EMBEDDINGS IN MATRIX RINGS

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For a fixed integer \( n \geq 1 \), and a given ring \( R \) there exists a homomorphism \( \rho: R \to M_n(K) \), \( K \) a commutative ring such that every homomorphism of \( R \) into an \( n \times n \) matrix ring \( M_n(H) \) over a commutative ring can be factored through \( \rho \) by a homomorphism induced by a mapping \( \gamma: K \to H \). The ring \( K \) is uniquely determined up to isomorphisms. Further properties of \( K \) are given.

1. Notations. Let \( R \) be an (associative) ring, \( M_n(R) \) will denote the ring of all \( n \times n \) matrices over \( R \). If \( \gamma: R \to S \) is a ring homomorphism then \( M_n(\gamma): M_n(R) \to M_n(S) \) denotes the homomorphism induced by \( \gamma \) on the matrix ring, i.e., \( M_n(\gamma)(r_{ik}) = (\gamma(r_{ik})) \).

If \( A \in M_n(R) \), we shall denote by \( (A)_{ik} \) the entry in the matrix \( A \) standing in the \((i, k)\) place.

Let \( k \) be a commutative ring with a unit (e.g., \( k = \mathbb{Z} \) the ring of integers). All rings considered henceforth will be assumed to be \( k \)-algebras on which \( 1 \in k \) acts as a unit, and all homomorphisms will be \( k \)-homomorphisms, and will be into unless stated otherwise.

Let \( \{x_i\} \) be a set (of high enough cardinality) of noncommutative indeterminates over \( k \), and put \( k[x] = k[\cdots, x_i, \cdots] \) the free ring generated over \( k \) with \( k \) commuting with the \( x_i \). We shall denote by \( k^0[x] \) the subring of \( k[x] \) containing all polynomials with free coefficient zero.

Denote by \( X_i = (\xi_{i, \alpha}) \alpha, \beta = 1, 2, \cdots, n \) the generic matrices of order \( n \) over \( k \), i.e., the elements \( \{\xi_{i, \alpha}\} \) are commutative indeterminates over \( k \). Let \( \Delta = k[\xi] = k[\cdots, \xi_{i, \alpha}, \cdots] \) denote the ring of all commutative polynomials in the \( \xi \)'s, then we have \( k^0[X] \subseteq k[X] \subseteq M_n(\Delta) \) where \( k[X] \) is the \( k \)-algebra generated by 1 and all the \( X_i \); \( k^0[X] \) is the \( k \)-algebra generated by the \( X_i \) (without the unit).

There is a canonical homomorphism \( \psi_0: k[x] \to k[X] \) which maps also \( k^0[x] \) onto \( k^0[X] \) given by \( \psi_0(x_i) = X_i \).

2. Main result. The object of this note is to prove the following:

**Theorem 1.** Let \( R \) be a \( k \)-algebra, then

(i) There exists a commutative \( k \)-algebra \( S \) and a homomorphism \( \rho: R \to M_n(S) \) such that:

(a) The entries \( \{[\rho(r)]_{ij}; r \in R \} \) generate together with 1, the ring \( S \).
(b) For any \( \sigma: R \rightarrow M_n(K) \), \( K \) a commutative \( k \)-algebra, with a unit, there exists a homomorphism \( \eta: S \rightarrow K \) such that for the induced map \( M_n(\eta): M_n(S) \rightarrow M_n(K) \), we have the relation \( M_n(\eta)\rho = \sigma \), i.e., \( \sigma \) is factored through \( \rho \) by a specialization \( M_n(\eta) \).

(ii) \( S \) is uniquely determined up to an isomorphism by properties (a) and (b); and similarly \( \rho \) is uniquely determined up to a multiple by an isomorphism of \( S \). Given \( S \), \( \rho \) and \( \sigma \) then \( M_n(\eta) \) is uniquely determined.

(iii) If \( R \) is a finitely generated \( k \)-algebra then so is \( S \). Thus if \( k \) is noetherian, \( S \) will also be noetherian.

Proof. Before proceeding with the proof of the existence of \( (S, \rho) \) we prove the uniqueness stated in (ii).

Let \( (S, \rho) \) \( (S_0, \rho_0) \) be two rings and homomorphisms satisfying (i), then by (b) it follows that there exist \( \eta: S \rightarrow S_0 \) and \( \eta_0: S_0 \rightarrow S \) such that \( M_n(\eta)\rho = \rho_0 \), \( M_n(\eta_0)\rho_0 = \rho \). Hence, \( M_n(\eta_0)M_n(\eta)\rho = \rho \). Clearly \( M_n(\eta_0)M_n(\eta) = M_n(\eta_0\eta) \) and \( \eta_0\eta: S \rightarrow S \). For every \( r \in R \), it follows that \( \rho(r) = M_n(\eta_0\eta)\rho(r) \) and so for every entry \( \rho(r)_{a\beta} \) we have

\[
\rho(r)_{a\beta} = (\eta_0\eta)[\rho(r)]_{a\beta}.
\]

Thus, \( \eta_0\eta \) is the identity on the entries of the matrices of \( \rho(R) \), and since \( \eta_0\eta \) are also \( k \)-homomorphism (by assumption stated in the introduction) and these entries generate \( S \) by (a)—we have \( \eta_0\eta \) = identity. Similarly \( \eta\eta_0 \) = identity on \( S \), and \( \eta, \eta_0 \) are isomorphism, and in particular it follows that \( \rho_0 = M_n(\eta_0)\rho \) which completes the proof of uniqueness of \( S \) and \( \rho \).

If \( \sigma: R \rightarrow M_n(K) \) is given and if there exist \( \eta, \eta': S \rightarrow K \) satisfying (i), i.e., \( M_n(\eta)\rho = M_n(\eta')\rho = \sigma \) then \( M_n(\eta)\rho(r) = M_n(\eta')\rho(r) \) for every \( r \in R \) and thus for every entry \( \rho(r)_{a\beta} \) we have \( \eta[\rho(r)_{a\beta}] = \eta'[\rho(r)_{a\beta}] \), and from the previous argument that all \( \rho(r)_{a\beta} \) generate \( S \) we have \( \eta = \eta' \).

Proof of (i). We define a homomorphism \( \rho \) and the ring \( S \) as follows: Let \( \{r_i\} \) be a set of \( k \)-generators of \( R \), and consider the homomorphism onto: \( \varphi: k^d[x] \rightarrow R \) given by \( \varphi(x_i) = r_i \), and let \( p = \text{Ker } \varphi \). Thus \( \varphi \) induces an isomorphism (denoted by \( \varphi \)) between \( k^d[x]/p \) and \( R \).

If \( \psi: k^d[x] \rightarrow k^d[X] \) given by \( \psi(x_i) = X_i \), then let \( P = \psi(p) \) the image of the ideal \( p \) under \( \psi \). Hence \( \psi \) induces a homomorphism (denoted by \( \psi \)) \( k^d[x]/p \rightarrow k^d[X]/P \).

The ring \( k^d[X] \) is a subalgebra of \( M_n(\Delta) \), so let \( \{P\} \) be the ideal in \( M_n(\Delta) \) generated by \( P \). Then \( \{P\} = M_n(I) \) for some ideal \( I \) in \( \Delta \), since \( \Delta \) contains a unit, \( I \) is the ideal generated by all entries of the matrices of \( \{P\} \). With this notation we put:
\( S = \Delta/I \) and \( \rho \) be the composite map:
\[
R \to k^0[x]p \to k^0[X]/P \to M_n(\Delta)/\{P\} \to M_n(\Delta/I) = M_n(S).
\]

Where the first map is \( \varphi^{-1} \), the second map is \( \psi \). The map
\[
\nu: k^0[X]/P \to M_n(\Delta)/\{P\}
\]
is the one induced by the inclusion \( k^0[X] \to M_n(\Delta) \) which maps, therefore, \( P \) into \( \{P\} \) and so \( \nu \) is well defined. The last map is the natural isomorphism of \( M_n(\Delta)/\{P\} = M_n(\Delta)/M_n(I) \cong M_n(\Delta/I) \), which correspond to a matrix \((u_{ik}) + M_n(I) \mapsto (u_{ik} + I)\).

Note that \( \Delta \) is generated by the \( \zeta_{a\beta} \) and 1, thus, so \( \Delta/I = S \) is generated by 1 and \( \zeta_{a\beta} + I \) but the latter are the \((\alpha\beta)\) entries of the matrices \( \rho(r_i) \). Indeed, \( \varphi^{-1}(r_i) = x_i + p \) so that \( \psi\varphi^{-1}(r_i) = X_i + P \) so that \( \rho(r_i)_{a\beta} = \zeta_{a\beta} + I \), which proves (a).

To prove (b) let \( \sigma: R \to M_n(K) \) a fixed homomorphism, then define \( \eta \) as follows:

Let \( \sigma(r_i) = (k_{a\beta}^i) \in M_n(K) \), then consider the specialization \( \gamma_0: \Delta = k[\xi] \to K \) given by \( \gamma_0(\xi_{a\beta}) = k_{a\beta}^i \). We have to show that the homomorphism \( \gamma_0 \) maps \( I \) into zero and \( \eta \) will be the induced map \( \Delta/I \to K \).

Consider the diagram:
\[
\begin{array}{ccc}
k^0[x] & \xrightarrow{\varphi_0} & k^0[X] \\
\downarrow{\varphi_0} & & \downarrow{\tau} \\
R & \xrightarrow{\tau} & M_n(K)
\end{array}
\]

where the second column is actually the composite
\[
k^0[x] \to M_n(\Delta) \to M_n(K),
\]
in which the first is the inclusion and the second is the map \( M_n(\gamma_0) \), we shall use the same notation \( M_n(\gamma_0) \) to denote also this map. This diagram is commutative since \( \tau\varphi_0(x_i) = \tau(r_i) = (k_{a\beta}^i) \) and also
\[
M_n(\gamma_0)\psi_0(x_i) = M_n(\gamma_0)X_i = (\gamma_0(\zeta_{a\beta}^i)) = (k_{a\beta}^i)
\]
by definition. Thus \( \tau\varphi_0 = M_n(\gamma_0)\psi_0 \) on the generators and hence on all \( k^0[x] \). In particular, if \( p[x] \in p = \ker \varphi_0 \), then
\[
0 = \tau\varphi_0(p[x]) = M_n(\gamma_0)\psi_0(p[x])
\]
which shows \( \psi_0(p[x]) \subseteq \ker M_n(\gamma_0) \) and thus \( P = \psi_0(p) \subseteq \ker M_n(\gamma_0) \). Consequently, the preceding diagram induces the commutative diagram (I):
Let $\tilde{\eta}: k^o[X]/p \to M_n(K)$ denote the second column homomorphism which is induced by $M_n(\eta_0)$. Observe that $\tilde{\eta}(X_i + P) = \tau(r_i)(= M_n(\eta)(X_i))$ since $\tilde{\eta}(X_i + P) = \tilde{\eta}\psi(x_i + p) = \tau\varphi(x_i + p) = \tau(r_i)$.

To obtain the final stage of our map $\rho$ we consider the diagram:

$$
\begin{array}{c}
k^o[X] \xrightarrow{\lambda_0} M_n(\Delta) \\
\cong \\
k^o[X]/P \xrightarrow{\bar{\eta}} M_n(K)
\end{array}
$$

where $\lambda_0$ is the injection, $r$ is the projection. This diagram is also commutative since

$$M_n(\eta_0)\lambda_0(X_i) = M_n(\eta_0)X_i = (\eta_0(\xi_{i,0})) = (k^i_{0,0}) = \tau(r_i),$$

and also $\bar{\eta}r(X_i) = \bar{\eta}(X_i + P) = \tau(r_i)$. This being true for the generators implies that $M_n(\eta_0)\lambda = \bar{\eta}r$.

Now $r(P) = 0$, hence $M_n(\eta_0)\lambda_0(P) = \bar{\eta}r(P) = 0$ and as $\lambda_0(P) = P$ (being the injection) it follows that $P \subseteq \ker M_n(\eta_0)$. The latter is an ideal in $M_n(\Delta)$, hence $\ker M_n(\eta_0) \supseteq \{P\}$. Consequently $M_n(\eta_0)$ induces a homomorphism $\tilde{\eta}: M_n(\Delta)/\{P\} \to M_n(K)$ and we have the commutative diagram (II):

$$
\begin{array}{ccc}
k^o[X]/P & \xrightarrow{\lambda_0} & M_n(\Delta)/\{P\} \\
\cong & & \\
& \xrightarrow{\bar{\eta}} & M_n(K)
\end{array}
$$

where $\lambda$ is the map induced by the injection $\lambda_0: k^o[X] \to M_n(\Delta)$, and $\lambda$ is well defined since $\lambda(P) \subseteq \{P\}$. The diagram is commutative, since

$$\tilde{\eta}\lambda(X_i + P) = \tilde{\eta}(X_i + \{P\}) = M_n(\eta_0)(X_i) = (\eta_0(\xi_{i,0})) = (k^i_{0,0}) = \tau(r_i)$$

and also $\tilde{\eta}(X_i + P) = \tau(r_i)$ as shown above.

Another consequence of the existence of $\tilde{\eta}$, is the fact that $\eta_0(I) = 0$ where $\{P\} = M_n(I)$. Indeed, as was shown $\{P\} \subseteq \ker M_n(\eta_0)$ so that $M_n(\eta_0)(\{P\}) = M_n(\eta_0)I = 0$. Thus $\eta_0: \Delta \to K$, induces a homomorphism $\eta: \Delta/I \to K$ and hence the homomorphism

$$M_n(\eta): M_n(\Delta/I) \to M_n(K)$$
and we have a third commutative diagram (III):

\[
\begin{array}{ccc}
M_n(\Delta)/\{P\} & \xrightarrow{\mu} & M_n(\Delta/I) \\
\downarrow & & \downarrow \\
M_n(\gamma) & \xrightarrow{\tau} & M_n(K)
\end{array}
\]

where \(\mu\) is the isomorphism \(M_n(\Delta)/P = M_n(\Delta)/M_n(I) \cong M_n(\Delta/I)\). This diagram is also commutative since \(\gamma_i(X_i + P) = M_n(\eta_i)(X_i) = \tau(r_i)\) as before, and \(M_n(\gamma)\mu(X_i + P) = M_n(\gamma)((\xi_{i\alpha} + I)) = (\gamma_i \xi_{i\alpha}) = \tau(r_i)\).

Combining the commutative diagrams (I), (II) and (III) and noting that \(\phi\) is an isomorphism, and that we have defined \(\rho = \mu \lambda \psi \varphi^{-1}\), we finally obtain

\[
M_n(\gamma)\rho = (M_n(\gamma)\mu)\lambda \psi \varphi^{-1} = (\gamma \lambda) \psi \varphi^{-1} = (\gamma \psi) \varphi^{-1} = \tau \varphi \varphi^{-1} = \tau
\]

and this completes the proof of our theorem.

Note that for this ring \(S = \Delta/I\), if \(R\) is finitely generated then we can choose the set \(\{x_i\}\) to be finite and, therefore, \(\Delta\) is a \(k\)-polynomial ring in a finite number of commutative indeterminate. Thus, \(S = \Delta/I\) is a finitely generated ring. This will prove (iii) of \((S, \rho)\) defined above will satisfy (i) and the uniqueness of (ii) shows that this property is independent on the definition of \(S\) and \(\rho\).

3. Other results. The proof of Theorem 1, can be carried over by replacing \(k^o[x], k^o[X]\) by the rings \(k[x], k[X]\) to the following situation.

Consider rings \(R\) with a unit, and unitary homomorphisms, i.e., homomorphisms which maps the unit onto the unit. Then

**Theorem 2.** There exists a commutative \(k\)-algebra \(S_u\) with a unit and a unitary homomorphism \(\rho_u: R \rightarrow M_n(S_u)\) which satisfies (i)-(iii) of Theorem 1 when restricted only to unitary homomorphisms \(\sigma_u: R \rightarrow M_n(K)\).

We remark that \(S_u\) is not necessarily the same as \(S\).

Another result which follows from the proof Theorem 1:

**Theorem 3.** \(R\) can be embedded in a matrix ring \(M_n(K)\) over some commutative ring \(K\), if and only if the morphism \(\rho: R \rightarrow M_n(S)\) of Theorem 1 is a monomorphism.

A necessary and sufficient condition that this holds, is that there exists a homomorphism \(\varphi\) of \(k^o[X]\) onto \(R\), and if \(P = \text{Ker} \varphi\) then \(\{P\} \cap k^o[X] = P\).

If this holds for one such presentation of \(R\) then it holds for
\textbf{Remark.} It goes without changes to show that Theorem 3 can be stated and shown for unitary embeddings.

The necessary and sufficient condition given in this theorem is actually included in the proof of Theorem 2.11 (Procesi, \textit{Non-commutative affine rings}, Accad. Lincei, v. VIII (1967), p. 250) which leads to the present result.

\textit{Proof.} If $\rho$ is a monomorphism then clearly $R$ can be embedded in a matrix ring over a commutative ring, e.g., in $M_n(S)$. Conversely, if there exist an embedding $\sigma: R \to M_n(K)$, then since $\sigma = M_n(\eta)\rho$ by Theorem 1 and $\sigma$ is a monomorphism, it follows that $\rho$ is a monomorphism.

The second part follows from the definition of $\rho$. Indeed $\rho = \mu \lambda \psi \varphi^{-1}$ where $\psi: k^\circ[x]/p \to k^\circ[X]/P$ is an epimorphism,

$$\lambda: k^\circ[X]/P \to M_n(\Delta)/\{P\}.$$ 

Thus, $\rho$ is a monomorphism if and only if $\psi$ is an isomorphism and $\lambda$ is a monomorphism. The fact that $\psi$ is an isomorphism means that $k^\circ[X]/P \cong k^\circ[x]/p \cong R$, and that $\lambda$ is a monomorphism is equivalent to saying that $\text{Ker} \lambda_0 = k^\circ[X] \cap \{P\} = P$.

Thus if the condition of our theorem holds for one representation, we can apply this representation to obtain the ring $S$ and so the given $\rho$ will be a monomorphism; but then by the uniqueness of $(S, \rho)$ this will hold in any other way we define an $S$ and an $\rho$. So the fact that $\rho$ is a monomorphism implies that $k^\circ[X] \cap \{P\} = P$ for any other representation of $R$.

A corollary of Theorem 1 (and a similar corollary of Theorem 2) is that

\textbf{Theorem 4.} Every $k$-algebra $R$ contains a unique ideal $Q$ such that $R/Q$ can be embedded in a matrix ring $M_n(K)$ over some commutative ring, and if $R/Q_0$ can be embedded in some $M_n(K)$ the $Q \subsetneq Q_0$.

\textit{Proof.} Let $\rho: R \to M_n(S)$ and set $Q = \text{Ker} \rho$. Then $\rho$ induces a monomorphism of $R/Q$ into $M_n(S)$. If $\sigma$ is any other homomorphism: $R \to M_n(K)$ then by Theorem 1 $M_n(\eta)\rho = \sigma$ so that

$$\text{Ker} (\sigma) \supseteq \text{Ker} (\rho) = Q$$

which proves Theorem 4.
4. Irreducible representations. Let $R$ be a $k$-algebra with a unit $1$ and $k$ be a field. A homomorphism $\varphi: R \to M_n(F)$, $F$ a commutative field, is called an irreducible representation if $\varphi(R)$ contains an $F$-base of $M_n(F)$, or equivalently $\varphi(R)F = M_n(F)$.

**Theorem 5.** Let $\rho: R \to M_n(S)$ be the unitary embedding of $R$ of Theorem 1, then $\rho(R)S = M_n(S)$ if and only if all irreducible representations of $R$ are of dimension $\geq n$, and then all representations of $R$ of dimension $n$ are irreducible.

**Proof.** In view of Theorem 1 it suffices to prove our result for a ring $S = \Delta/I$ obtained by a fixed presentation of $R = k^0[X]/P$ and with $\{P\} = M_n(I)$.

Let $\Omega$ be the field of all rational functions on the $\xi$'s, i.e., the quotient field of $k[\xi] = \Delta$. By a result of Procesi (ibid.), $k^0[X]\Omega = M_n(\Omega)$. Actually it was shown that $k[X]\Omega = M_n(\Omega)$, but since

$$k^0[X]\Omega \subseteq M_n(\Omega)$$

and any identity which holds in $k^0[X]$ will hold also in $M_n(\Omega)$ as such an identity is a relation in generic matrices, it follows that $k^0[X]\Omega$ cannot be a proper subalgebra of $M_n(\Omega)$ since these have different identities. Hence, since every element of $\Omega$ is a quotient of two polynomials in $\xi$ it follows that there exists $0 \neq h$ in $\Delta$ such that $he_{ik} \in k^0[X]\Delta$ where $e_{ik}$ is a matrix base of $M_n(\Omega)$. In particular this implies that $k^0[X]\Delta \supseteq M_n(T)$ for some ideal $T$ in $\Delta$ and, in fact, we choose $T$ to be the maximal with this property.

Next we show that in our case $T + I = \Delta$:

Indeed, if it were not so, then let $m \neq \Delta$ be a maximal ideal in $\Delta$, $m \supseteq T + I$. Let $F = \Delta/m$ and $\sigma$ be the composite homomorphism $\sigma: R \to M_n(\Delta/I) \to M_n(F)$. This representation must be irreducible, otherwise $\sigma(R)F$ is a proper subalgebra (with a unit) of $M_n(F)$ and, therefore, it has an irreducible representation of dimension $< n^2$, or else $\sigma(R)$ is nilpotent but $\sigma(R) = \sigma(R^2)$, thus, $R$ will have representations which contradict our assumption. Hence, $\sigma(R)F = M_n(F)$.

Consider the commutative diagram

$$
\begin{array}{ccc}
\kappa^0[X] & \longrightarrow & M_n(\Delta) \\
\downarrow & & \downarrow \\
R & \longrightarrow & M_n(\Delta/I) \to M_n(F)
\end{array}
$$

where $\sigma$ is the composite of the lower row, and denote by $\tau$ the composite $\tau: k^0[X] \to M_n(\Delta) \to M_n(\Delta/I) \to M_n(F)$. The first vertical

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1 It is sufficient to assume that $R^2 = R$. 
map is an epimorphism hence $\tau(k^\circ[x]) = \sigma(R)$. Consequently, since
$\sigma(R)F = M_n(F)$, there exists a set of polynomials $f_i[X] \in k^\circ[X]$, $\lambda = 1, 2, \cdots, n^2$ such that $\tau(f_i)$ are a base of $M_n(F)$. This is equivalent

 to the statement that the discriminant $\delta = \det(\text{tr}(\tau(f_i)\tau(f_\mu))) \neq 0$,

where $\text{tr} (\cdot)$ is the reduced trace of $M_n(F)$.

Considering $f_i$ as elements of $M_n(A)$ and noting that the reduced
$\text{tr} (\cdot)$ commutes with the specialization $\tau_0: \Delta \to \Delta/I \to F$, it follows that

$$0 \neq \delta = \det(\text{tr}(f_jf_\mu)) = \tau_0(\det(\text{tr}(f_jf_\mu)))$$

and so $\det[\text{tr}(f_jf_\mu)] = D \neq 0$ in $M_n(\Delta) \subseteq M_n(\Omega)$. Hence $\{f_j\}$ is an $\Omega$-
base of $M_n(\Omega)$.

In particular $e_{ik} = \sum f_j[X]u_{i,jk}$ with $u_{i,jk} \in \Omega$. By multiplying each

equation by $f_\mu$ and taking the trace we obtain:

$$\sum \text{tr}(f_\mu f_j)u_{i,jk} = h_{\mu,ik} \in \Delta.$$  

Eliminating these equations by Cramer’s rule we obtain $Du_{i,jk} \in \Delta$

where $D = \det[\text{tr}(f_jf_\mu)]$ which implies

$$De_{ik} = \sum f_j[X] \cdot Du_{j,ik} \in k[X] \Delta.$$  

Namely $D \in T$. This leads to a contradiction, since then $D \in T + I \subseteq m$

and so $D \equiv 0 \pmod{m}$, and so $\tau_0(D) = 0$ under the mapping

$$\tau_0: \Delta \to \Delta/I \to \Delta/m = F$$

but on the other hand $\tau_0(D) = \sigma \neq 0$.

This completes the proof that $T + I = \Delta$. And so

$$M_n(\Delta) = M_n(T) + M_n(I) \subseteq k[X] \Delta + \{P\} \subseteq M_n(\Delta) .$$

Applying $M_n(\eta): M_n(\Delta) \to M_n(S)$ to this equality we obtain

$$M_n(S) = M_n(\eta)M_n(\Delta) = M_n(\eta)(k[X]\Delta) = \rho(R)S$$

since $\eta(I) = 0$, $\eta(\Delta) = S$ and $M_n(k[X]) = \rho(R)$. Thus $M_n(S) = \rho(R)S$.

Conversely, if $M_n(S) = \rho(R)S$, then any homomorphism $\tau: R \to M_n(H)$ is irreducible. Indeed, $\tau = M_n(\eta)\rho$ for some $\eta: S \to H$. Hence

$\tau(R)H \supseteq M_n(\eta)[\rho(R)S] = M_n(\eta)S$. Consequently, $M_n(H) \subseteq M_n(\eta)S \subseteq \tau(R)H \subseteq M_n(H)$ which proves that $\tau$ is irreducible. The rest follows

from the fact that any representation $\tau: R \to M_n(H)$ $m \leq n$ could be followed by an embedding $M_n(H) \to M_n(H)$ and since the composite

$R \to M_n(H)$ must be irreducible we obtain that $n = m$, as required.

**Corollary 6.** If $R$ satisfies an identity of degree $\leq 2n$, then
all irreducible representations of $R$ are exactly of dimension $n$ — if
and only if \( \rho(R)S = M_n(S) \).

Indeed, since the irreducible representation of such an algebra will satisfy identities of the same degree, hence their dimension is anyway \( \leq n^2 \). Thus Theorem 5 yields in this case our corollary.

Another equivalent condition to Theorem 5, is the following:

**Theorem 7.** A ring \( R \) has all its irreducible representation of dimension \( \geq n - 1 \) if and only if there exists a polynomial \( f[x_1, \cdots, x_k] \) such that \( f[x] = 0 \) holds identically in \( M_{n-1}(k) \) and \( f[r_1, \cdots, r_k] = 1 \) for some \( r_i \in R \).

Indeed, let \( R \cong k[k]/\mathfrak{p} \) and let \( \mathfrak{m}_{n-1} \) be the ideal of identities of \( M_{n-1}(k) \). Then \( \mathfrak{p} + \mathfrak{m}_{n-1} = k[x] \), otherwise, there exist a maximal ideal \( \mathfrak{m} \supseteq \mathfrak{p} + \mathfrak{m}_{n-1} \), \( \mathfrak{m} \neq k[x] \). Hence \( k[x]/\mathfrak{m} \) is a simple ring and satisfies all identities of \( M_{n-1}(k) \) so it is central simple of dimension \( < n^2 \). But it yields also an irreducible representation of \( R \) of the same degree, which contradicts our assumption. Thus \( k[x] = \mathfrak{p} + \mathfrak{m}_{n-1} \) and so \( 1 \equiv f[x] \mod \mathfrak{p} \) with \( f \in \mathfrak{m}_{n-1} \) and \( f \) satisfies our theorem.

The converse, is evident, since under any map \( \sigma : M_m(H) \to M_n(k) \), \( m < n \) we must have
\[
\sigma(f[r_1, \cdots, r_k]) = f[\sigma(r_1), \cdots, \sigma(r_k)] = 0
\]
but \( f[r_1, \cdots, r_k] = 1 \). Hence, \( m \geq n \).

**Remark.** Examples of rings satisfying Theorem 5 are central simple algebras of dimension \( n^2 \) over their center, and then \( \rho \) is a monomorphism. Hence the relation \( \rho(R)S = M_n(S) \) means that \( S \) is a splitting ring of \( R \), and in view of Theorem 1, it follows that \( S \) is the uniquely determined splitting ring of \( R \).

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Norman Larrabee Alling, *Analytic and harmonic obstruction on nonorientable Klein surfaces* .................................................. 1

Shimshon A. Amitsur, *Embeddings in matrix rings* ......................... 21

William Louis Armacost, *The Frobenius reciprocity theorem and essentially bounded induced representations* ............................................. 31

Kenneth Paul Baclawski and Kenneth Kapp, *Topisms and induced non-associative systems* ...................................................... 45

George M. Bergman, *The index of a group in a semigroup* ............... 55

Simeon M. Berman, *Excursions above high levels for stationary Gaussian processes* ................................................................. 63

Peter Southcott Bullen, *A criterion for n-convexity* ....................... 81

W. Homer Carlisle, III, *Residual finiteness of finitely generated commutative semigroups* ................................................................. 99

Roger Clement Crocker, *On the sum of a prime and of two powers of two* ......................................................................................... 103

David Eisenbud and Phillip Alan Griffith, *The structure of serial rings* . 109

Timothy V. Fossum, *Characters and orthogonality in Frobenius algebras* ..................................................................................... 123

Hugh Gordon, *Rings of functions determined by zero-sets* ............... 133

William Ray Hare, Jr. and John Willis Kenelly, *Characterizations of Radon partitions* ................................................................. 159

Philip Hartman, *On third order, nonlinear, singular boundary value problems* ............................................................................... 165

David Michael Henry, *Conditions for countable bases in spaces of countable and point-countable type* ......................................... 181

James R. Holub, *Hilbertian operators and reflexive tensor products* .... 185


Erwin Kreyszig, *On Bergman operators for partial differential equations in two variables* ............................................................. 201

Chin-pi Lu, *Local rings with noetherian filtrations* ......................... 209

Louis Edward Narens, *A nonstandard proof of the Jordan curve theorem* ...................................................................................... 219


Hoyt D. Warner, *Finite primes in simple algebras* ............................ 245

Horst Günter Zimmer, *An elementary proof of the Riemann hypothesis for an elliptic curve over a finite field* ......................... 267