ON COMMUTATIVE ENDMORPHISM RINGS

WOLMER V. VASCONCELOS

This note deals with a finitely generated faithful module $E$ over a commutative semi-prime noetherian ring $R$, with commutative endomorphism ring $\text{Hom}_R(E, E) = \Omega(E)$. It is shown that $E$ is identifiable to an ideal of $R$ whenever $\Omega(E)$ lacks nilpotent elements; a class of examples with $\Omega(E)$ commutative but not semi-prime is discussed.

1. Main result. Throughout $R$ will denote a commutative noetherian ring and modules will be finitely generated. In order to use the full measure of the ring, we shall consider mostly faithful modules. As for notation, unadorned $\otimes$ and $\text{Hom}$ are taken over the base ring.

In case $R$ is semi-prime (meaning here: no nilpotent elements distinct from 0) we recall that its total ring of quotients $K$ is semi-simple, and thus a direct sum of fields $K = \bigoplus \sum K_i, 1 \leq i \leq n$. Any ideal $I$ of $R$ has the property that $\text{Hom}(I, I)$ is commutative and semi-prime: for if $S$ denotes the set of regular elements of $R$,

$$\text{Hom}(I, I) \cong \text{Hom}(I, I)_S = \text{Hom}_{R_S}(I_S, I_S).$$

But this last is a subring of $K$. The content of the next theorem is precisely a converse to this observation.

**Theorem 1.1.** Let $E$ be a finitely generated faithful module over the semi-prime ring $R$. Then, if $\text{Hom}(E, E)$ is commutative and semi-prime, $E$ is isomorphic to an ideal of $R$.

**Proof.** Denote by $T$ the torsion submodule of $E$, i.e., let $T$ be the set of elements of $E$ annihilated by a regular element of $R$. If $T = 0$, then $\text{Hom}(E, E) \subseteq \text{Hom}(E, E)_S = \text{Hom}_{R_S}(E_S, E_S)$; using the decomposition of $R_S = K$ as a direct sum of fields,

$$\text{Hom}_K(E \otimes K, E \otimes K) = \bigoplus \sum \text{Hom}_{K_i}(E \otimes K_i, E \otimes K_i).$$

Since $\text{Hom}_K(E \otimes K, E \otimes K)$ is commutative, we must have, for each $i$, $E \otimes K_i = 0$ or isomorphic to $K_i$. This allows identification of $E_S$ to a submodule of $K$ and consequently of $E$ to an ideal of $R$, since $E$ is finitely generated.

Assume then, by way of contradiction, $T \neq 0$ and consider the exact sequence

$$0 \longrightarrow T \longrightarrow E \xrightarrow{\pi} F \longrightarrow 0.$$
It yields
\[(1) \quad 0 \longrightarrow \text{Hom}(E, T) \longrightarrow \text{Hom}(E, E) \xrightarrow{\pi_*} \text{Hom}(F, F)\]
as $T$ is a characteristic submodule of $E$; observe also that $\pi_*$ is an $R$-algebra homomorphism. Let $P$ be a prime ideal of $R$ minimal over the annihilator $J$ of $T$. Then $T_P \neq 0$ and can be viewed as a $R_P/J_P$-module; by the choice of $P$ this last ring is artinian [2; Chap. IV, p. 147] and $T_P$ has finite length as an $R_P$-module. On the other hand, localization at $P$ does not introduce nilpotent elements in either $R_P$ or $\Omega = \text{Hom}_{R_P}(E_P, E_P) (= \text{Hom}(E, E)_P)$. Let $I$ denote $\text{Hom}(E, T)_P$; since $T_P$ has finite length, $I$ also has finite length and the sequence
\[I \supseteq I^2 \supseteq \cdots \supseteq I^n \supseteq \cdots\]
must eventually become stationary. Say $I^n = I^{n+1}$ for some $n$; by [2; Chap. I, p. 83] $I^n$ is generated by an idempotent $e$ of $\Omega$. Actually, $I = \Omega e$, for $\Omega$ lacks nilpotent elements and $(I(1 - e))^n = 0$. The idempotent $e$ induces the direct sum decomposition $M = eM \oplus (1 - e)M$, with $M = E_P$. Thus
\[
\Omega = \begin{bmatrix}
\text{Hom}_{R_P}(eM, eM) & \text{Hom}_{R_P}(eM, (1 - e)M) \\
\text{Hom}_{R_P}((1 - e)M, eM) & \text{Hom}_{R_P}(1 - e)eM, (1 - e)M)
\end{bmatrix}.
\]
Since is semi-prime, $\text{Hom}_{R_P}((1 - e)M, eM) = 0$. Observe that $eM \subseteq T_P$ and thus $(1 - e)M$ is a faithful $R_P$-module. To conclude we need the

**Lemma 1.2.** If $A$ is a finitely generated faithful module over the commutative ring $R$, then every simple $R$-module is a homomorphic image of $A$.

**Proof.** Just note that for each maximal ideal $P$, $PA \neq A$ [2; Chap. I, p. 83 again].

Returning to the proof of the theorem, observe that $eM$ must contain a simple submodule, unless $e = 0$. Then $I = 0$ and again by the lemma, $T_P = 0$.

2. **Examples.** In order to construct examples of faithful modules $E$ with commutative $\Omega(E)$ but not isomorphic to ideals, by the preceding it will be necessary to waive the requirement that $\Omega(E)$ be semi-prime.

We shall need a special case of the following result, which has various amusing consequences. Let $R$, as before, be a commutative noetherian ring and $E$ a finitely generated $R$-module. Assume that $E$ is faithful; then $R$ can be viewed as a subring of the center $C$ of
Hom \((E, E)\). \(E\) is said to be balanced if \(R = C\). A mild homological hypothesis will imply that torsion-less modules (i.e., submodules of direct products of \(R\)) are, very often, balanced.

To state this condition we recall the notion of grade of an ideal \(I\): it is the smallest integer \(n\) such that \(I\) contains no \(R\)-sequence of length \(n + 1\) [3].

**Proposition 2.1.** Let \(E\) be a finitely generated, torsion-less, faithful \(R\)-module. Then if \(E_p\) is \(R_p\)-free for each prime ideal \(P\) with grade \(PR_p \leq 1\) (as \(R_p\) ideal), then \(E\) is balanced.

**Proof.** Consider the exact sequence
\[(2) 0 \rightarrow R \rightarrow C \rightarrow L \rightarrow 0\]
induced by the inclusion of \(R\) into \(C\). With the present finiteness conditions, "\(C\) localizes", i.e., for each prime ideal \(P\), \(C_p\) is the center of \(\text{Hom}(E, E)_p = \text{Hom}_{E_p}(E_p, E_p)\). Thus for each prime ideal \(P\), with grade \(PR_p \leq 1\), \(L_p = 0\) as \(E_p\) is then \(R_p\)-free. Let \(J\) be the annihilator of \(L\). The preceding says that \(J\) has grade \(\geq 2\). Applying \(\text{Hom}(R/J, -)\) to the sequence (2) we get
\[0 \rightarrow \text{Hom}(R/J, R) \rightarrow \text{Hom}(R/J, C) \rightarrow \text{Hom}(R/J, L) \rightarrow \text{Ext}(R/J, R) \rightarrow 0\] Since \(C\) is torsion-free, \(\text{Hom}(R/J, C) = 0\), while by [3]
\[\text{Ext}(R/J, R) = 0\].

Thus \(\text{Hom}(R/J, L) = 0\), which evidently leads to \(L = 0\).

The following are cases where the proposition applies:
\[(i) \text{ } I \text{ ideal of } R \text{ of grade } 2; \text{ then } \text{Hom}(I, I) = R.\]
\[(ii) \text{ Serre's normality criterion } [4; III-13].\]
\[(iii) \text{ } E \text{ is a finitely generated, torsion-less, faithful } R\text{-module of finite projective dimension; then } E \text{ is balanced.}\]
\[(iv) \text{ Commutative noetherian rings of finite global dimension are integrally closed.}\]

**Example 2.3.** Let \(P\) be a maximal ideal of a commutative domain \(R\), such that grade \(P \geq 2\). Then \(\text{Ext}(P, R/P) \neq 0\), as otherwise \(R_p\) would be a discrete valuation ring, which is not the case [1]. Let \(E\) be a nontrivial extension of \(P\) by \(R/P\), that is, consider a non-splitting sequence
\[(3) 0 \rightarrow R/P \rightarrow E \rightarrow P \rightarrow 0\].
The exact sequence corresponding to (1) is
\[ 0 \to \text{Hom}(E, R/P) \to \text{Hom}(E, E) \xrightarrow{\pi_*} \text{Hom}(P, P) . \]

By (i) above, \( \text{Hom}(P, P) = R \) and \( \pi_* \) is actually a surjection with the endomorphisms of \( E \) induced by multiplication by elements of \( R \) mapping injectively onto \( \text{Hom}(P, P) \). Thus
\[ \text{Hom}(E, E) = R + I \]
with \( I = \text{Hom}(E, R/P) \). By Lemma 1.2 we know that \( I \neq 0 \). \( \text{Hom}(E, E) \) will be commutative if \( I^2 = 0 \). If \( I^2 \neq 0 \), there would be \( f, g \in I \), with \( f \circ g \neq 0 \). This however says that \( f: E \to R/P \) is non-trivial on \( R/P \). We could then modify \( f \) by multiplication by an element in \( R - P \), and thus accomplish a splitting of (3), against the assumption.

In the example above the projective dimension of \( E \) is at least 2; it would be interesting to find an example with similar properties but lower projective dimension (=1).

If \( R \) is no longer noetherian, then Theorem 1.1 looks still plausible if \( E \) is assumed of finite presentation.

As a final remark, in a lighter vein, it should be of interest to determine all commutative rings \( R \) in which endomorphism rings of ideals are always commutative. In the noetherian case, we conjecture that the total ring of quotients of \( R \) is quasi-Frobenius.

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REFERENCES


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