite number of open sets $\{G_i\}_{i=1}^n$ such that $G_i \cap G_{i+1} \neq \emptyset$ (Take $G_{n+1} = G_1$), x is on the boundary of each G_i and $G_i \cap G_{i+1}$, $U = \{x\} \cup \bigcup_{i=1}^n G_i$ and f has a differential from the G_i side for each i .*

This is a corollary to Theorems 7 and 9.

REFERENCES

1. Casper Goffman, *Calculus of Several Variables*, Harper & Row, Publisher, New York, 1965.
2. I. M. Gilfand and S. V. Fomin, *Calculus of Variations*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963.

Received December 10, 1969.

EAST TEXAS STATE UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON
Stanford University
Stanford, California 94305

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

RICHARD PIERCE
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLE

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

* * *

AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
TRW SYSTEMS
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

John D. Arrison and Michael Rich, <i>On nearly commutative degree one algebras</i>	533
Bruce Alan Barnes, <i>Algebras with minimal left ideals which are Hilbert spaces</i>	537
Robert F. Brown, <i>An elementary proof of the uniqueness of the fixed point index</i>	549
Ronn L. Carpenter, <i>Principal ideals in F-algebras</i>	559
Chen Chung Chang and Yiannis (John) Nicolas Moschovakis, <i>The Suslin-Kleene theorem for V_κ with cofinality $(\kappa) = \omega$</i>	565
Theodore Seio Chihara, <i>The derived set of the spectrum of a distribution function</i>	571
Tae Geun Cho, <i>On the Choquet boundary for a nonclosed subspace of $C(S)$</i>	575
Richard Brian Darst, <i>The Lebesgue decomposition, Radon-Nikodym derivative, conditional expectation, and martingale convergence for lattices of sets</i>	581
David E. Fields, <i>Dimension theory in power series rings</i>	601
Michael Lawrence Fredman, <i>Congruence formulas obtained by counting irreducibles</i>	613
John Eric Gilbert, <i>On the ideal structure of some algebras of analytic functions</i>	625
G. Goss and Giovanni Viglino, <i>Some topological properties weaker than compactness</i>	635
George Grätzer and J. Sichler, <i>On the endomorphism semigroup (and category) of bounded lattices</i>	639
R. C. Lacher, <i>Cell-like mappings. II</i>	649
Shiva Narain Lal, <i>On a theorem of M. Izumi and S. Izumi</i>	661
Howard Barrow Lambert, <i>Differential mappings on a vector space</i>	669
Richard G. Levin and Takayuki Tamura, <i>Notes on commutative power joined semigroups</i>	673
Robert Edward Lewand and Kevin Mor McCrimmon, <i>Macdonald's theorem for quadratic Jordan algebras</i>	681
J. A. Marti, <i>On some types of completeness in topological vector spaces</i>	707
Walter J. Meyer, <i>Characterization of the Steiner point</i>	717
Saad H. Mohamed, <i>Rings whose homomorphic images are q-rings</i>	727
Thomas V. O'Brien and William Lawrence Reddy, <i>Each compact orientable surface of positive genus admits an expansive homeomorphism</i>	737
Robert James Plemmons and M. T. West, <i>On the semigroup of binary relations</i>	743
Calvin R. Putnam, <i>Unbounded inverses of hyponormal operators</i>	755
William T. Reid, <i>Some remarks on special disconjugacy criteria for differential systems</i>	763
C. Ambrose Rogers, <i>The convex generation of convex Borel sets in euclidean space</i>	773
S. Saran, <i>A general theorem for bilinear generating functions</i>	783
S. W. Smith, <i>Cone relationships of biorthogonal systems</i>	787
Wolmer Vasconcelos, <i>On commutative endomorphism rings</i>	795
Vernon Emil Zander, <i>Products of finitely additive set functions from Orlicz spaces</i>	799
G. Sankaranarayanan and C. Suyambulingom, <i>Correction to: "Some renewal theorems concerning a sequence of correlated random variables"</i>	805
Joseph Zaks, <i>Correction to: "Trivially extending decompositions of E^n"</i>	805
Dong Hoon Lee, <i>Correction to: "The adjoint group of Lie groups"</i>	805
James Edward Ward, <i>Correction to: "Two-groups and Jordan algebras"</i>	806