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EMBEDDINGS IN MATRIX RINGS

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For a fixed integer $n \geqq 1$, and a given ring R there exists a homomorphism $\rho; R \to M_n(K)$, K a commutative ring such that every homomorphism of R into an $n \times n$ matrix ring $M_n(H)$ over a commutative ring can be factored through ρ by a homomorphism induced by a mapping $\eta: K \to H$. The ring K is uniquely determined up to isomorphisms. Further properties of K are given.

1. Notations. Let R be an (associative) ring, $M_n(R)$ will denote the ring of all $n \times n$ matrices over R. If $\eta: R \to S$ is a ring homomorphism then $M_n(\eta): M_n(R) \to M_n(S)$ denotes the homomorphism induced by η on the matrix ring, i.e., $M_n(\eta)(r_{ik}) = (\eta(r_{ik}))$.

If $A \in M_n(R)$, we shall denote by $(A)_{ik}$ the entry in the matrix A standing in the (i, k) place.

Let k be a commutative ring with a unit (e.g., $k = \mathbf{Z}$ the ring of integers). All rings considered henceforth will be assumed to be k-algebras on which $1 \in k$ acts as a unit, and all homomorphisms will be k-homomorphisms, and will be into unless stated otherwise.

Let $\{x_i\}$ be a set (of high enough cardinality) of noncommutative indeterminates over k, and put $k[x] = k[\cdots, x_i, \cdots]$ the free ring generated over k with k commuting with the x_i. We shall denote by $k^0[x]$ the subring of $k[x]$ containing all polynomials with free coefficient zero.

Denote by $X_i = (\xi^i_{\alpha,\beta})$ $\alpha, \beta = 1, 2, \cdots, n$ the generic matrices of order n over k, i.e., the elements $\{\xi^i_{\alpha,\beta}\}$ are commutative indeterminates over k. Let $\Delta = k[\xi] = k[\cdots, \xi^i_{\alpha,\beta}, \cdots]$ denote the ring of all commutative polynomials in the ξ's, then we have $k^0[X] \subseteqq k[X] \subseteqq M_n(\Delta)$ where $k[X]$ is the k-algebra generated by 1 and all the X_i; $k^0[X]$ is the k-algebra generated by the X_i (without the unit).

There is a canonical homomorphism $\psi_0: k[x] \to k[X]$ which maps also $k^0[x]$ onto $k^0[X]$ given by $\psi_0(x_i) = X_i$.

2. Main result. The object of this note is to prove the following:

THEOREM 1. *Let R be a k-algebra, then*

(i) *There exists a commutative k-algebra S and a homomorphism $\rho: R \to M_n(S)$ such that:*

(a) *The entries $\{[\rho(r)]_{\alpha\beta}; r \in R\}$ generate together with 1, the ring S.*

(b) *For any* $\sigma: R \to M_n(K)$, *K a commutative k-algebra, with a unit, (but with the same n) there exists a homomorphism* $\eta: S \to K$ *such that for the induced map* $M_n(\eta): M_n(S) \to M_n(K)$, *we have the relation* $M_n(\eta)\rho = \sigma$, i.e., *σ is factored through ρ by a specialization* $M_n(\eta)$.

(ii) *S is uniquely determined up to an isomorphism by properties* (a) *and* (b); *and similarly ρ is uniquely determined up to a multiple by an isomorphism of S. Given S, ρ and σ then $M_n(\eta)$ is uniquely determined.*

(iii) *If R is a finitely generated k-algebra then so is S. Thus if k is noetherian, S will also be noetherian.*

Proof. Before proceeding with the proof of the existence of (S, ρ) we prove the uniqueness stated in (ii).

Let (S, ρ) (S_0, ρ_0) be two rings and homomorphisms satisfying (i), then by (b) it follows that there exist $\eta: S \to S_0$ and $\eta_0: S_0 \to S$ such that $M_n(\eta)\rho = \rho_0$, $M_n(\eta_0)\rho_0 = \rho$. Hence, $M_n(\eta_0)M_n(\eta)\rho = \rho$. Clearly $M_n(\eta_0)M_n(\eta) = M_n(\eta_0\eta)$ and $\eta_0\eta: S \to S$. For every $r \in R$, it follows that $\rho(r) = M_n(\eta_0\eta)\rho(r)$ and so for every entry $\rho(r)_{\alpha\beta}$ we have

$$\rho(r)_{\alpha,\beta} = (\eta_0\eta)[\rho(r)]_{\alpha\beta} .$$

Thus, $\eta_0\eta$ is the identity on the entries of the matrices of $\rho(R)$, and since $\eta_0\eta$ are also k-homomorphism (by assumption stated in the introduction) and these entries generate S by (a)—we have $\eta_0\eta =$ identity. Similarly $\eta\eta_0 =$ identity on S_0 and η, η_0 are isomorphism, and in particular it follows that $\rho_0 = M_n(\eta_0)\rho$ which completes the proof of uniqueness of S and ρ.

If $\sigma: R \to M_n(K)$ is given and if there exist $\eta, \eta': S \to K$ satisfying (i), i.e., $M_n(\eta)\rho = M_n(\eta')\rho = \sigma$ then $M_n(\eta)\rho(r) = M_n(\eta')\rho(r)$ for every $r \in R$ and thus for every entry $\rho(r)_{\alpha\beta}$ we have $\eta[\rho(r)_{\alpha\beta}] = \eta'[\rho(r)_{\alpha\beta}]$, and from the previous argument that all $\rho(r)_{\alpha\beta}$ generate S we have $\eta = \eta'$.

Proof of (i). We define a homomorphism ρ and the ring S as follows: Let $\{r_i\}$ be a set of k-generators of R, and consider the homomorphism onto: $\varphi_0: k^0[x] \to R$ given by $\varphi_0(x_i) = r_i$, and let $\mathfrak{p} = \operatorname{Ker}\varphi$. Thus φ induces an isomorphism (denoted by φ) between $k^0[x]/\mathfrak{p}$ and R.

If $\psi_0: k^0[x] \to k^0[X]$ given by $\psi(x_i) = X_i$, then let $P = \psi_0(\mathfrak{p})$ the image of the ideal $\mathfrak{p}$ under ψ_0. Hence ψ_0 induces a homomorphism (denoted by ψ) $k^0[x]/\mathfrak{p} \to k^0[X]/P$.

The ring $k^0[X]$ is a subalgebra of $M_n(\Delta)$, so let $\{P\}$ be the ideal in $M_n(\Delta)$ generated by P. Then $\{P\} = M_n(I)$ for some ideal I in Δ, since Δ contains a unit, I is the ideal generated by all entries of the matrices of $\{P\}$. With this notation we put:

$$S = \Delta/I \text{ and } \rho \text{ be the composite map:}$$

$$R \to k^0[x]\mathfrak{p} \to k^0[X]/P \to M_n(\Delta)/\{P\} \to M_n(\Delta/I) = M_n(S).$$

Where the first map is φ^{-1}, the second map is ψ. The map

$$\nu\colon k^0[X]/P \to M_n(\Delta)/\{P\}$$

is the one induced by the inclusion $k^0[X] \to M_n(\Delta)$ which maps, therefore, P into $\{P\}$ and so ν is well defined. The last map is the natural isomorphism of $M_n(\Delta)/\{P\} = M_n(\Delta)/M_n(I) \cong M_n(\Delta/I)$, which correspond to a matrix $(u_{ik}) + M_n(I) \mapsto (u_{ik} + I)$.

Note that Δ is generated by the $\xi^i_{\alpha\beta}$ and 1, thus, so $\Delta/I = S$ is generated by 1 and $\xi^i_{\alpha\beta} + I$ but the latter are the $(\alpha\beta)$ entries of the matrices $\rho(r_i)$. Indeed, $\varphi^{-1}(r_i) = x_i + \mathfrak{p}$ so that $\psi\varphi^{-1}(r_i) = X_i + P$ so that $\rho(r_i)_{\alpha\beta} = \xi^i_{\alpha\beta} + I$, which proves (a).

To prove (b) let $\sigma\colon R \to M_n(K)$ a fixed homomorphism, then define η as follows:

Let $\sigma(r_i) = (k^i_{\alpha\beta}) \in M_n(K)$, then consider the specialization $\eta_0\colon \Delta = k[\xi] \to K$ given by $\eta_0(\xi^i_{\alpha\beta}) = k^i_{\alpha\beta}$. We have to show that the homomorphism η_0 maps I into zero and η will be the induced map $\Delta/I \to K$.

Consider the diagram:

$$\begin{array}{ccc} k^0[x] & \xrightarrow{\psi_0} & k^0[X] \\ {\scriptstyle\varphi_0}\downarrow & & \downarrow \\ R & \xrightarrow[\tau]{} & M_n(K) \end{array}$$

where the second column is actually the composite

$$k^0[x] \to M_n(\Delta) \to M_n(K) ,$$

in which the first is the inclusion and the second is the map $M_n(\eta_0)$, we shall use the same notation $M_n(\eta_0)$ to denote also this map. This diagram is commutative since $\tau\varphi_0(x_i) = \tau(r_i) = (k^i_{\alpha\beta})$ and also

$$M_n(\eta_0)\psi_0(x_i) = M_n(\eta_0)X_i = (\eta_0(\xi^i_{\alpha\beta})) = (k^i_{\alpha\beta})$$

by definition. Thus $\tau\varphi_0 = M_n(\eta_0)\psi_0$ on the generators and hence on all $k^0[x]$. In particular, if $p[x] \in \mathfrak{p} = \ker \varphi_0$, then

$$0 = \tau\varphi_0(p[x]) = M_n(\eta_0)\psi_0(p[x])$$

which shows $\psi_0(p[x]) \subseteq \operatorname{Ker} M_n(\eta_0)$ and thus $P = \psi_0(\mathfrak{p}) \subseteq \operatorname{Ker} M_n(\eta_0)$. Consequently, the preceding diagram induces the commutative diagram (I):

$$\begin{array}{ccc} k^0[x]/\mathfrak{p} & \xrightarrow{\psi} & k^0[X]/P \\ {\scriptstyle\varphi}\downarrow & & \downarrow \\ R & \xrightarrow[\tau]{} & M_n(K)\ . \end{array}$$

Let $\bar{\eta}: k^0[X]/P \to M_n(K)$ denote the second column homomorphism which is induced by $M_n(\eta_0)$. Observe that $\bar{\eta}(X_i + P) = \tau(r_i)(= M_n(\eta)(X_i))$ since $\bar{\eta}(X_i + P) = \bar{\eta}\psi(x_i + \mathfrak{p}) = \tau\varphi(x_i + \mathfrak{p}) = \tau(r_i)$.

To obtain the final stage of our map ρ we consider the diagram:

$$\begin{array}{ccc} k^0[X] & \xrightarrow{\lambda_0} & M_n(\Delta) \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle M_n(\eta_0)} \\ k^0[X]/P & \xrightarrow{\bar{\eta}} & M_n(K) \end{array}$$

where λ_0 is the injection, r is the projection. This diagram is also commutative since

$$M_n(\eta_0)\lambda_0(X_i) = M_n(\eta_0)X_i = (\eta_0(\xi^i_{\alpha\beta})) = (k^i_{\alpha\beta}) = \tau(r_i)\ ,$$

and also $\bar{\eta}r(X_i) = \bar{\eta}(X_i + P) = \tau(r_i)$. This being true for the generators implies that $M_n(\eta_0)\lambda = \bar{\eta}r$.

Now $r(P) = 0$, hence $M_n(\eta_0)\lambda_0(P) = \bar{\eta}r(P) = 0$ and as $\lambda_0(P) = P$ (being the injection) it follows that $P \subseteqq \operatorname{Ker} M_n(\eta_0)$. The latter is an ideal in $M_n(\Delta)$, hence $\operatorname{Ker} M_n(\eta_0) \supseteqq \{P\}$. Consequently $M_n(\eta_0)$ induces a homomorphism $\tilde{\eta}: M_n(\Delta)/\{P\} \to M_n(K)$ and we have the commutative diagram (II):

$$\begin{array}{ccc} k^0[X]/P & \xrightarrow{\lambda} & M_n(\Delta)/\{P\} \\ & {\scriptstyle\bar{\eta}}\searrow \quad \swarrow{\scriptstyle\tilde{\eta}} & \\ & M_n(K) & \end{array}$$

where λ is the map induced by the injection $\lambda_0: k^0[X] \to M_n(\Delta)$, and λ is well defined since $\lambda(P) \subseteqq \{P\}$. The diagram is commutative, since

$$\tilde{\eta}\lambda(X_i + P) = \tilde{\eta}(X_i + \{P\}) = M_n(\eta_0)(X_i) = (\eta_0\xi^i_{\alpha\beta}) = (k^i_{\alpha\beta}) = \tau(r_i)$$

and also $\bar{\eta}(X_i + P) = \tau(r_i)$ as shown above.

Another consequence of the existence of $\tilde{\eta}$, is the fact that $\eta_0(I) = 0$ where $\{P\} = M_n(I)$. Indeed, as was shown $\{P\} \subseteqq \operatorname{Ker} M_n(\eta_0)$ so that $M_n(\eta_0)(\{P\}) = M_n(\eta_0 I) = 0$. Thus $\eta_0: \Delta \to K$, induces a homomorphism $\eta: \Delta/I \to K$ and hence the homomorphism

$$M_n(\eta): M_n(\Delta/I) \to M_n(K)$$

and we have a third commutative diagram (III):

$$\begin{array}{ccc} M_n(\Delta)/\{P\} & \xrightarrow{\mu} & M_n(\Delta/I) \\ & \tilde{\eta}\searrow \quad \swarrow M_n(\eta) & \\ & M_n(K) & \end{array}$$

where μ is the isomorphism $M_n(\Delta)/P = M_n(\Delta)/M_n(I) \cong M_n(\Delta/I)$. This diagram is also commutative since $\tilde{\eta}(X_i + P) = M_n(\eta_0)(X_i) = \tau(r_i)$ as before, and $M_n(\eta)\mu(X_i + P) = M_n(\eta)((\xi^i_{\alpha\beta} + I)) = (\eta_0\xi^i_{\alpha\beta}) = \tau(r_i)$.

Combining the commutative diagrams (I), (II) and (III) and noting that φ is an isomorphism, and that we have defined ρ to be $\rho = \mu\lambda\psi\varphi^{-1}$, we finally obtain

$$M_n(\eta)\rho = (M_n(\eta)\mu)\lambda\psi\varphi^{-1} = (\tilde{\eta}\lambda)\psi\varphi^{-1} = (\bar{\eta}\psi)\varphi^{-1} = \tau\varphi\varphi^{-1} = \tau$$

and this completes the proof of our theorem.

Note that for this ring $S = \Delta/I$, if R is finitely generated then we can choose the set $\{x_i\}$ to be finite and, therefore, Δ is a k-polynomial ring in a finite number of commutative indeterminate. Thus, $S = \Delta/I$ is a finitely generated ring. This will prove (iii) of (S, ρ) defined above will satisfy (i) and the uniqueness of (ii) shows that this property is independent on the definition of S and ρ.

3. Other results. The proof of Theorem 1, can be carried over by replacing $k^0[x], k^0[X]$ by the rings $k[x], k[X]$ to the following situation.

Consider rings R with a unit, and unitary homomorphisms, i.e., homomorphisms which maps the unit onto the unit. Then

THEOREM 2. *There exists a commutative k-algebra S_u with a unit and a unitary homomorphism $\rho_u\colon R \to M_n(S_u)$ which satisfies* (i)–(iii) *of Theorem* 1 *when restricted only to unitary homomorphisms $\sigma_u\colon R \to M_n(K)$.*

We remark that S_u is not necessarily the same as S.

Another result which follows from the proof Theorem 1:

THEOREM 3. *R can be embedded in a matrix ring $M_n(K)$ over some commutative ring K, if and only if the morphism $\rho\colon R \to M_n(S)$ of Theorem* 1 *is a monomorphism.*

A necessary and sufficient condition that this holds, is that there exists a homomorphism φ of $k^0[X]$ onto R, and if $P = \operatorname{Ker}\varphi$ then $\{P\} \cap k^0[X] = P$.

If this holds for one such presentation of R then it holds for *all of them.*

REMARK. It goes without changes to show that Theorem 3 can be stated and shown for unitary embeddings.

The necessary and sufficient condition given in this theorem is actually included in the proof of Theorem 2.11 (Procesi, *Non-commutative affine rings*, Accad. Lincei, v. VIII (1967), p. 250) which leads to the present result.

Proof. If ρ is a monomorphism then clearly R can be embedded in a matrix ring over a commutative ring, e.g., in $M_n(S)$. Conversely, if there exist an embedding $\sigma: R \to M_n(K)$, then since $\sigma = M_n(\eta)\rho$ by Theorem 1 and σ is a monomorphism, it follows that ρ is a monomorphism.

The second part follows from the definition of ρ. Indeed $\rho = \mu\lambda\psi\varphi^{-1}$ where $\psi: k^0[x]/\mathfrak{p} \to k^0[X]/P$ is an epimorphism,

$$\lambda: k^0[X]/P \to M_n(\Delta)/\{P\} .$$

Thus, ρ is a monomorphism if and only if ψ is an isomorphism and λ is a monomorphism. The fact that ψ is an isomorphism means that $k^0[X]/P \cong k^0[x]/\mathfrak{p} \cong R$, and that λ is a monomorphism is equivalent to saying that $\operatorname{Ker} \lambda_0 = k^0[X] \cap \{P\} = P$.

Thus if the condition of our theorem holds for one representation, we can apply this representation to obtain the ring S and so the given ρ will be a monomorphism; but then by the uniqueness of (S, ρ) this will hold in any other way we define an S and an ρ. So the fact that ρ is a monomorphism implies that $k^0[X] \cap \{P\} = P$ for any other representation of R.

A corollary of Theorem 1 (and a similar corollary of Theorem 2) is that

THEOREM 4. *Every k-algebra R contains a unique ideal Q such that R/Q can be embedded in a matrix ring $M_n(K)$ over some commutative ring, and if R/Q_0 can be embedded in some $M_n(K)$ the $Q \subseteqq Q_0$.*

Proof. Let $\rho: R \to M_n(S)$ and set $Q = \operatorname{Ker} \rho$. Then ρ induces a monomorphism of R/Q into $M_n(S)$. If σ is any other homomorphism: $R \to M_n(K)$ then by Theorem 1 $M_n(\eta)\rho = \sigma$ so that

$$\operatorname{Ker}(\sigma) \supseteqq \operatorname{Ker}(\rho) = Q$$

which proves Theorem 4.

4. Irreducible representations. Let R be a k-algebra with a unit[1] and k be a field. A homomorphism $\varphi: R \to M_n(F)$, F a commutative field, is called an irreducible representation if $\varphi(R)$ contains an F-base of $M_n(F)$, or equivalently $\varphi(R)F = M_n(F)$.

THEOREM 5. *Let $\rho: R \to M_n(S)$ be the unitary embedding of R of Theorem 1, then $\rho(R)S = M_n(S)$ if and only if all irreducible representations of R are of dimension $\geqq n$, and then all representations of R of dimension n are irreducible.*

Proof. In view of Theorem 1 it suffices to prove our result for a ring $S = \Delta/I$ obtained by a fixed presentation of $R = k^0[X]/P$ and with $\{P\} = M_n(I)$.

Let Ω be the field of all rational functions on the $\xi's$, i.e., the quotient field of $k[\xi] = \Delta$. By a result of Procesi (ibid.), $k^0[X]\Omega = M_n(\Omega)$. Actually it was shown that $k[X]\Omega = M_n(\Omega)$, but since

$$k^0[X]\Omega \subseteqq M_n(\Omega)$$

and any identity which holds in $k^0[X]$ will hold also in $M_n(\Omega)$ as such an identity is a relation in generic matrices, it follows that $k^0[X]\Omega$ cannot be a proper subalgebra of $M_n(\Omega)$ since these have different identities. Hence, since every element of Ω is a quotient of two polynomials in ξ it follows that there exists $0 \neq h$ in Δ such that $he_{ik} \in k^0[X]\Delta$ where e_{ik} is a matrix base of $M_n(\Omega)$. In particular this implies that $k^0[X]\Delta \supseteqq M_n(T)$ for some ideal T in Δ and, in fact, we choose T to be the maximal with this property.

Next we show that in our case $T + I = \Delta$:

Indeed, if it were not so, then let $\mathfrak{m} \neq \Delta$ be a maximal ideal in Δ, $\mathfrak{m} \supseteqq T + I$. Let $F = \Delta/\mathfrak{m}$ and σ be the composite homomorphism $\sigma: R \to M_n(\Delta/I) \to M_n(F)$. This representation must be irreducible, otherwise $\sigma(R)F$ is a proper subalgebra (with a unit) of $M_n(F)$ and, therefore, it has an irreducible representation of dimension $< n^2$, or else $\sigma(R)$ is nilpotent but $\sigma(R) = \sigma(R^2)$, thus, R will have representations which contradict our assumption. Hence, $\sigma(R)F = M_n(F)$.

Consider the commutative diagram

$$\begin{array}{ccc} k^0[X] & \longrightarrow & M_n(\Delta) \\ \downarrow & & \downarrow \\ R & \longrightarrow & M_n(\Delta/I) \to M_n(F) \end{array}$$

where σ is the composite of the lower row, and denote by τ the composite $\tau: k^0[X] \to M_n(\Delta) \to M_n(\Delta/I) \to M_n(F)$. The first vertical

[1] It is sufficient to assume that $R^2 = R$.

map is an epimorphism hence $\tau(k^0[x]) = \sigma(R)$. Consequently, since $\sigma(R)F = M_n(F)$, there exists a set of polynomials $f_\tau[X] \in k^0[X]$, $\lambda = 1, 2, \cdots, n^2$ such that $\tau(f_\lambda)$ are a base of $M_n(F)$. This is equivalent to the statement that the discriminant $\delta = \det(\mathrm{tr}\,[\tau(f_\lambda)\tau(f_\mu)] \neq 0$, where $\mathrm{tr}\,(\cdot)$ is the reduced trace of $M_n(F)$.

Considering f_λ as elements of $M_n(\Delta)$ and noting that the reduced $\mathrm{tr}\,(\cdot)$ commutes with the specialization $\tau_0\colon \Delta \longrightarrow \Delta/I \longrightarrow F$, it follows that

$$0 \neq \delta = \det(\mathrm{tr}\,(f_\lambda f_\mu)]) = \det(\tau_0[f_\lambda f_\mu)]) = \tau_0[\det(\mathrm{tr}(f_\lambda f_\mu))]$$

and so $\det[\mathrm{tr}(f_\lambda f_\mu)] = D \neq 0$ in $M_n(\Delta) \subseteqq M_n(\Omega)$. Hence $\{f_\lambda\}$ is an Ω-base of $M_n(\Omega)$.

In particular $e_{ik} = \sum f_\lambda[X]u_{\lambda,ik}$ with $u_{\lambda,ik} \in \Omega$. By multiplying each equation by f_μ and taking the trace we obtain:

$$\sum \mathrm{tr}(f_\mu f_\lambda)u_{\lambda,ik} = h_{\mu,ik} \in \Delta\ .$$

Eliminating these equations by Cramer's rule we obtain $Du_{\lambda,ik} \in \Delta$ where $D = \det[\mathrm{tr}(f_\lambda f_\mu)]$ which implies

$$De_{ik} = \sum f_\lambda[X]\cdot Du_{\lambda,ik} \in k[X]\Delta\ .$$

Namely $D \in T$. This leads to a contradiction, since then $D \in T + I \subseteqq \mathfrak{m}$ and so $D \equiv 0 \pmod{\mathfrak{m}}$, and so $\tau_0(D) = 0$ under the mapping

$$\tau_0\colon \Delta \longrightarrow \Delta/I \longrightarrow \Delta/\mathfrak{m} = F$$

but on the other hand $\tau_0(D) = \sigma \neq 0$.

This completes the proof that $T + I = \Delta$. And so

$$M_n(\Delta) = M_n(T) + M_n(I) \subseteqq k[X]\Delta + \{P\} \subseteqq M_n(\Delta)\ .$$

Applying $M_n(\eta)\colon M_n(\Delta) \longrightarrow M_n(S)$ to this equality we obtain

$$M_n(S) = M_n(\eta)M_n(\Delta) = M_n(\eta)(k[X]\Delta) = \rho(R)S$$

since $\eta(I) = 0$, $\eta(\Delta) = S$ and $M_n(k[X]) = \rho(R)$. Thus $M_n(S) = \rho(R)S$.

Conversely, if $M_n(S) = \rho(R)S$, then any homomorphism $\tau\colon R \longrightarrow M_n(H)$ is irreducible. Indeed, $\tau = M_n(\eta)\rho$ for some $\eta\colon S \longrightarrow H$. Hence $\tau(R)H \supseteqq M_n(\eta)[\rho(R)S] = M_n(\eta S)$. Consequently, $M_n(H) \subseteqq M_n(\eta S)H \subseteqq \tau(R)H \subseteqq M_n(H)$ which proves that τ is irreducible. The rest follows from the fact that any representation $\tau\colon R \longrightarrow M_m(H)$ $m \leqq n$ could be followed by an embedding $M_m(H) \longrightarrow M_n(H)$ and since the composite $R \longrightarrow M_n(H)$ must be irreducible we obtain that $n = m$, as required.

COROLLARY 6. *If R satisfies an identity of degree $\leqq 2n$, then all irreducible representations of R are exactly of dimension n – if* *and only if* $\rho(R)S = M_n(S)$.

Indeed, since the irreducible representation of such an algebra will satisfy identities of the same degree, hence their dimension is anyway $\leqq n^2$. Thus Theorem 5 yields in this case our corollary.

Another equivalent condition to Theorem 5, is the following:

THEOREM 7. *A ring R has all its irreducible representation of dimension $\geqq n$ — if and only if there exists a polynomial $f[x_1, \cdots, x_k]$ such that $f[x] \equiv 0$ holds identically in $M_{n-1}(k)$ and $f[r_1 \cdots, r_k] = 1$ for some $r_i \in R$.*

Indeed, let $R \cong k[k]/\mathfrak{p}$ and let $\mathfrak{m}_{n-1}$ be the ideal of identities of $M_{n-1}(k)$. Then $\mathfrak{p} + \mathfrak{m}_{n-1} = k[x]$, otherwise, there exist a maximal ideal $\mathfrak{m} \supseteqq \mathfrak{p} + \mathfrak{m}_{n-1}$, $\mathfrak{m} \neq k[x]$. Hence $k[x]/\mathfrak{m}$ is a simple ring and satisfies all identities of $M_{n-1}(k)$ so it is central simple of dimension $< n^2$. But it yields also an irreducible representation of R of the same degree, which contradicts our assumption. Thus $k[x] = \mathfrak{p} + \mathfrak{m}_{n-1}$ and so $1 \equiv f[x] \pmod{\mathfrak{p}}$ with $f \in \mathfrak{m}_{n-1}$ and f satisfies our theorem.

The converse, is evident, since under any map $\sigma \to M_m(H)$, $m < n$ we must have

$$\sigma(f[r_1, \cdots, r_k]) = f[\sigma(r_1), \cdots, \sigma(r_k)] = 0$$

but $f[r_1, \cdots, r_k] = 1$. Hence, $m \geqq n$.

REMARK. Examples of rings satisfying Theorem 5 are central simple algebras of dimension n^2 over their center, and then ρ is a monomorphism. Hence the relation $\rho(R)S = M_n(S)$ means that S is a splitting ring of R, and in view of Theorem 1, it follows that S is the uniquely determined *splitting ring of R*.

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