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The development of the P^n-integral of R. D. James and W. H. Gage is based on certain properties of n-convex functions. In order to develop this integral systematically a more detailed study of n-convex functions is needed. In the second section of this paper various derivatives are defined and some of their properties given; in the third and last sections properties of n-convex functions are developed.

2. **Definitions and some simple properties of generalized derivatives.** Suppose F is a real-valued function defined on the bounded closed interval $[a, b]$ then if it is true that for $x_0 \in]a, b[$

$$(1) \qquad \frac{F(x_0 + h) + F(x_0 - h)}{2} = \sum_{k=0}^{r} \beta_{2k} \frac{h^{2k}}{(2k)!} + o(h^{2r}), \quad \text{as } h \to 0$$

where $\beta_0, \beta_2 \cdots, \beta_{2r}$ depend on x_0 only, and not on h, then $\beta_{2k}, 0 \leqq k \leqq r$, is called the *de la Vallée Poussin derivative of order* $2k$ *of* F *at* x_0, and we write $\beta_{2k} = D_{2k}F(x_0)$.

If F possesses derivatives $D_{2k}F(x_0), 0 \leqq k \leqq r - 1$, write

$$(2) \qquad \frac{h^{2r}}{(2r)!} \theta_{2r}(F; x_0, h) = \frac{F(x_0 + h) + F(x_0 - h)}{2} - \sum_{k=0}^{r-1} \frac{h^{2k}}{(2k)!} D_{2k}F(x_0)$$

and define

$$(3) \qquad \begin{aligned} \bar{D}_{2r}F(x_0) &= \limsup_{h \to 0} \theta_{2r}(F; x_0, h) , \\ \underline{D}_{2r}F(x_0) &= \liminf_{h \to 0} \theta_{2r}(F; x_0, h) . \end{aligned}$$

F will be said to satisfy *Condition* C_{2r} in $[a, b]$ if and only if

(4)
- (a) F is continuous in $]a, b[$,
- (b) $D_{2k}F$ exists, is finite, and has no simple discontinuities in $]a, b[\ 0 \leqq k \leqq r - 1$,
- (c) $\lim_{h \to 0} h\theta_{2r}(F; x, h) = 0, x \in]a, b[\sim E$, where E is countable.

In particular C_2 requires F to be continuous in $]a, b[$ and smooth in $]a, b[\sim E$.

In a similar way *the de la Vallée Poussin derivatives of odd order* can be defined by replacing (1) by

$$(1)' \qquad \frac{F(x_0 + h) - F(x_0 - h)}{2} = \sum_{k=0}^{r} \beta_{2k+1} \frac{h^{2k+1}}{(2k + 1)!} + o(h^{2r+1}) ,$$

as $h \to 0$, with similar changes in (2), (3) and (4).

If it is true that

$$(5) \qquad F(x_0 + h) - F(x_0) = \sum_{k=1}^{r} \alpha_k \frac{h^k}{k!} + o(h^r), \quad \text{as} \quad h \to 0$$

where $\alpha_1, \cdots, \alpha_r$ depend on x_0 only, and not on h, then $\alpha_k, 1 \leqq k \leqq r$, is called *the Peano derivative of order k of F at x_0*, and we write $\alpha_k = F_{(k)}(x_0)$. If F possesses derivatives $F_{(k)}(x_0), 1 \leqq k \leqq r-1$, write

$$(6) \qquad \frac{h^r}{r!}\gamma_r(F; x_0, h) = F(x_0 + h) - F(x_0) - \sum_{k=1}^{r-1} \frac{h^k}{k!} F_{(k)}(x_0) ,$$

then proceeding as in (3) we define $\bar{F}_{(r)}(x_0)$ and $\underline{F}_{(r)}(x_0)$. Further by restricting h to be positive, or negative, in (5), or (6) we can define *one-sided Peano derivatives*, written $F_{(k),+}(x_0)$, $F_{(k),-}(x_0)$, $\bar{F}_{(k),+}(x_0)$, etc. It is easily seen, [3], that if $F_{(k)}(x_0), 1 \leqq k \leqq r$, exists then

$$(7) \qquad F_{(r)}(x_0) = \lim_{k \to 0} \frac{1}{h^r} \sum_{k=0}^{r} (-1)^k \binom{r}{k} F(x + (r-k)h) .$$

It is shown in [7] that the condition C_n, $n = 2r$ or $2r + 1$, holds automatically for the Peano derivatives. If we say $F_{(k)}, 1 \leqq k \leqq r$, exists in an (a, b) we will mean that $F_{(k)}$ exists in $]a, b[$ and that the appropriate one sided derivates exists at those of the points a and b that are in (a, b).

Let $x_0, \cdots, x_r$ be $(r + 1)$ distinct points from $[a, b]$ then the rth *divided difference of F at these $(r + 1)$ points* is defined by

$$(8) \qquad \begin{aligned} V_r(F) &= V_r(F; x_r) = V_r(F; \{x_k\}) = V_r(F; x_0, \cdots, x_r) \\ &= \sum_{k=0}^{r} \frac{F(x_k)}{w'(x_k)} , \end{aligned}$$

where

$$(9) \qquad \begin{aligned} w(x) &= w_r(x) = w_r(x; x_k), \quad \text{etc.} \\ &= \prod_{k=0}^{r} (x - x_k) . \end{aligned}$$

This rth divided difference has the following properties, which we collect for reference in

LEMMA 1. (a) $V_r(F; x_k) = 0$ *for all choices of points $x_0, \cdots, x_r$ if and only if F is a polynomial of degree at most $r - 1$.*

(b) *If F is a polynomial of degree r then for all $x_0, \cdots, x_r$, $V_r(F; x_k) =$ coefficient of x^r.*

(c) *$V_r(F; x_0, \cdots, x_r)$ is independent of the order of the points $x_0, \cdots, x_r$.*

(d) *There is a simple relation between successive divided differences given by*

$$(10)\qquad \begin{aligned}&(x_0 - x_r)V_r(F; x_0, \cdots, x_r)\\ &= V_{r-1}(F; x_0, \cdots, x_{r-1}) - V_{r-1}(F; x_1, \cdots, x_r) .\end{aligned}$$

(e) *For any F we have the Newton Interpolation Formula,*

$$(11)\qquad \begin{aligned}F(x) = F(x_1) &+ \sum_{k=1}^{r-1} V_k(F; x_1, \cdots, x_{k+1})w_{k-1}(x; x_i)\\ &+ V_r(F; x, x_1, \cdots, x_r)w_{r-1}(x; x_k) .\end{aligned}$$

This last formula can be written differently as follows. Given the $(r+1)$ points P_k, $0 \leqq k \leqq r$, with coordinates $(x_k, F(x_k))$, $0 \leqq k \leqq r$, respectively, there is a unique polynomial of degree at most r passing through these points given by

$$(12)\qquad \begin{aligned}\pi_r(F; x; P_k) &= \pi_r(x; P_k) = \pi_r(x; x_0, x_0, \cdots, x_r), \quad \text{etc.}\\ &= \sum_{k=0}^{r} F(x_k) \prod_{\substack{j=0\\ j\neq k}}^{r} \frac{(x - x_j)}{(x_k - x_j)} .\end{aligned}$$

This formula (12) is known as *the Lagrange Interpolation Formula.* It is easily seen that for all $(r+1)$ distinct $y_0, \cdots, y_r$

$$(13)\qquad V_r(\pi_r; y_k) = V_r(F; x_k) .$$

Then (11) can be written

$$(14)\qquad F(x) = \pi_{r-1}(F; x; x_k) + V_r(F; x, x_1, \cdots, x_r)w_{r-1}(x; x_k) .$$

Using the divided difference we now define another derivative. Suppose all of $x, x_0, \cdots, x_r$ are in $[a, b]$ and

$$(15)\qquad \begin{aligned}&x_k = x + h_k,\ 0 \leqq k \leqq r, \quad \text{with}\\ &0 \leqq |h_0| < \cdots < |h_r| ,\end{aligned}$$

then *the* rth *Riemann derivative of F at x* is defined by

$$(16)\qquad D^r F(x) = \lim_{h_r \to 0} \cdots \lim_{h_0 \to 0} r!\, V_r(F; x_k)$$

if this iterated limit exists independently of the manner in which the h_k tend to zero, subject only to (15). In a similar manner we define the upper and lower derivatives; and if the h_k all have the same sign the one-sided derivatives; these will be written $\bar{D}^r F(x)$, $\bar{D}^r_+ F(x)$, etc. If we say $D^r F$ exists in (a, b) we make the same gloss as for $F_{(r)}$.

Since we can let $h_0, \cdots, h_s$ very first and then $h_{s+1}, \cdots, h_r$ the above definition and (10) imply that if $D^r F(x)$ exists then so does $D^k F(x)$, $1 \leqq k \leqq r$; or more generally if $\bar{D}^r_+ F(x)$ is finite then $\bar{D}^k_+ F(x)$ is finite,

$1 \leqq k \leqq r$. Remark however that even if $D^r_+F(x)$ and $D^r_-F(x)$ exist, are finite and equal, this does not imply that $D^rF(x)$ exists, [15, p. 26].

If instead of (15) and (16) we have

$$(15)' \qquad h_k = (r - 2k)h, 0 \leqq k \leqq r ,$$

$$(16)' \qquad D^r_sF(x) = \lim r!\ V_r(F; x_k) ,$$

(with obvious modifications for the upper and lower derivatives), this is called *the* r^{th} *symmetric Riemann derivative*. In particular the cases $r = 1, 2$ coincide with definitions of D_1F, D_2F respectively. In general if $\bar{D}^r_sF < \infty$ in $]a, b[$ then $F_{(r)}$ exists and equals $\bar{D}^r_sF$ almost everywhere, [12].

The usual rth order derivative of F at x, $x \in (a, b)$, will be written $F^{(r)}(x)$.

THEOREM 2. *If* $x \in [a, b[$ *then* $D^r_+F(x) = F_{(r),+}(x)$, *provided one side exists.*

Proof. Suppose first that $F_{(r),+}(x)$ exists; then taking the rth divided difference of $F(x + h)$, (considered as a function of h) at the points $h_0, h_1, \cdots, h_r$, $0 \leqq h_0 < \cdots < h_r$, using (5) and Lemma 1 (a), (b) we see that

$$r!\ V_r(F; x + h_k) = F_{(r),+}(x) + V_r(o(h^r); h_k) .$$

Letting $h_0, \cdots, h_r$ tend to 0 successively we get that $D^r_+F(x)$ exists and equals $F_{(r),+}(x)$.

If now we suppose that $D^r_+F(x)$ exists then the rest of the theorem follows using Lemma 1(e).

A similar result obviously holds for lefthanded and two-sided derivatives; the latter is due to Denjoy [6] and Corominas [4], who give different proofs.

COROLLARY 3. (a) *If* $x \in [a, b]$ *and* $F_{(k),+}(x)$ *exists* $1 \leqq k \leqq r - 1$ *then* $\bar{F}_{(r)+}(x) = \bar{D}^r_+F(x)$, *and* $\underline{F}_{(r),+}(x) = \underline{D}^r_+F(x)$.

(b) *If* $x \in\ ]a, b[$ *and* $D^kF(x)$ *exists* $1 \leqq k \leqq r - 1$ *and* $D^r_+F(x)$, $D^r_-F(x)$ *exist and are equal then* $D^rF(x)$ *exists, and is equal to this common rule.*

Proof. (a) is proved by a simple adaption of the proof of Theorem 2. (b) holds since the similar result holds for Peano derivatives.

The following results due to Burkill [3], Corominas [4], and Olivier [14] should be noted.

THEOREM 4. (a) *If $F_{(r-1)}$ exists, in $[a, b]$ and if*

$$\inf [\underline{F}_{(r),+}, \underline{F}_{(r),-}] > A > -\infty ,$$

then $F_{(r-1)}$ is continuous.

(b) *If $F_{(r)}$ is continuous in $[a, b]$ then $F^{(r)}$ exists, and $F^{(r)} = F_{(r)}$.*

(c) *If $F_{(r)}$ exists at all points of $[a, b]$ then $F_{(r)}$, possesses both the Darboux property and the mean-value property.*

The definitions of the terms used in (c) can be found in [14].

3. n-convex functions. A real-valued function F defined on the closed bounded interval $[a, b]$ is said to be *n-convex on* $[a, b]$ if and only if for all choices of $(n + 1)$ distinct points, $x_0, \cdots, x_n$, in $[a, b]$, $V_n(F; x_k) \geqq 0$, [4, 7, 15]. If $-F$ is n-convex then F is said to be *n-concave*. The only functions that are both n-convex and n-concave are polynomials of degree at most $n - 1$, (Lemma 1).

If $n = 1$ this is just the class of monotonic increasing functions and $n = 2$ is the class of convex functions; (the class $n = 0$ is just the class of nonnegative functions, but we will usually only be interested in $n \geqq 1$).

THEOREM 5. *Let*

$$P_k = (x_k, y_k), 1 \leqq k \leqq n, n \geqq 2, a \leqq x_1 < \cdots < x_n \leqq b ,$$

be any n distinct points on the graph of the function F. Then F is n-convex if and only if for all such sets of n distinct points, the graph lies alternately above and below the curve $y = \pi_{n-1}(F; x; P_k)$, lying below if $x_{n-1} \leqq x \leqq x_n$. Further $\pi_{n-1}(x; P_k) \leqq F(x)$, $x_n \leqq x \leqq b$; and $\pi_{n-1}(x; P_k) \leqq F(x) (\geqq F(x))$ if $a \leqq x < x_1$, n being even (odd).

Proof. Let $x_0 \neq x_k$, $1 \leqq k \leqq n$, $x_1 < x_0 < x_n$ and suppose in fact $x_j < x_0 < x_{j+1}$. If F is n-convex then $V_n(F; x_0, \cdots, x_n) \geqq 0$; i.e.,

$$\sum_{k=1}^{n} \frac{F(x_k)}{w'_n(x_k)} \geqq -\frac{F(x_0)}{w'_n(x_0)} ,$$

or $F(x_0) \geqq -\sum_{k=1}^{n} F(x_k)[w'_n(x_0)/w'_n(x_k)] = \pi_{n-1}(x_0, P_k)$, if $(n - j)$ is even, but $F(x_0) \leqq \pi_{n-1}(x_0, P_k)$ if $(n - j)$ is odd. This proves the necessity; the sufficiently is immediate by reversing the argument. The last remark follows in a similar way by considering $x_n < x_0 < b$, and $a \leqq x_0 < x_1$.

This theorem generalizes the property that a convex function always lies below its chord.

THEOREM 6. *If F is an n-convex function on $[a, b]$ and*

$$a \leqq x_1 < \cdots < x_n \leqq b,\ a \leqq z_1 < \cdots < z_n \leqq b,\ z_k \leqq x_k,\ 1 \leqq k \leqq n\ ,$$

then $V_{n-1}(F; z_k) \leqq V_{n-1}(F; x_k)$.

Proof. The following particular case suffices to prove this result.

$$x_k = z_k,\ k \neq j+1,\ x_j < z_{j+1} < x_{j+1}\ .$$

Then, as in Theorem 5,

$$\operatorname{sign}[F(z_{j+1}) - \pi_{n-1}(z_{j+1}; x_k)] = (-1)^{n-j}\ .$$

Hence, with this π_{n-1},

$$V_{n-1}(F; z_k) - V_{n-n}(\pi_{n-1}; z_k) = \frac{F(z_{j+1}) - \pi_{n-1}(z_{j+1}; x_k)}{\prod_{\substack{k=1\\ k\neq j+1}}^{n} (z_{j+1} - x_k)} \leqq 0\ .$$

That is

$$\begin{aligned} V_{n-1}(F; z_k) &\leqq V_{n-1}(\pi_{n-1}; z_k) \\ &= V_{n-1}(F; x_r), \quad \text{by (13)}\ . \end{aligned}$$

THEOREM 7. *If F is n-convex in $[a, b]$ then*

(a) *$F^{(r)}$ exists and is continuous in $[a, b]$, $1 \leqq r \leqq n-2$,*

(b) *both $F_{(n-1),-}$, $F_{(n-1),+}$ are monotonic increasing and if*

$$a \leqq x_1 < \cdots < x_n \leqq x \leqq y_1 < \cdots < y_n \leqq b$$

then

$$\begin{aligned} (18) \qquad (n-1)!\ V_{n-1}(F; x_k) &\leqq F_{(n-1),-}(x) \\ &\leqq F_{(n-1),+}(x) \leqq (n-1)!\ V_{n-1}(F; y_k)\ , \end{aligned}$$

(c) $F_{(n-1),+} = (F^{(n-2)})'_+$, $F_{(n-1),-} = (F^{(n-2)})'_-$,

(d) *$F^{(n-1)}$ exists at all except a countable set of points.*

Proof. Using Theorem 2, it is an immediate consequence of Theorem 6 that $F_{(r),+}$ exists in $[a, b[$, $F_{(r),-}$ exists in $]a, b]$, $1 \leqq r \leqq n-1$ and that (b) holds.

From (b) we get that both $F_{(n-1),+}$, $F_{(n-1),-}$ are continuous except on a countable set. Then, again from (b), we have that $F_{(n-1),+} = F_{(n-1),-}$ except on a countable set.

Then if we prove (a) and (c), (d) is immediate.

Suppose $a \leqq x_1 < \cdots < x_n \leqq b$ then repeated application of (10) gives

$$V_{n-1}(F; x_1, \cdots, x_n) = \frac{\dfrac{\dfrac{\dfrac{V_1(F; x_1, x_2) - V_1(F; x_2, x_3)}{x_1 - x_3} - V_2(F; x_2, x_3, x_4)}{(x_1 - x_4)} \cdots}{\cdots\cdots\cdots\cdots}}{(x_1 - x_n)} .$$

Now let $x_1 \to x_2$, then by Theorem 6 the left-hand side of this expression tends to a finite limit, K_1 say: i.e.,

$$K_1(x_2, \cdots, x_n) = \frac{\dfrac{\dfrac{\dfrac{D^1F(x_2) - V_1(F; x_2, x_3)}{(x_2 - x_3)} - V_2(F; x_2, x_3, x_4)}{(x_2 - x_4)} \cdots}{\cdots\cdots\cdots\cdots}}{(x_2 - x_n)} .$$

If now $x_3 \to x_2$ we get a finite limit on l.h.s. of this last expression: hence $D^1_-F(x_2) = D^1_+F(x_2)$; that is $DF(x_2)$ exists. A similar argument shows DF is continuous in $]a, b[$.

In a similar way, expressing V_{n-1} in terms of $V_2, V_3, \cdots$ we show that $D^2_+F(x_3) = D^2_-F(x_3)$ and so by Corollary 3(b), $D^2F(x_3)$ exists then as above D^2F exists and is continuous in $]a, b[$.

In this way we show D^rF exists and is continuous in $]a, b[, 1 \leqq r \leqq n - 2$. Hence, by Theorem 2, $F_{(r)}$ exists and is continuous in $]a, b[, 1 \leqq r \leqq n - 2$ and so finally, by Theorem 4(b), the same is true of $F^{(a)}$. This proves (a).

For the proof of (c) let $x_0 < \cdots < x_{2n-3}$ then repeated application of (10) gives

$$\sum_{k=0}^{n-2} (x_k - x_{k+n-1}) V_{n-1}(F; x_k, \ldots, x_{k+n-1})$$
$$= V_{n-2}(F; x_0, \ldots, x_{n-2}) - V_{n-2}(F; x_{n-0}, \ldots, x_{2n-3}) .$$

Let $x_k \to x_0, 1 \leqq k \leqq n - 2, x_k \to x_{n-1}, n \leqq k \leqq 2n - 3$ then by Theorem 6 the limit on the left hand side exists, and the value limit on the right hand side follows from (a). Thus we get an expression of the form

$$(n - 1)(x_0 - x_{n-1})K(x_0, x_{n-1}) = \frac{1}{(n - 2)!}\{F^{(n-2)}_{(x_0)} - F^{(n-2)}_{(x_{n-1})}\} .$$

Now dividing and letting $x_{n-1} \to x_0$ we get

$$(n - 1)! \lim_{x_{n-1} \to x_0+} K(x_0, x_{n-1}) = (F^{(n-2)})'_+(x_0) ;$$

a simple application of (11) shows that the left hand side of this last expression is equal to $F_{(n-1),+}(x_0)$. This completes the proof of the first part of (c), the rest follows using a similar argument.

Formula (18) is due to James [7, Lemma 10.4], who however assumes the existence of $F_{(n-1)}$ in $]a, b[$.

COROLLARY 8. (a) *F is n-convex on $[a, b]$ if and only if F differs by a polynomial of degree at most $(n-1)$ from $\int_a^x (x-t)^{n-1}\mu(dt)$, for some Lebesgue-Stieltjes measure μ. In particular if and only if F is the $(n-1)$st integral of a monotonic function.*

(b) *If F is n-convex in $[a, b]$, $|F| \leqq k$, then $|F_{(k)}(x)| \leqq AK \sup \{1/(b-x)^k, 1/(x-a)^k\}$, $0 \leqq k \leqq n-1$ where A is a constant independent of k, F and x, and where if $k = n-1$ the derivative is to be interpreted as* $\sup(|F_{(n-1),+}(x)|, |F_{(n-1),-}(x)|)$.

(c) *If F is n-convex on $[a, b]$, $a \leqq x \leqq y \leqq b$, $a \leqq x + h \leqq y$, and $x \leqq y + k \leqq b$ then*

$$\gamma_{n-1}(F; x; h) \leqq F_{(n-1),-}(y) \quad \textit{and} \quad F_{(n-1),+}(x) \leqq \gamma_{n-1}(F; y; k) .$$

Proof. (a) This is immediate from Theorem 7 (b).

(b) From (18) we have that

$$\frac{1}{(n-1)!}\sum_{k=0}^{n-1}\frac{F(x_k)}{w'(x_k)} \leqq \sup\{F_{(n-1),+}(x), F_{(n-1),-}(x)\} \leqq \frac{1}{(n-1)!}\sum_{k=0}^{n-1}\frac{F(y_k)}{w'(y_k)}$$

from which (b) in the case $k = n-1$ is easily deduced. The rest follows by integration, using, (a).

(c) Immediate using (18), (11), (6) Theorems 2 and 4.

The definition, (12), of $\pi_r(x; P_k)$ can be extended to cover the case when not all of the P_k are distinct. Thus if only s of these points are distinct then besides giving the values at the s points, a total of $r+1-s$ derivatives must also be given—either $r+1-s$ derivatives all at one point, or $r+1-s$ first derivatives at $r+1-s$ distinct points, (when $r+1-s \leqq s$), etc. Theorem 5 can be extended, using Theorems 6, 7 and taking limits; thus as an example of many possible extensions we state

THEOREM 9. *Let $P_k = (x_k, y_k)$, $1 \leqq k \leqq r$, $a \leqq x_1 < \cdots < x_r \leqq b$, be r distinct points on the graph of the function F. Suppose that $F_{(s),+}(x_1)$ exists, $1 \leqq s \leqq n-r$. Then Theorem 5 holds if $\pi_{n-1}(x; P_k)$ is taken to have $\pi_{n-1}(x_s; P_k) = F(x_s)$, $1 \leqq s \leqq r$, $\pi_{n-1}^{(r)}(x_1; P_k) = F_{(s),+}(x_1)$, $1 \leqq s \leqq n-r$, and if P_1 is considered as $n-r+1$ points at and to the right of P_1 but to the left of P_2.*

THEOREM 10. (a) *If F is n-convex on $[a, b]$ and $P_k = (x_k, y_k)$, $1 \leqq k \leqq n$ are n distinct points on the graph of F, $a \leqq x_1 < b$, let*

$x_k = x_1 + \varepsilon_k h$, $0 < \varepsilon_2 < \cdots < \varepsilon_n$; *then as* $h \to 0+$, $\pi_{n-1}(x; P_k)$ *converges uniformly to the right tangent polynomial at* x_1,

$$(19)\qquad \begin{aligned} \tau_{n,+}(F; x; x_1) = \tau_+(x) = F(x_1) + \sum_{k=1}^{n-2} \frac{(x - x_1)^k}{k!} F^{(k)}(x_1) \\ + \frac{(x - x_1)^{n-1}}{(n-1)!} F_{(n-1),+}(x_1),\ x_1 \leqq x \leqq b\ . \end{aligned}$$

Further on the right of x_1, $\tau_+ \leqq F$.

(b) *A similar result holds for the left tangent polynomial at* x_1, $\tau_-(x; x_1)$, $a \leqq x \leqq x_1$, $a < x_1 \leqq b$. *However in this case if* n *is even (odd) then on the left of* x_1, $\tau_- \leqq F (\geqq F)$.

(c) *At all but a countable set of points* x_1, *a similar result holds for the tangent polynomial at* x_1, $\tau(x_1; x)$, $a < x < b$, $a < x_1 < b$. *However if* n *is even the graph of* τ *lies below that of* F, *whereas if* n *is odd the graphs cross,* τ *being above on the left of* x_1, *and below on the right of* x_1.

Proof. It suffices to consider (a). But (a) is a simple consequence of Theorems 5, 7, (11), and (14).

COROLLARY 11. (a) *If* F *is* n-*convex in* $[a, b]$ *then*

$$\inf \{\underline{F}_{(n),+}, \underline{F}_{(n),-}\} \geqq 0\ .$$

(b) *If* F *is* n-*convex in* $[a, b]$ *and* $F_{(n-1)}$ *exists in* $[a, b]$ *then it is continuous.*

(c) *If* F *is* n-*convex in* $[a, b]$ *then* $F_{(n-1),+}$ *is upper-semi continuous* (u.s.c.), $F_{(n-1),-}$ *is lower semi-continuous* (l.s.c.).

Proof. (a) Suppose in Theorem 10, for simplicity, that $x_1 = 0$. Then F lies above the right tangent polynomial at $x = 0$, i.e.,

$$\frac{F(x) - \tau_+(x)}{x^n} \geqq 0\ ,$$

in some interval $[0, h]$. Hence $\underline{F}_{(n),+}(0) \geqq 0$: in a similar way $\underline{F}_{(n),-}(0) \geqq 0$.

(b) Immediate from (a), Theorem 4(a), Theorem 7(a).

(a) This is just Theorem 3.2 [3], adapted to one sided derivatives.

The following theorem generalizes a result well known when $n = 1$, [13, Corollary 32.3] and $n = 2$ [7, Th. 4].

THEOREM 12. *If* F *is* n-*convex on* $[a, b]$, $a < \alpha < \beta < b$, $E_k = \{x; \alpha \leqq x \leqq \beta$ *and* $\bar{F}_{(n)}(x) \geqq k\}$ *then*

$$(20)\qquad km^*(E_k) \leqq 2n\{F_{(n-1),-}(\beta) - F_{(n-1),+}(\alpha)\}$$

(*where* m^* *denotes the outer Lebesgue measure*).

Proof. For simplicity we will ignore the countable set where $F_{(n-1)}$ may not exist and suppose that $k > 0$. Further let E_k^+ be as E_k but with $\bar{F}_{(n),+}$ instead of $\bar{F}_{(n)}$ and suppose $m^*E_k^+ > 0$; with a similar definition for E_k^-.

If then $\varepsilon > 0$, $x \in E_k^+$ there is an $h > 0$ such that

$$\gamma_n(F; x; h) \geqq \bar{F}_{(n),+}(x) - \varepsilon \geqq k - \varepsilon .$$

So, by [20], there is a finite family of nonoverlapping intervals $[x_i, x_i + h_i]$, $i = 1, \cdots, p$ such that $x_p + h_p \leqq \beta$,

$$\gamma_n(F; x_i, h_i) \geqq k - \varepsilon, i = 1, \cdots, p ,$$

and

$$\sum_{i=1}^{p} h_i \geqq m^*E_k^+ - \varepsilon .$$

Thus

$$\sum_{i=1}^{p} h_i\gamma_n(F; x_i, h_i) \geqq (k - \varepsilon)(m^*E_k^+ - \varepsilon) ;$$

but since

$$h\gamma_n(F; x, h) = n\{\gamma_{n-1}(F: x, h) - F_{(n-1)}(x)\} \tag{21}$$

we have that

$$\sum_{i=1}^{p} \{\gamma_{n-1}(F; x_i, h_i) - F_{(n-1)}(x_i)\} \geqq \frac{k - \varepsilon}{n}(m^*E_k^+ - \varepsilon) .$$

However by Corollary 8(c)

$$\sum_{i=1}^{p-1} \{F_{(n-1)}(x_{i+1}) - \gamma_{n-1}(F; x_i, h_i)\} \geqq 0 ,$$
$$F_{(n-1)}(x_i) - F_{(n-1)}(\alpha) \geqq 0,$$
$$F_{(n-1)}(\beta) - \gamma_{n-1}(F; x_p, h_p) \geqq 0 .$$

Adding the last four inequalities we get that

$$F_{(n-1)}(\beta) - F_{(n-1)}(\alpha) \geqq \frac{k - \varepsilon}{n}(m^*E_k^+ - \varepsilon) .$$

This together with a similar inequality for E_k^-, implies (20).

A function that is the difference of two n-convex functions will be called *δ-n-convex*; as in the cases $n = 1$ and $n = 2$, [16], such functions can be characterized by their variational properties.

If F is defined on $[a, b]$ as well as $F_{(k)}, 1 \leqq k \leqq n-1$, let us write

$$\omega_n(F; a, b) = \omega_n(a, b) = \max\{\sup_{a<x<b} |(x-a)\gamma_n(F; a; x-a)|, \sup_{a<x<b} |(b-x)\gamma_n(F; a; b-x)|\} ;$$

this quantity was introduced by Sargent [19].

THEOREM 13. *A function F defined on $[a, b]$ is δ-n-convex if and only if either of the following conditions is satisfied.*

(a) $\sum_{k=1}^{m} \omega_n(F; a_k, b_k) < K$ *for all finite sets of nonoverlapping intervals,* $[a_k, b_k], 1 \leqq k \leqq m$.

(b) $\sum_{k=0}^{m} |(x_k - x_{k+n}) V_n(F; x_k, \cdots, x_{k+n})| < K$ *for all finite sets of distinct points* $x_0, \cdots, x_{m+n}$.

Proof. The discussion of (*b*) is similar to the case $n = 2$ in [16] but using Corollary 8(a).

If (a) is satisfied then $F_{(n-1)}$ is of bounded-variation [19, Lemma 1], and so by Corollary 8(a) F is δ-n-convex.

If F is n-convex then by (21) and Corollary 8(c),

$$(x-a)\gamma_n(F; a; x-a) = n\{\gamma_{n-1}(F; a; x-a) - F_{(n-1)}(a)\} \geqq 0$$

and so by Corollary 8(c)

$$\omega_n(F; a, b) \leqq n\{F_{n-1}(b) - F_{(n-1)}(\mathrm{a})\} . \tag{22}$$

From this it easily follows that if F is δ-n-convex then (a) holds.

4. Sufficient conditions for n-convexity. In this section we obtain some sufficient conditions for a function to be n-convex. First we prove the following generalization of a well-known property of convex functions.

THEOREM 14. (a) *If F is n-convex in $[a, b]$ then $F^{(n-2)}$ has no proper maximum in $]a, b[$.*

(b) *A function F with continuous derivative of order $(n-2)$ is n-convex if and only if no function of the form $F(x) + \sum_{k=0}^{n-1} a_k x^k$ has its derivative of order $(n-2)$ attaining a maximum in $]a, b[$.*

Proof. (a) Suppose $F^{(n-2)}$ has a proper maximum at x_0, then consider $G(x) = F(x) - \pi_{n-2}(x; P_k)$, where the polynomial π_{n-2} is determined uniquely by the conditions

$$G(x_0) = G'(x_0) = \cdots = G^{(n-2)}(x_0) = 0 .$$

Now consider $\pi_{n-2}(x; Q_k)$ where $Q_k = (x_k, G(x_k)), 0 \leqq k \leqq n-2$, $x_0 < \cdots < x_{n-2}$. Then by Theorem III [4], (13), and Lemma 1(b), the coefficient of x^{n-2} in $\pi_{n-2}(x; Q_k)$ is $G^{(n-2)}(x_0 + \delta)$, $x_0 + \delta$ being some point in $]x_0, x_{n-2}[$. Hence, using Theorem 7(a), since x_0 is a proper maximum of $G^{(n-2)}$ and $G^{(n-2)}(x_0) = 0$, if $x_0, \cdots, x_{n-2}$ are close enough together this coefficient is not positive.

Let $x_k \to x_0$, $1 \leqq k \leqq n-3$ then $\pi_{n-2}(x; Q_k)$ becomes a polynomial of degree $n-2$ with its value and that of its first $(n-3)$ derivatives at x_0 being zero; it's $(n-2)$nd derivative is nonpositive. Hence, by Theorem 9, $G \leqq 0$ in $[x_0, x_{n-2}]$.

In a similar way $G \geqq 0 (\leqq 0)$ in some interval to the left of x_0 when n is odd (even). Further in every such interval around x_0 there are points where these inequalities are strict.

Now consider the $(n+1)$ points $z_0, \cdots, z_n$ where

$$z_0 < z_1 \cdots < z_{[n/2]} = x_0 < \cdots < z_n .$$

Then

$$V_n(F; z_k) = V_n(G; z_k) = \frac{G(z_0)}{w_n'(z_0)} + F + \frac{G(z_n)}{w_n'(z_n)} \geqq 0 .$$

If then $z_1, \cdots z_{n-1}$ tend to x_0 then $K \to 0$ and we get

$$\frac{G(z_0)}{(z_0 - x_0)^{n-1}(z_0 - z_n)} + \frac{G(z_n)}{(z_n - x_0)^{n-1}(z_n - z_0)} \geqq 0 .$$

But whether n is even, or odd both terms on the l.h.s. of this expression can be chosen to be negative-which contradiction completes the proof of (a).

(b) The necessity is evident. Suppose then that F is not n-convex. Then by Theorem 5 there exists a polynomial $\pi_{n-1}(x; P_k)$ such that the two curves $y = F(x)$, $y = \pi_{n-1}(x; P_k)$ do not intertwine correctly.

Consider $G(x) = F(x) - \pi_{n-1}(x; P_k)$; then $G(x_1) = \cdots = G(x_n) = 0$ and G changes sign at most $(n-2)$ times. Hence $G^{(n-2)}$ has three zeros and so has a local maximum. This completes the proof.

COROLLARY 15. (a) *If F is n-convex then $F^{(r)}$ is $(n-r)$-convex,* $1 \leqq r \leqq n-2$.

(b) *If F is n-convex then $F^{(n)}$ exist a.e.*

Proof. (a) The case $r = n-2$ is just Theorem 14(b). In general $F^{(k)}$, $1 \leqq k \leqq n-3$, has a continuous derivative of order $n-k-2$ satisfying the hypotheses of Theorem 14(b), and hence $F^{(k)}$ is $(n-k)$-convex.

(b) Since $F^{(n-2)}$ is convex this follows immediately from well known properties of convex functions.

Note that the case $r = n - 1$ of Corollary 15(a) is just the last part of Theorem 7(b).

We now wish to prove a converse of Corollary 11(a). Because of applications to symmetric Perron integral, [7], this converse will be obtained in terms of de la Vallée Poussin derivatives and the results in terms of Peano derivatives will be simple corollaries. A direct proof could be constructed from the proof of the more general results.

THEOREM 16. *If F satisfies C_{2m}, $m \geqq 1$, in $]a, b[$ and*

(a) $\bar{D}_{2m}F(x) \geqq 0$, $x \in]a, b[\sim E$, $|E| = 0$,

(b) $\bar{D}_{2m}F(x) > -\infty$, $x \in]a, b[\sim S$, *S a scattered set,*

(c) $\limsup_{h\to 0} h\theta_{2m}(F; x; h) \geqq 0 \geqq \liminf_{h\to 0} h\theta_{2m}(F; x; h)$, $x \in S$ *then F is $2m$-convex.* (*A set is said to be scattered if it contains no subsets that are dense in themselves.*)

Proof. If $E = S$ then by Theorem 6.1, [9], (a), (b), (c) imply $\bar{D}_{2m}F \geqq 0$ in $]a, b[$ and so the result follows from Theorem 4.2, [8].

Given $\varepsilon > 0$, T, $|T| = 0$, $T \in G_\delta$, $T \neq \varnothing$ let $\chi_{\varepsilon,T} = \chi$ be a function on $[a, b]$ such that

(i) χ is absolutely continuous,

(ii) χ is differentiable,

(iii) $\chi'(x) = \infty$, $x \in T$,

(iv) $0 \leqq \chi'(x) < \infty$, $x \notin T$,

(v) $\chi(a) = 0$, $0 \leqq \chi(b) \leqq \varepsilon/(b - a)^{2m-1}$. That such a function exists is well known, [21]. Then define

$$\Psi_{\varepsilon,T,2m}(x) = \Psi(x) = \frac{1}{(2m-2)!}\int_a^x (x-t)^{2m-2}\chi(t)dt\ , \tag{23}$$

the $(2m-1)st$ integral of χ. Then $\Psi^{(2m-1)}(x) = \chi(x)$ and, using (2), we have on integrating by parts that

$$\begin{aligned} \frac{h^{2m}}{2m!}\theta_{2m}(\Psi; x; h) &= \frac{1}{2(2m-2)!}\int_0^h (h-t)^{2m-2}\{\chi(x+t) - \chi(x-t)\}dt \\ &\geqq \frac{1}{2(2m-1)!}\chi'(x)\cdot h^{2m}\ , \end{aligned} \tag{24}$$

so

$$\underline{D}_{2m}\Psi(x) \geqq m\chi'(x) \geqq 0\ .$$

If now $E \subset T$ then we easily see that (i) Ψ is C_{2m}, and $2m$-convex, (ii)

$\underline{D}_{2m}\Psi(x) \geqq 0$, (iii) $\underline{D}_{2m}\Psi(x) = \infty$, $x \in E$, (iv) $0 \leqq \Psi \leqq \varepsilon$.

Hence if we write $\Psi_n = \Psi_\varepsilon$, with $\varepsilon = 1/n$, and put $G_n = F + \Psi_n$ then G_n satisfies the conditions of the theorem with $E = S$, and so by the above is $2m$-convex. Letting $n \to \infty$ we then get that F is $2m$-convex.

The case of $m = 1$, $E = \varnothing$, S countable is a classic result about convex functions, [22].

COROLLARY 17. *If* F, G *are defined in* $[a, b]$ *and* (a) $F - G$ *is* C_{2m}, (b) $\bar{D}_{2m}(F - G)(x) \geqq 0 \geqq \underline{D}_{2m}(F - G)(x)$ *for* $x \in]a, b[\sim E$, $|E| = 0$, (c) $\underline{D}_{2m}(F - G)(x) < \infty$, $\bar{D}_{2m}(F - G)(x) > -\infty$, $x \in]a, b[\sim S$, *S scattered*, (d) $\limsup_{h \to 0} h\theta_{2m}(F - G; x; h) \geqq 0 \geqq \liminf_{h \to 0} h\theta_{2m}(F - G; x; h)$ *for* $x \in S$ *then for all sets* $x_1, \cdots, x_{2m}$ *of* $2m$ *distinct points in* $[a, b]$, *if* $P_k = (x_k, F(x_k))$, $Q_k = (x_k, G(x_k))$, $1 \leqq k \leqq 2m$

$$F(x) - \pi_{2m-1}(x; P_k) = G(x) - \pi_{2m-1}(x; Q_k) . \tag{25}$$

Proof. If F_1, G_1, denote the l.h.s., r.h.s., of (25) respectively then $F_1 - G_1$ is both $2m$-convex and $2m$-concave, by Theorem 16. So being a polynomial of degree at most $2m - 1$ and vanishing at x_k, $1 \leqq k \leqq 2m$, is identically zero.

This result is well known in the case $m = 1$ when it implies that if $F - G$ is continuous, $D_2(F - G) = 0$ then F, G differ by a linear function, [10]. Kassimatis [11] pointed out that the requirement $F - G$ continuous is not sufficient in the general case; the condition required is that of Corollary 17.

COROLLARY 18. (a) *If* $n \geqq 2$ (i) $\bar{F}_{(n)}(x) \geqq 0$, $x \in]a, b[\sim E$, $|E| = 0$, (ii) $\bar{F}_{(n)}(x) > -\infty$, $x \in]a, b[\sim S$, *S a scattered set, then F is n-convex.*

(b) *If* $n \geqq 2$ (i) $\overline{(F - G)}_{(n)}(x) \geqq 0 \geqq \underline{(F - G)}_{(n)}(x)$, $x \in]a, b[\sim E$, $|E| = 0$, (ii) $\underline{(F - G)}_{(n)}(x) < \infty$, $\overline{(F - G)}_{(n)}(x) > -\infty$, $x \in]a, b[\sim S$, *S scattered, then* (25) *holds.*

Proof. This is an immediate corollary of Theorem 16, Corollary 17, the analogous results for the odd-ordered derivatives and the remark made earlier that C_n is satisfied.

This result generalizes the classic case, when $n = 1$, see for instance, [17, p. 203]. But this can be still further extended as follows.

THEOREM 19. *If* $n \geqq 2$, *and* (i) $F_{(n-1)}$ *exists in* $[a, b]$, (ii) $\bar{F}_{(n),+}(x) \geqq 0$, $x \in [a, b] \sim E$, $|E| = 0$, (iii) $\bar{F}_{(n),+}(x) > -\infty$, $x \in [a, b] \sim C$, *C countable, then F is n-convex.*

Proof. As in the proof of Theorem 16 we can assume that $E = C$ and so suppose $\bar{F}_{(n),+}(x) \geqq 0$ except when $x = x_0, x_1, \cdots$. We may assume that for all $k \in N$, $x_k \neq b$.

Adopting a procedure due to Bosanquet [1] and Sargent [18] we exhibit for each $k \in N$ a monotonic n-convex function Z_k with the following properties

(i) $Z_k^{(r)}(a) = 0$, $Z_k^{(r)}(b) \leqq [(b-a)^{n-r-1}/(n-r-1)!]2^{-(k+1)}\varepsilon$, $0 \leqq r \leqq n-1$,

(ii) $\overline{(F+Z_k)}_{(n),+}(x_k) \geqq 0$,

(iii) $V_n(Z_k; y_r) \leqq K2^{-(k+1)}\varepsilon$, for all $(n+1)$ distinct points $y_0, \cdots, y_n$.

Then if we define $G(x) = F(x) + \sum_{k \in N} Z_k(x)$, $G_{(n),+}(x) \geqq 0$ everywhere and so is n-convex, by usual arguments; but

$$V_n(G; y_r) = V_n(F; y_r) + \sum_{k \in N} V_n(Z_k; y_r)$$

and so $V_n(F; y_r) \geqq -K\varepsilon$, which implies F is n-convex.

It remains to define the function Z_k. Since C_n is satisfied, we have, by (4) and (6), $\lim_{n\to 0} h\gamma_n(F; x_k; h) = 0$ so we can find a sequence $y_1, y_2, \cdots$ in $[x_k, b[$ such that $0 < y_{s+1} - x_k = h_{s+1} < \frac{1}{2}(y_s - x_k) = h_s/2$, and $h_s\gamma_n(F; x_k; h_s) > -\varepsilon \cdot 2^{-(k+s)}$. Now define the function z_k in such a way as to be continuous and

$$\begin{aligned} z_k(x) &= 0,\ a \leqq x \leqq x_k, \\ &= 2^{-(k+1)}\varepsilon,\ y_1 < x \leqq b, \\ &= 2^{-(k+s)}\varepsilon,\ x = y_s,\ s = 1, 2, \cdots, \\ &= \text{linear in } [y_{s+1}, y_s],\ s = 1, 2, \cdots . \end{aligned}$$

Then z_k is continuous, increasing on $[a, b]$, $z_k(a) = 0$, $z_k(b) = 2^{-(k+1)}\varepsilon$, $z_k(x_k) = 0$, $z_k(x)/x - x_k$ decreases in $]x_k, b[$. It is then easily checked that

$$\int_0^{h_s} (h_s - t)^{n-2} z_k(x_k + t)dt \geqq \frac{z_k(y_s)h_s^{n-1}}{n(n-1)} = \frac{2^{-(k+s)}h_s^{n-1}\varepsilon}{n(n-1)} .$$

Define then,

$$Z_k(x) = \frac{1}{(n-2)!}\int_a^x (x-t)^{n-2} z_k(t)dt ,$$

the $(n-1)$st integral of z_k. Then $Z_k^{(n-1)} = z_k$ and using Theorem 7, and Corollary 8, Z_k clearly has all properties wanted except possibly (ii). This we now check. First note that by (21)

$$h_s\gamma_n(Z_k; x_k, h_s) = n\gamma_{n-1}(Z_k; x_k, h_s) .$$

So as in the proof of (23),

$$h_s\gamma_n(Z_k; x_k, h_s) = n\frac{(n-1)}{h_s^{n-1}}\int_0^{h_s}(h_s - t)^{n-2}z_k(x_k + t)dt \geqq 2^{-(k+s)}\varepsilon .$$

Hence,

$$h_s\gamma_n(Z_k + F; x_k, h_s) \geqq 0$$

which completes the proof.

A theorem of a slightly different form can be obtained using the symmetric Riemann derivatives.

Let us say a real valued function F on $[a, b]$ is of *type* D_r if for all sets of $(r+1)$ distinct points $x_0, \cdots, x_r$ in $[a, b]$

$$\inf_{a<x<b} \bar{D}_s^r F(x) \leqq r!\, V_r(F; x_k) \leqq \sup_{a<x<b} \underline{D}_s^r F(x) . \tag{26}$$

The following simple lemmas will be useful.

LEMMA 20. *If* $F^{(r-2)}$ *exists and is continuous in* $[a, b]$ *then for sets of* $(r+1)$ *distinct points* $x_0, \cdots, x_r$ *in* $[a, b]$

$$\inf_{a<x<b} \bar{D}_s^2 F^{(r-2)}(x) \leqq r!\, V_r(F; x_k) \leqq \sup_{a<x<b} \underline{D}_s^2 F^{(r-2)}(x) .$$

In particular if $F^{(r)}$ *exists in* $[a, b]$ *then* F *is of type* D_r.

Proof. Let $G(x) = F(x) - \pi_{r-1}(F; x_0, \cdots, x_{r-1}) - \lambda P(x)$ where P is a polynomial of degree r, λ a constant determined by requiring that $G(x_k) = 0$, $0 \leqq k \leqq r$ and $V_r(F; x_k) = \lambda$.

Then since G has at least $(r+1)$ zeros $G^{(r-2)}$ has at least 3 zeros and so has a nonnegative maximum; that is for some y $V_2(G^{(r-2)}; y_1, y, y_2) \leqq 0$ for all y_1, y_2 near enough to y; that is

$$2 \cdot V_2(G^{(r-2)}; y_1, y, y_2) = 2V_2(F^{(r-2)}; y_1, y, y_2,) - r!\,\lambda \leqq 0 .$$

The proof now follows that in [6].

LEMMA 21. *If* F *is of type* D_n *then*

$$\inf_{a<x<b} \bar{D}_s^n F(x) = \inf_{a<x<b} \underline{D}_s^n F(x),\ \sup_{a<x<b} \bar{D}_s^n F(x) = \sup_{a<x<b} \underline{D}_s^n F(x) .$$

Proof. The case $n = 2$ and more is proved in [6, p. 9]. The proof of the general case is the same.

THEOREM 22. *If* F *is of type* D_n *and* (a) $\bar{D}_s^n F(x) \geqq 0$, $x \in]a, b[\sim E$, $|E| = 0$, (b) $\bar{D}_s^n F > -\infty$, *then* F *is* n*-convex.*

Proof. Since the $2m$-convex function Ψ of Theorem 16 is, using

Lemma 20, of type D_{2m} we can, as in Theorem 16, assume $E = \emptyset$. The result is then a trivial consequence of (26).

COROLLARY 23. *If* F, G *are such that* (a) $F - G$ *is of type* D_n, (b) $\bar{D}_s^n(F-G)(x) \geqq 0 \geqq \underline{D}(F-G)(x)$, $x \in]a, b[\sim E$, $|E| = 0$, (c) $\bar{D}_s^n(F-G) > -\infty$, $\underline{D}_s^n(F-G) < \infty$, *then* (24) *holds.*

It would be of interest to produce some reasonable conditions on F that ensure it is of type D_r. It is known, [15], that if F is continuous then F is of type D_2, but Kassimatis, [10], has pointed out that if $r > 2$ this is false. One would expect the existence and continuity of $F^{(r-2)}$ to imply F is of type D_r but this has not been proved. Let us say F is of *type* d_r when

$$\inf_{a<x<b} \underline{D}_s^r F(x) \leqq r!\ V_r(F; x_k) \leqq \sup_{a<x<b} \bar{D}_s^r F(x) .$$

If in Theorem 22 and Corollary 23 we weaken our hypothesis to F being of type d_n, obvious modifications of the other conditions will produce analogous theorems. It has been proved in [2] that if $F^{(r-2)}$ exists and is continuous, $r = 2, 3, 4$, then F is d_r; unfortunately the method fails if $r > 4$.

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