PRODUCTS OF FINITELY ADDITIVE SET FUNCTIONS FROM ORLICZ SPACES

Vernon Emil Zander
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This note establishes two results on products of finitely additive vector-valued set functions from Orlicz spaces. A triple \((\Omega, \Sigma, \mu)\) is called a charge space if \(\Sigma\) is a ring of subsets of a set \(\Omega\) and the charge \(\mu\) is a finitely additive, non-negative, finite-valued function with domain \(\Sigma\).

Theorem. For \((\Omega_i, \Sigma_i, \mu_i)\) \((i = 1, \ldots, n)\) a family of charge spaces and \((\Omega, \Sigma, \mu)\) the corresponding product charge space, for \(u\) an \(n\)-linear continuous operator from the product of the Banach spaces \(Z_i, \ldots, Z_n\) into a Banach space \(W\), the function \(v\) defined by \(v(A) = u(v_1(A), \ldots, v_n(A))\) for \(A \in \Sigma\) and \(v_i\) from the Orlicz space \(A^\varphi(\Omega_i, \Sigma_i, \mu_i, Z_i)\) belongs to the Orlicz space \(A^\varphi(\Omega, \Sigma, \mu, W)\).

For the infinite product case the following result holds:

Theorem. For \((\Omega_t, \Sigma_t, \mu_t)\) \((t \in T)\) a family of probability charge spaces and \((\Omega, \Sigma, \mu)\) the product probability charge space, for \(u\) an infinitely linear bounded operator on the multiplicative product space \(P_t(A^\varphi(\Omega_t, \Sigma_t, \mu_t, Z_t), v^t)\) the function \(v^0\) defined by \(v^0(A) = u(v(A))\) for \(A \in \Sigma\) belongs to the Orlicz space \(A^\varphi(\Omega, \Sigma, \mu, W)\).

These results allow one to develop an integral determined by a product of charges from Orlicz spaces.

In a recent paper by Uhl [5] the Orlicz space \(A^\varphi\) of vector-valued finitely additive set functions is investigated. The present paper presents results concerning finite and infinite products of finitely additive vector-valued set functions from the Orlicz spaces

\[ A^\varphi(\Omega_r, \Sigma_r, \mu_r, X_r) \]

for \(r\) ranging through an index set. The results for a finite product of set functions resemble a generalization of a result by Bogdanowicz [2, Th. 1] for the \(L^r\)-spaces of Lebesgue-Bochner summable functions; and the results for an infinite product of set functions resemble a generalization of a result for the Lebesgue space \(L\), by Bogdanowicz and Zander [3, Proposition 5], but the techniques used for the results in the present paper are different.

We shall assume throughout that \(\Phi\) is a convex, nondecreasing function defined on the positive real line such that \(\Phi(0) = 0\) and \(\Phi\) is continuous except for at most one point, after which the function must be identically infinite. We shall also assume throughout that
the function $\Phi$ satisfies the following growth condition:

$$\Phi(xy) \leq M\Phi(x)\Phi(y)$$

for all $x, y \geq 0$, where $M$ is a positive constant. (This growth condition is called the $A'$-condition (see [4, p. 29]).)

**Remark 1.** Each of the functions $f(x) = x^q$ with $q \geq 1$ and

$$g(x) = x^a(\log^+ x + 1)$$

with $\alpha > 1$ satisfies the $A'$-condition with constant 1, and each is a candidate for the function $\Phi$.

Let $\Omega$ be any set, $\Sigma$ a ring of subsets of $\Omega$, and $\mu$ a nonnegative, real-valued, finitely additive function with domain $\Sigma$. The associated triple $(\Omega, \Sigma, \mu)$ shall be called a charge space. If, further, $\Omega \in \Sigma$ and $\mu(\Omega) = 1$ then the triple $(\Omega, \Sigma, \mu)$ shall be called a probability charge space.

For $(\Omega, \Sigma, \mu)$ a charge space and $Z$ a Banach space denote by $A^\Phi(\mu, Z)$ the space of all finitely additive functions $v$ from the ring $\Sigma$ into the space $Z$ which satisfy the following conditions: (i) $v$ vanishes on the $\mu$-null sets, (ii) $I_\Phi(v/k) \leq 1$ for some positive $k$, where the function $I_\Phi$ is defined by the following expression:

$$I_\Phi(v) = \sup \Sigma \{\Phi(|v(A)|/\mu(A))/\mu(A): A \in \mathcal{F}\}$$

where the supremum is taken over all finite families $\mathcal{F}$ of disjoint sets from the ring $\Sigma$. This definition is recently given by Uhl [5, p. 24], and is due originally to Bochner [1, p. 778]. By Theorem 11 of [5] the functional $N_\Phi$ defined by

$$N_\Phi(v) = \inf \{k > 0: I_\Phi(v/k) \leq 1\}$$

is a norm under which the space $A^\Phi(\mu, Z)$ is Banach space.

We shall use the following notation: Let $T_i$ be a family of subsets from an abstract set $X_i$, for $i = 1, \ldots, n$. By $T_1 \times \cdots \times T_n$ we shall mean the family of all sets of the form $A_1 \times \cdots \times A_n$ where $A_i \in T_i$ for $i = 1, \ldots, n$.

For charge spaces $(\Omega_i, \Sigma_i, \mu_i)$ ($i = 1, \ldots, n$) define the triple $(\Omega, \Sigma_0, \mu)$ by $\Omega = \Omega_1 \times \cdots \times \Omega_n$, $\Sigma = \Sigma_1 \times \cdots \times \Sigma_n$, $\Sigma_0$ is the ring generated from the prering $\Sigma$, and $\mu(A) = \mu_1(A_1) \cdots \mu_n(A_n)$ for $A \in \Sigma$. It is easy to see that $(\Omega, \Sigma_0, \mu)$ is a charge space.

**Definition.** For $\mathcal{F}$ a finite family of disjoint sets from the prering $\Sigma$ and for $i$ between 1 and $n$ let $\mathcal{F}_i$ denote the family of all $i$-th coordinate sets $A_i$, where $A_1 \times \cdots \times A_i \times \cdots \times A_n \in \mathcal{F}$, and let
\( \mathcal{F}_i \) denote the refinement of \( \mathcal{F}_i \) consisting of the finite family of disjoint sets from \( \Sigma \), whose union is the same as that of all sets from \( \mathcal{F}_i \). That is, if \( \mathcal{F}_i = \{B_1, \ldots, B_n\} \), and if \( B' \cap B_j = \emptyset \) then \( \mathcal{F}_i \) is the set of all intersections \( C_1 \cap \cdots \cap C_m \) where each \( C_j \) is \( B_j \) or \( B'_j \) and at least one \( C_j \) is \( B_j \). By the \( \mathcal{F} \)-product we shall mean the corresponding finite family of disjoint sets \( \mathcal{F}_i \times \cdots \times \mathcal{F}_n \) from the pre-\( \mathcal{F} \). We shall denote this family by \( \mathcal{F} \).

**Lemma 1.** For \( v \in A^\sigma(\mu, Z) \) we have the relation

\[
I_\phi(v) = \sup \Sigma \{ \Phi(|v(A)|/\mu(A)) \mu(A) : A \in (\mathcal{F}) \}
\]

where the supremum is taken over all finite families \( \mathcal{F} \) of disjoint sets from \( \Sigma \).

**Proof.** Let \( \alpha \) denote the right side of the above relation. Then \( \alpha \) since an \( \mathcal{F} \)-product is a particular finite disjoint family from \( \Sigma \). Let \( B, C \) be disjoint sets from \( \Sigma \), and put \( A = B \cup C \). Then the relation \( \alpha \) follows from the positivity of values of the function \( \Phi \) and from the relation

\[
\Phi(|v(A)|/\mu(A)) \mu(A) \leq \Phi(|v(B)|/\mu(B)) \mu(B) + \Phi(|v(C)|/\mu(C)) \mu(C)
\]

which in turn follows from the convexity of the function \( \Phi \).

Let \( Z_i, W \) be Banach spaces and let \( u \) be an \( n \)-linear continuous operator from the product of the spaces \( Z_1, \ldots, Z_n \) into the space \( W \). Denote the norms in the above spaces by \( | \cdot | \).

**Theorem 2.** If \( v_i \in A^\sigma(\mu, Z_i) \) for \( i = 1, \ldots, n \) then \( v \in A^\sigma(\mu, W) \), where

\[
v(A_1 \times \cdots \times A_n) = u(v_1(A_1), \ldots, v_n(A_n)) \quad \text{for } A \in \Sigma.
\]

**Proof.** We shall establish that \( I_\phi(v/k) \leq 1 \) for some positive \( k \). Assume throughout that the \( \mathcal{F} \)-constant \( M \geq 1 \). Put

\[
w_j(\cdot) = |v_j(\cdot)|/N_\phi(v_j)
\]

and \( b = M^n |u| N_\phi(v_1) \cdots N_\phi(v_n) \), and let \( \mathcal{F} \) be a finite family of disjoint sets from \( \Sigma \). Then we have the following estimates:

\[
\Sigma \{ \Phi(|v(A)|/b\mu(A)) \mu(A) : A \in (\mathcal{F}) \}
\]

\[
\leq \Sigma \left\{ \Phi \left( \prod_{k=1}^n w_k(A_k)/M\mu_k(A_k) \right) \mu(A) : A \in (\mathcal{F}) \right\}
\]

\[
\leq \Sigma \left\{ \left[ \prod_{k=1}^n M\Phi(w_k(A_k)/M\mu_k(A_k)) \right] \mu(A) : A \in (\mathcal{F}) \right\}.
\]
Noting that \( \Phi(a/M) \leq \Phi(a)/M \) for \( a \geq 0 \) since \( \Phi \) is convex and \( M \geq 1 \), we conclude that
\[
\sum \{ \Phi(|v(A)/b\mu(A))\mu(A): A \in (\mathcal{F}) \} 
\leq \prod_{k=1}^{n} \{ \sum \Phi(w_k(A)/\mu_k(A))\mu_k(A): A \in \mathcal{F}_k \}
\leq I_\phi(w_1) \cdots I_\phi(w_n) \leq 1.
\]

By taking the appropriate supremum, we see that \( I_\phi(v/b) \leq 1 \). From this it can easily be seen that \( v \in A^\phi(\mu, W) \).

We shall now consider infinite products of vector-valued finitely additive set functions.

Let \( T \) be an arbitrary index set, let \( (\Omega_i, \Sigma_i, \mu_i) \) \( (i \in T) \) be a family of probability charge spaces and let \( (\Omega, \Sigma, \mu) \) be the product space (i.e., \( \Omega = \times_i \Omega_i \), \( \Sigma \) is the family of all finite disjoint unions of sets of the form \( A = \times_i A_i \) where \( A_i \in \Sigma_i \) and \( A_i \neq \Omega_i \) for at most a finite number of indices \( i \in T \), and \( \mu(A) = \prod_i \mu_i(A_i) \) for all \( A \in \Sigma \).) It is clear that the triple \((\Omega, \Sigma, \mu)\) is a probability charge space.

Let \( Y, Z_i \) \( (i \in T) \), \( W \) be Banach spaces, let \( z_i \in Z_i \) each be of unit norm, and let \( Z' = \prod T Z_i \) be the corresponding multiplicative product space. That is, \( \prod T Z_i \) is the set of all \( z = (z_i)_T \) such that \( z_i \in Z_i \), for \( i \in T \) and \( \sum_T |z_i - z'_i| < \infty \). Define a functional \( d \) by
\[
d(y, z) = \sum_T |y_i - z_i|
\]
for all \( y, z \in Z' \). In [3] it is shown that the space \((Z', d)\) is a complete metric space. It is easy to see that the space \( Z' \) is also an affine subspace of the linear space \( \times_i Z_i \), and if \( T \) is uncountable the metric \( d \) cannot be extended to a metric on the space \( \times_i Z_i \).

An operator \( u \) mapping the space \( Z' \) into the space \( W \) shall be called infinitely linear and bounded if \( u \) is linear on each coordinate space \( Z_i \) separately, for all \( i \in T \), and if there is a positive constant \( c \) such that \( |u(z)| \leq c ||z|| \) for all \( z \in Z' \). Let \( ||u|| \) denote the smallest of all such constants. Let \( L(Z'; W) \) denote the space of all such bounded infinitely linear operators. In [3] it is shown that the space \( L(Z'; W) \) under the functional \( || \cdot || \) is a Banach space.

In order to make the next definition we must further restrict the function \( \Phi \) so that \( \Phi(1) = 1 \) and the \( \lambda' \)-constant \( M = 1 \). These conditions are the results of applying a version of Lemma 13 of [5], modified to assume the \( \lambda' \)-condition. Since the functions \( g \) and \( f \) from Remark 1 are nontrivial examples of functions which satisfy these conditions, the additional conditions are not as restrictive as they originally appeared to be.

Select \( v'_i \in A^\phi(\mu_i, Z_i) \) such that \( v'_i(\Omega_i) = z'_i \) and \( N_\phi(v'_i) = 1 \), and let
$P_\tau(A^\phi(\mu_t, Z_t), v'_t)$ be the corresponding multiplicative product space. We remark that from Lemma 13 of [5] it follows that $v(\Omega) \in Z'$ for all $v \in P_\tau(A^\phi(\mu_t, Z_t), v'_t)$.

For the infinite product ring $\Sigma$ we need some notation similar to the $\mathcal{F}$-product concept for the finite case. With this in mind let $\mathcal{F}$ be a finite family of disjoint sets from the infinite product ring $\Sigma$, let the families $\mathcal{F}_1$ and $\mathcal{F}_1^*$ be defined as in the finite product case, and define the $\mathcal{F}$-product as the corresponding finite family of disjoint sets

$$X_\tau \mathcal{F}_1^* = \{ A = X_\tau A_t; A_t \in \mathcal{F}_1^* \text{ for all } t \in T \}$$

from the ring $\Sigma$. Again we shall denote the $\mathcal{F}$-product by $(\mathcal{F})$.

(It follows from the definition of $\Sigma$ and properties of a ring that $(\mathcal{F})$ consists of a finite number of disjoint sets from $\Sigma$.)

**Theorem 3.** If $v \in P_\tau(A^\phi(\mu_t, Z_t), v'_t)$ and $u \in L(Z'; W)$ then the function $v^0$ defined by $v^0(A) = u(v(A))$ for all $A \in \Sigma$ belongs to the space $A^\phi(\mu, W)$.

**Proof.** It is clear that the function $v^0$ is finitely additive on $\Sigma$, and from the estimate $|v^0(A)| \leq ||u||I_\tau |v_t(A)|$ for all $A \in \Sigma$ it is clear that $v(A) = 0$ when $\mu(A) = 0$. To establish that $I_\phi(v/k) \leq 1$ for some constant $k$, put $w_t(\cdot) = |v_t(\cdot)|/N_\phi(v_t)$ and let $\mathcal{F}$ be a finite family of disjoint sets from $\Sigma$. Applying Lemma 1, which is clearly valid here, we get the following estimates:

$$\Sigma \{ \Phi(\frac{|v^0(A)|\mu(A)}{b\mu(A)})\mu(A); A \in (\mathcal{F}) \}$$

$$\leq \Sigma \{ \Phi(\Pi_s w_t(\Omega_t)\Pi_s (w_t(A_t)/\mu_t(A_t)))\mu(A); A \in (\mathcal{F}) \}$$

$$\leq \Sigma \{ \Pi_s \Phi(w_t(A_t)/\mu_t(A_t))\mu(A); A \in (\mathcal{F}) \}$$

$$\leq \Pi_s I_\phi(w_t) \leq 1.$$ 

Hence $v^0 \in A^\phi(\mu, W)$ and $N_\phi(v^0) \leq ||u||I_\tau N_\phi(v_t)$.

**Remark 2.** The results in this note remain valid if all rings are relaxed to prerings, with no changes needed in the above proofs to establish this.

**Remark 3.** Forthcoming papers will discuss the other formulations of the Fubini theorems for Orlicz spaces in terms of a countably additive nonnegative finite valued set function (called a volume) defined on a prering. The prering, volume combination appears to be the most natural context in which to discuss the Fubini theorems; that it is also a valid context follows from the work of Bogdanowicz,
which develops an integration theory based upon the prering and volume. See [2] for a suitable bibliography.

REFERENCES


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