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ON COMMUTATIVE ENDOMORPHISM RINGS

WOLMER VASCONCELOS

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This note deals with a finitely generated faithful module E over a commutative semi-prime noetherian ring R , with commutative endomorphism ring $\text{Hom}_R(E, E) = \Omega(E)$. It is shown that E is identifiable to an ideal of R whenever $\Omega(E)$ lacks nilpotent elements; a class of examples with $\Omega(E)$ commutative but not semi-prime is discussed.

1. **Main result.** Throughout R will denote a commutative noetherian ring and modules will be finitely generated. In order to use the full measure of the ring, we shall consider mostly faithful modules. As for notation, unadorned \otimes and Hom are taken over the base ring.

In case R is semi-prime (meaning here: no nilpotent elements distinct from 0) we recall that its total ring of quotients K is semi-simple, and thus a direct sum of fields $K = \bigoplus \sum K_i, 1 \leq i \leq n$. Any ideal I of R has the property that $\text{Hom}(I, I)$ is commutative and semi-prime: for if S denotes the set of regular elements of R ,

$$\text{Hom}(I, I) \cong \text{Hom}(I, I)_S = \text{Hom}_{R_S}(I_S, I_S).$$

But this last is a subring of K . The content of the next theorem is precisely a converse to this observation.

THEOREM 1.1. *Let E be a finitely generated faithful module over the semi-prime ring R . Then, if $\text{Hom}(E, E)$ is commutative and semi-prime, E is isomorphic to an ideal of R .*

Proof. Denote by T the torsion submodule of E , i.e., let T be the set of elements of E annihilated by a regular element of R . If $T = 0$, then $\text{Hom}(E, E) \cong \text{Hom}(E, E)_S = \text{Hom}_{R_S}(E_S, E_S)$; using the decomposition of $R_S = K$ as a direct sum of fields,

$$\text{Hom}_K(E \otimes K, E \otimes K) = \bigoplus \sum \text{Hom}_{K_i}(E \otimes K_i, E \otimes K_i).$$

Since $\text{Hom}_K(E \otimes K, E \otimes K)$ is commutative, we must have, for each i , $E \otimes K_i = 0$ or isomorphic to K_i . This allows identification of E_S to a submodule of K and consequently of E to an ideal of R , since E is finitely generated.

Assume then, by way of contradiction, $T \neq 0$ and consider the exact sequence

$$0 \longrightarrow T \longrightarrow E \xrightarrow{\pi} F \longrightarrow 0.$$

It yields

$$(1) \quad 0 \text{ --- Hom}(E, T) \text{ --- Hom}(E, E) \xrightarrow{\pi_*} \text{Hom}(F, F)$$

as T is a characteristic submodule of E ; observe also that π_* is an R -algebra homomorphism. Let P be a prime ideal of R minimal over the annihilator J of T . Then $T_P \neq 0$ and can be viewed as a R_P/J_P -module; by the choice of P this last ring is artinian [2; Chap. IV, p. 147] and T_P has finite length as an R_P -module. On the other hand, localization at P does not introduce nilpotent elements in either R_P or $\Omega = \text{Hom}_{R_P}(E_P, E_P) (= \text{Hom}(E, E)_P)$. Let I denote $\text{Hom}(E, T)_P$; since T_P has finite length, I also has finite length and the sequence

$$I \supseteq I^2 \supseteq \dots \supseteq I^n \supseteq \dots$$

must eventually become stationary. Say $I^n = I^{2n}$ for some n ; by [2; Chap. I, p. 83] I^n is generated by an idempotent e of Ω . Actually, $I = \Omega e$, for Ω lacks nilpotent elements and $(I(1 - e))^n = 0$. The idempotent e induces the direct sum decomposition $M = eM \oplus (1 - e)M$, with $M = E_P$. Thus

$$\Omega = \begin{bmatrix} \text{Hom}_{R_P}(eM, eM) & \text{Hom}_{R_P}(eM, (1 - e)M) \\ \text{Hom}_{R_P}((1 - e)M, eM) & \text{Hom}_{R_P}((1 - e)M, (1 - e)M) \end{bmatrix}.$$

Since Ω is semi-prime, $\text{Hom}_{R_P}((1 - e)M, eM) = 0$. Observe that $eM \subseteq T_P$ and thus $(1 - e)M$ is a faithful R_P -module. To conclude we need the

LEMMA 1.2. *If A is a finitely generated faithful module over the commutative ring R , then every simple R -module is a homomorphic image of A .*

Proof. Just note that for each maximal ideal P , $PA \neq A$ [2; Chap. I, p. 83 again].

Returning to the proof of the theorem, observe that eM must contain a simple submodule, unless $e = 0$. Then $I = 0$ and again by the lemma, $T_P = 0$.

2. Examples. In order to construct examples of faithful modules E with commutative $\Omega(E)$ but not isomorphic to ideals, by the preceding it will be necessary to waive the requirement that $\Omega(E)$ be semi-prime.

We shall need a special case of the following result, which has various amusing consequences. Let R , as before, be a commutative noetherian ring and E a finitely generated R -module. Assume that E is faithful; then R can be viewed as a subring of the center C of

$\text{Hom}(E, E)$. E is said to be *balanced* if $R = C$. A mild homological hypothesis will imply that torsion-less modules (i.e., submodules of direct products of R) are, very often, balanced.

To state this condition we recall the notion of grade of an ideal I : it is the smallest integer n such that I contains no R -sequence of length $n + 1$ [3].

PROPOSITION 2.1. *Let E be a finitely generated, torsion-less, faithful R -module. Then if E_P is R_P -free for each prime ideal P with grade $PR_P \leq 1$ (as R_P ideal), then E is balanced.*

Proof. Consider the exact sequence

$$(2) \quad 0 \longrightarrow R \longrightarrow C \longrightarrow L \longrightarrow 0$$

induced by the inclusion of R into C . With the present finiteness conditions, " C localizes", i.e., for each prime ideal P , C_P is the center of $\text{Hom}(E, E)_P = \text{Hom}_{R_P}(E_P, E_P)$. Thus for each prime ideal P , with grade $PR_P \leq 1$, $L_P = 0$ as E_P is then R_P -free. Let J be the annihilator of L . The preceding says that J has grade ≥ 2 . Applying $\text{Hom}(R/J, -)$ to the sequence (2) we get

$$\begin{aligned} 0 \longrightarrow \text{Hom}(R/J, R) &\longrightarrow \text{Hom}(R/J, C) \\ &\longrightarrow \text{Hom}(R/J, L) \longrightarrow \text{Ext}(R/J, R) . \end{aligned}$$

Since C is torsion-free, $\text{Hom}(R/J, C) = 0$, while by [3]

$$\text{Ext}(R/J, R) = 0 .$$

Thus $\text{Hom}(R/J, L) = 0$, which evidently leads to $L = 0$.

The following are cases where the proposition applies:

- (i) I ideal of R of grade 2; then $\text{Hom}(I, I) = R$.
- (ii) Serre's normality criterion [4; III-13].
- (iii) E is a finitely generated, torsion-less, faithful R -module of finite projective dimension; then E is balanced.
- (iv) Commutative noetherian rings of finite global dimension are integrally closed.

EXAMPLE 2.3. Let P be a maximal ideal of a commutative domain R , such that grade $P \geq 2$. Then $\text{Ext}(P, R/P) \neq 0$, as otherwise R_P would be a discrete valuation ring, which is not the case [1]. Let E be a nontrivial extension of P by R/P , that is, consider a non-splitting sequence

$$(3) \quad 0 \longrightarrow R/P \longrightarrow E \xrightarrow{\pi} P \longrightarrow 0 .$$

The exact sequence corresponding to (1) is

$$0 \longrightarrow \text{Hom}(E, R/P) \longrightarrow \text{Hom}(E, E) \xrightarrow{\pi_*} \text{Hom}(P, P).$$

By (i) above, $\text{Hom}(P, P) = R$ and π_* is actually a surjection with the endomorphisms of E induced by multiplication by elements of R mapping injectively onto $\text{Hom}(P, P)$. Thus

$$\text{Hom}(E, E) = R + I$$

with $I = \text{Hom}(E, R/P)$. By Lemma 1.2 we know that $I \neq 0$. $\text{Hom}(E, E)$ will be commutative if $I^2 = 0$. If $I^2 \neq 0$, there would be $f, g \in I$, with $f \circ g \neq 0$. This however says that $f: E \rightarrow R/P$ is non-trivial on R/P . We could then modify f by multiplication by an element in $R - P$, and thus accomplish a splitting of (3), against the assumption.

In the example above the projective dimension of E is at least 2; it would be interesting to find an example with similar properties but lower projective dimension (=1).

If R is no longer noetherian, then Theorem 1.1 looks still plausible if E is assumed of finite presentation.

As a final remark, in a lighter vein, it should be of interest to determine all commutative rings R in which endomorphism rings of ideals are always commutative. In the noetherian case, we conjecture that the total ring of quotients of R is quasi-Frobenius.

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