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NOTES ON COMMUTATIVE POWER JOINED SEMIGROUPS

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NOTES ON COMMUTATIVE POWER JOINED SEMIGROUPS

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Let S be a commutative semigroup. The main theorem in this paper is to prove that the following two conditions are equivalent: (1) For all $a, b \in S$ there are positive integers m, n such that $a^m = b^n$. (2) For all $a, b \in S$, $a^l = a^m b^n$, $b^r = b^s a^t$ for some l, m, n, r, s, t. As a consequence of the theorem, the authors prove that a commutative archimedean semigroup S without idempotent is power joined if and only if the structure group of S is a torsion group.

Let S be a commutative archimedean semigroup without idempotent. Consider the following question: "Under what condition on the structure group (defined below) of S will S be power joined?" Levin proved in [4] that if S is finitely generated, equivalently if the structure group of S is finite, then S is power joined. Also he obtained a necessary and sufficient condition for S to be power joined. The following is Theorem 2 in [4]:

THEOREM 1. *Let S be a commutative, archimedean semigroup without idempotent. Let $G_a = S/\rho_a$ be the structure group of S determined by a. Then S is power joined if and only if G_a is periodic and the congruence class containing a modulo ρ_a is power joined.*

If we assume that S is additionally cancellative, that is, S is an $\mathfrak{N}$-semigroup, then the answer is simple. The following is due to Chrislock [1, 2].

THEOREM 2. *An $\mathfrak{N}$-semigroup S is power joined if and only if G_a is periodic for some $a \in S$, equivalently for all $a \in S$.*

Naturally the following question is raised: Can Theorem 1 be improved such that Theorem 2 is extended to S in Theorem 1? The question is affirmative. In this paper we study the problem for more general case, i.e., for commutative archimedean semigroups. The main theorem of this paper asserts that a commutative semigroup S is power joined if and only if it is archimedean and its group homomorphic images are periodic. As a corollary we can answer the above question.

Semigroups are assumed to be commutative throughout this paper.

DEFINITION 1. A semigroup S is called power joined if and only if for all $a, b \in S$, there are positive integers n, m such that

$$a^n = b^m .$$

DEFINITION 2. A semigroup S is called archimedean if and only if for all $a, b \in S$, there exist $u, v \in S$ and positive integers n, m such that

$$a^n = bu \quad \text{and} \quad b^m = av .$$

DEFINITION 3. Let S be an archimedean semigroup without idempotent. We define a congruence ρ_b on S for fixed $b \in S$ as follows. We define $x\rho_b y$ if and only if there are positive integers n and m such that

$$b^n x = b^m y .$$

REMARK. More information on commutative, archimedean semigroups without idempotent can be found in [1], [6] and [7]. In particular a proof that ρ_b (as defined above) is a congruence relation and that $S/\rho_b = G_b$ is a group can be found in [7]. S/ρ_b is called the structure group of S determined by b. Also notice $xy \neq y$ for all $x, y \in S$.

THEOREM 3. *The following statements are equivalent.*

(3.1) *The semigroup S is power joined.*

(3.2) *The semigroup S is archimedean and its group homomorphic images are periodic.*

(3.3) *The semigroup S satisfies the conditions: for all pairs $a, b \in S$, there are positive integers l, m, n, s, t, p such that*

$$a^l = a^m b^n \quad \textit{and} \quad b^s = b^t a^p .$$

Proof. We will prove: (3.1) $\Rightarrow$ (3.2) $\Rightarrow$ (3.3) $\Rightarrow$ (3.1). Let S be a power joined semigroup. It is trivial to show that S is archimedean. Let G be a group homomorphic image of S with $\varphi: S \to G$ the homomorphism. We will show that G is a periodic group. Let $a \in G$ and let e be the identity of G. There exist $x, y \in S$ such that $\varphi(x) = a$, $\varphi(y) = e$. Since S is power joined, there exist positive integers n, m such that $x^n = y^m$. Then

$$a^n = [\varphi(x)]^n = \varphi(x^n) = \varphi(y^m) = [\varphi(y)]^m = e^m = e .$$

We see that G is periodic and this completes the proof that (3.1) $\Rightarrow$ (3.2).

We next prove that (3.2) $\Rightarrow$ (3.3). Let S be an archimedean semigroup whose group homomorphic images are periodic.

Case 1. Assume that S has an idempotent e. Then the set Se is a group and is the homomorphic image of S (see [3] or [5]). Let $a, b \in S$. Then ae and be are elements of Se. Since Se is a periodic group with e as its identity element, there exist positive integers n and m such that

$$(ae)^n = e \quad \text{and} \quad (be)^m = e .$$

That is,

$$a^n e = e = b^m e . \tag{1}$$

Since S is archimedean, there exist positive integers k and t and $u, v \in S$ such that

$$a^k = ev \quad \text{and} \quad b^t = eu . \tag{2}$$

From equations (1) and (2) we derive

$$\begin{aligned} a^n eu &= b^m eu, \\ \text{or } a^n b^t &= b^m b^t, \\ \text{or } a^n b^t &= b^r \quad \text{where} \quad \mathrm{r} = m + t . \end{aligned}$$

Similarly, we derive $a^l = a^k b^m$ for some positive integers l, k and m.

Case 2. Assume that S does not have an idempotent. Let $a, b \in S$. Consider the congruence ρ_a of Definition 3. Then S/ρ_a is a group homomorphic image of S and, therefore, is a periodic group. Also

$$S = \bigcup_{\lambda \in S/\rho_a} S_\lambda$$

and $a \in S_\varepsilon$, where ε is the identity of S/ρ_a. There is $\lambda \in S/\rho_a$ such that $b \in S_\lambda$. There exists a positive integer k such that $\lambda^k = \varepsilon$. Thus,

$$b^k \in S_{\lambda^k} = S_\varepsilon .$$

That is, a and b^k are ρ_a related. By definition of ρ_a, there are positive integers n and m such that

$$\begin{aligned} a^n a &= a^m b^k , \\ \text{or } a^l &= a^m b^k , \quad \text{where} \quad l = n + 1 . \end{aligned}$$

Similarly, we can derive the equation

$$b^s = b^t a^p .$$

The proof that $(3.2) \Rightarrow (3.3)$ is now complete.

We now prove that $(3.3) \Rightarrow (3.1)$.

Case 1. Assume that S has an idempotent e. Let $a \in S$. Then there are positive integers l, m, n, s, t and p such that

$$(3) \qquad e^l = e^m a^n \quad \text{and} \quad a^s = a^t e^p ,$$

$$(4) \qquad \text{or } e = ea^n \quad \text{and} \quad a^s = a^t e .$$

Using the equations of (4) we derive

$$e = e^t = (ea^n)^t = e^t(a^t)^n = (ea^t)^n = (a^s)^n .$$

Thus, we have $e = a^r$ for a positive integer r.
It is now obvious that if $a, b \in S$, there are positive integers u and v such that $a^u = b^v$. Therefore S is power joined.

Case 2. Assume that S has no idempotent. Again we have for any pair $a, b \in S$, positive integers l, m, n, s, t and p such that

$$(5) \qquad a^l = a^m b^n \quad \text{and} \quad b^s = b^t a^p .$$

We will prove that there are positive integers l' and n' such that $a^{l'} = a^m b^{n'}$, and $n'p \geqq mt$. Since S does not have an idempotent, $l > m$ in (5). Then

$$a^{2l-m} = a^{l-m}a^l = a^{l-m}a^m b^n = a^l b^n = (a^m b^n)b^n = a^m b^{2n} .$$

Now assume that for some integer $k \geqq 1$, we have

$$a^{kl-(k-1)m} = a^m b^{kn} .$$

We will prove that

$$a^{(k+1)l-km} = a^m b^{(k+1)n} .$$

Now we have

$$\begin{aligned} a^{(k+1)l-km} &= a^{kl-km}a^l = a^{kl-km}(a^m b^n) = (a^{kl-km}a^m)b^n \\ &= a^{kl-(k-1)m}b^n = (a^m b^{kn})b^n = a^m b^{(k+1)n} . \end{aligned}$$

Thus, by induction we have the relation: for every $k \geqq 1$

$$a^{kl-(k-1)m} = a^m b^{kn} .$$

Now choose k such that $knp \geqq mt$. Set $n' = kn$, $l' = kl - (k-1)m$. We replace the equations of (5) by the equations

$$(6) \qquad a^{l'} = a^m b^{n'} \quad \text{and} \quad b^s = b^t a^p .$$

From (6) we derive

$$\begin{aligned} a^{l'tp} &= (b^{n'})^{tp}(a^m)^{tp} = (b^t)^{n'p}(a^p)^{mt}\ , \\ \text{or } a^{l'tp} &= (b^t)^{mt+(n'p-mt)}(a^p)^{mt} \\ &= (b^t)^{mt}(b^t)^{n'p-mt}(a^p)^{mt} \\ &= (b^ta^p)^{mt}(b^t)^{n'p-mt} \\ &= (b^s)^{mt}(b^t)^{n'p-mt}\ . \end{aligned}$$

Set $u = l'tp$ and $v = smt + t(n'p - mt)$. We see that we have derived the equation $a^u = b^v$. Therefore S is power joined. This concludes the proof that (3.3) $\Rightarrow$ (3.1).

REMARK. Each of (3.1), (3.2) and (3.3) is equivalent to one of (3.4) and (3.5) below:

(3.4) The semigroup S satisfies the following condition: there is an element a_0 of S such that for all $b \in S$ there are positive integers l, m, n, s, t, p satisfying

$$a_0^l = a_0^m b^n \text{ and } b^s = b^t a_0^p\ .$$

(3.5) The semigroup S satisfies the condition: for all pairs $a, b \in S$ there are positive integers l, m, s, t such that

$$a^l = (ab)^m \quad\text{and}\quad b^s = (ba)^t\ .$$

Proof. We define a relation τ on S as follows:
$a\tau b$ if and only if $a^l = a^m b^n$ and $b^s = b^t a^p$ for some l, m, n, s, t, p. Then τ is an equivalence on S. Reflexivity and symmetry are obvious. Transitivity is proved as follows: suppose $a^l = a^m b^n$ and $b^k = b^q c^v$. First we have

$$a^{lk} = a^{mk}b^{nk} = a^{mk}b^{nq}c^{nv}$$

and then

$$a^{l'} = a^{m'}(a^{mq}b^{nq})c^{nv} = a^{m'+lq}c^{nv}$$

where

$$l' = lk,\ m' = mk - mq \quad\text{if}\quad k \geqq q$$

$$l' = lk = mq - mk,\ m' = 0 \quad\text{if}\quad k < q\ .$$

Therefore (3.4) $\Rightarrow$ (3.3) is obtained as an immediate consequence; (3.3) $\rightarrow$ (3.4), (3.1) $\rightarrow$ (3.5) and (3.5) $\rightarrow$ (3.3) are obvious.

If S is a nil-semigroup, i.e., a semigroup in which some power of every element is zero, Theorem 3 is trivial since every nil-semigroup is power joined.

If S is an archimedean semigroup whose idempotent is not zero, then $G = Se$ is the kernel, i.e., the minimal ideal and the unique maximal subgroup. Then we have

COROLLARY 5. *S is power joined if and only if the kernel G is periodic.*

The essense of Theorem 3 is in the case where S is an archimedean semigroup without idempotent.

THEOREM 6. *An archimedean semigroup without idempotent is power joined if and only if the structure group $G_a = S/\rho_a$ of S is periodic for some $a \in S$, equivalently for all $a \in S$.*

Proof. Let S be an archimedean semigroup without idempotent. Then the statement (3.3) is equivalent to:

$$S/\rho_a \text{ is periodic for all } a \in S .$$

(3.4) is equivalent to:

$$S/\rho_a \text{ is periodic for some } a \in S .$$

The first statement is obvious. To see the second we will prove the following:

If S/ρ_{a_0} is periodic, then for all $b \in S$ there are positive integers l, m, n, s, t, p such that

$$a_0^l = a_0^m b^n , \quad b^s = b^t a_0^p . \tag{7}$$

The first of (7) is immediately obtained. Since S is archimedean there is a positive integer k and an element $c \in S$ such that

$$b^k = a_0 c$$

which implies $b^{kl} = a_0^l c^l$. Since S has no idempotent, $l > m$ in the first of (7). Now we have

$$b^{kl} a_0^{l-m} = a_0^{l-m} a_0^l c^l = a_0^{l-m} a_0^m b^n c^l = b^n a_0^l c^l = b^n b^{kl} = b^{n+kl} .$$

This completes the proof.

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