A CRITERION FOR $n$-CONVEXITY

Peter Southcott Bullen
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P. S. BULLEN

The development of the \( P^n\)-integral of R. D. James and W. H. Gage is based on certain properties of \( n\)-convex functions. In order to develop this integral systematically a more detailed study of \( n\)-convex functions is needed. In the second section of this paper various derivatives are defined and some of their properties given, in the third and last sections properties of \( n\)-convex functions are developed.

2. Definitions and some simple properties of generalized derivatives. Suppose \( F \) is a real-valued function defined on the bounded closed interval \([a, b]\) then if it is true that for \( x_0 \in ]a, b[\)

\[
\frac{F(x_0 + h) + F(x_0 - h)}{2} = \sum_{k=0}^{r} \beta_{2k} \frac{h^{2k}}{(2k)!} + o(h^{2r}), \quad \text{as } h \to 0
\]

where \( \beta_0, \beta_1, \ldots, \beta_{2r} \) depend on \( x_0 \) only, and not on \( h \), then \( \beta_{2k}, 0 \leq k \leq r \), is called the \textit{de la Vallée Poussin derivative of order} \( 2k \) of \( F \) at \( x_0 \), and we write \( \beta_{2k} = D_{2k}F(x_0) \).

If \( F \) possesses derivatives \( D_{2k}F(x_0), 0 \leq k \leq r - 1 \), write

\[
\frac{h^{2r}}{(2r)!} \theta_r(F; x_0, h) = \frac{F(x_0 + h) + F(x_0 - h)}{2} - \sum_{k=0}^{r-1} \frac{h^{2k}}{(2k)!} D_{2k}F(x_0)
\]

and define

\[
\bar{D}_{2r}F(x_0) = \limsup_{h \to 0} \theta_2(F; x_0, h),
\]

\[
D_{2r}F(x_0) = \liminf_{h \to 0} \theta_2(F; x_0, h).
\]

\( F \) will be said to satisfy \textit{Condition} \( C_{2r} \) in \([a, b]\) if and only if

\begin{enumerate}
  \item \( F \) is continuous in \([a, b]\),
  \item \( D_{2k}F \) exists, is finite, and has no simple discontinuities in \([a, b]\), \( 0 \leq k \leq r - 1 \),
  \item \( \lim_{k \to 0} \theta_{2r}(F; x, h) = 0 \), \( x \in ]a, b[ \sim E \), where \( E \) is countable.
\end{enumerate}

In particular \( C_2 \) requires \( F \) to be continuous in \([a, b]\) and smooth in \([a, b]\sim E \).

In a similar way \textit{the de la Vallée Poussin derivatives of odd order} can be defined by replacing (1) by

\[
\frac{F(x_0 + h) - F(x_0 - h)}{2} = \sum_{k=0}^{r} \beta_{2k+1} \frac{h^{2k+1}}{(2k + 1)!} + o(h^{2r+1}),
\]
as $h \to 0$, with similar changes in (2), (3) and (4).

If it is true that

\[ F(x_0 + h) - F(x_0) = \sum_{k=1}^{r} \alpha_k \frac{h^k}{k!} + o(h^r), \quad \text{as } h \to 0 \]

where $\alpha_1, \ldots, \alpha_r$ depend on $x_0$ only, and not on $h$, then $\alpha_k, 1 \leq k \leq r$, is called the Peano derivative of order $k$ of $F$ at $x_0$, and we write $\alpha_k = F_{(k)}(x_0)$. If $F$ possesses derivatives $F_{(k)}(x_0), 1 \leq k \leq r - 1$, write

\[ \frac{h^r}{r!} \gamma_r(F; x_0, h) = F(x_0 + h) - F(x_0) - \sum_{k=1}^{r-1} \frac{h^k}{k!} F_{(k)}(x_0), \]

then proceeding as in (3) we define $F_{(r)}(x_0)$ and $F_{(r)}(x_0)$. Further by restricting $h$ to be positive, or negative, in (5), or (6) we can define one-sided Peano derivatives, written $F_{(k),+}(x_0), F_{(k),-}(x_0), F_{(k),+}(x_0), etc.\, It\, is\, easily\, seen, \,[3],\, that\, if\, F_{(k)}(x_0), 1 \leq k \leq r, \, exists\, then

\[ F_{(r)}(x_0) = \lim_{h \to 0} \frac{1}{h^r} \sum_{k=0}^{r} (-1)^k \binom{r}{k} F(x + (r - k)h). \]

It is shown in [7] that the condition $C_n, n = 2r$ or $2r + 1$, holds automatically for the Peano derivatives. If we say $F_{(k)}(x_0), 1 \leq k \leq r$, exists in an $(a, b)$ we will mean that $F_{(k)}$ exists in $[a, b]$ and that the appropriate one sided derivatives exist at those of the points $a$ and $b$ that are in $(a, b)$.

Let $x_0, \ldots, x_r$ be $(r + 1)$ distinct points from $[a, b]$ then the $r$th divided difference of $F$ at these $(r + 1)$ points is defined by

\[ V_r(F) = V_r(F; x_0, \ldots, x_r) = \frac{F(x_0)}{w(x_0)} \]

where

\[ w(x) = w_r(x) = w_r(x; x_k), \quad \text{etc.} \]

\[ = \prod_{k=0}^{r} (x - x_k). \]

This $r$th divided difference has the following properties, which we collect for reference in

**Lemma 1.** (a) $V_r(F; x_k) = 0$ for all choices of points $x_0, \ldots, x_r$ if and only if $F$ is a polynomial of degree at most $r - 1$.

(b) If $F$ is a polynomial of degree $r$ then for all $x_0, \ldots, x_r, V_r(F; x_k) = \text{coefficient of } x^r$.

(c) $V_r(F; x_0, \ldots, x_r)$ is independent of the order of the points $x_0, \ldots, x_r$. 

(d) There is a simple relation between successive divided differences given by

\[(x_0 - x_r) V_r(F; x_0, \ldots, x_r) = V_{r-1}(F; x_0, \ldots, x_{r-1}) - V_{r-1}(F; x_1, \ldots, x_r).\]

(e) For any \(F\) we have the Newton Interpolation Formula,

\[F(x) = F(x_0) + \sum_{k=1}^{r-1} V_k(F; x_1, \ldots, x_{k+1}) w_{k-1}(x; x) + V_r(F; x, x_1, \ldots, x_r) w_{r-1}(x; x_k).\]

This last formula can be written differently as follows. Given the \((r + 1)\) points \(P_k, 0 \leq k \leq r,\) with coordinates \((x_k, F(x_k)), 0 \leq k \leq r,\) respectively, there is a unique polynomial of degree at most \(r\) passing through these points given by

\[\pi_r(F; x; P_k) = \pi_r(x; P_k) = \tau_r(\alpha; a, \ldots, \alpha), \text{ etc.}\]

This formula (12) is known as the Lagrange Interpolation Formula. It is easily seen that for all \((r + 1)\) distinct \(y_0, \ldots, y_r\)

\[V_r(\pi_r; y_k) = V_r(F; x_k).\]

Then (11) can be written

\[F(x) = \pi_{r-1}(F; x; x_k) + V_r(F; x, x_1, \ldots, x_r) w_{r-1}(x; x_k).\]

Using the divided difference we now define another derivative. Suppose all of \(x, x_0, \ldots, x_r\) are in \([a, b]\) and

\[x_k = x + h_k, 0 \leq k \leq r, \text{ with} \]

\[0 \leq |h_0| < \cdots < |h_r|,\]

then the \(r\)th Riemann derivative of \(F\) at \(x\) is defined by

\[D^r F(x) = \lim_{h_k \to 0} \cdots \lim_{h_{r-1} \to 0} r! V_r(F; x_k)\]

if this iterated limit exists independently of the manner in which the \(h_k\) tend to zero, subject only to (15). In a similar manner we define the upper and lower derivatives; and if the \(h_k\) all have the same sign the one-sided derivatives; these will be written \(\bar{D}^r F(x), \tilde{D}^r F(x),\) etc. If we say \(D^r F\) exists in \((a, b)\) we make the same gloss as for \(F_{(r)}\).

Since we can let \(h_0, \ldots, h_1\) very first and then \(h_{r+1}, \ldots, h_r\) the above definition and (10) imply that if \(D^r F(x)\) exists then so does \(D^k F(x), 1 \leq k \leq r;\) or more generally if \(\bar{D}^r F(x)\) is finite then \(\tilde{D}^k F(x)\) is finite.
1 \leq k \leq r$. Remark however that even if $D_r^k F(x)$ and $D_{-r}^k F(x)$ exist, are finite and equal, this does not imply that $D^r F(x)$ exists, [15, p. 26].

If instead of (15) and (16) we have

\begin{align*}
(15)' & \quad h_k = (r - 2k)h, \ 0 \leq k \leq r, \\
(16)' & \quad D_r^k F(x) = \lim r! V_r(F; x_k),
\end{align*}

(with obvious modifications for the upper and lower derivatives), this is called the $r^{th}$ symmetric Riemann derivative. In particular the cases $r = 1, 2$ coincide with definitions of $D_1F, D_2F$ respectively. In general if $\bar{D}_r F < \infty$ in $]a, b[$ then $F_{(r)}$ exists and equals $\bar{D}_r F$ almost everywhere, [12].

The usual $r$th order derivative of $F$ at $x, x \in (a, b)$, will be written $F^{(r)}(x)$.

**Theorem 2.** If $x \in [a, b]$ then $D_+^r F(x) = F_{(r),+}(x)$, provided one side exists.

**Proof.** Suppose first that $F_{(r),+}(x)$ exists; then taking the $r$th divided difference of $F(x + h)$, (considered as a function of $h$) at the points $h_0, h_1, \cdots, h_r, 0 \leq h_0 < \cdots < h_r$, using (5) and Lemma 1 (a), (b) we see that

$$r! V_r(F; x + h_k) = F_{(r),+}(x) + V_r(o(h^r); h_k).$$

Letting $h_0, \cdots, h_r$ tend to 0 successively we get that $D^r_+ F(x)$ exists and equals $F_{(r),+}(x)$.

If now we suppose that $D^r_+ F(x)$ exists then the rest of the theorem follows using Lemma 1(e).

A similar result obviously holds for lefthanded and two-sided derivatives; the latter is due to Denjoy [6] and Corominas [4], who give different proofs.

**Corollary 3.** (a) If $x \in [a, b]$ and $F_{(k),+}(x)$ exists $1 \leq k \leq r - 1$ then $\bar{F}_{(r),+}(x) = \bar{D}_r F(x)$, and $\bar{F}_{(r),+}(x) = D_{-r} F(x)$.

(b) If $x \in ]a, b[$ and $D^k F(x)$ exists $1 \leq k \leq r - 1$ and $D_+^r F(x)$, $D_-^r F(x)$ exist and are equal then $D^r F(x)$ exists, and is equal to this common rule.

**Proof.** (a) is proved by a simple adaption of the proof of Theorem 2. (b) holds since the similar result holds for Peano derivatives.

The following results due to Burkill [3], Corominas [4], and Olivier [14] should be noted.
THEOREM 4. (a) If $F_{(r-1)}$ exists, in $[a,b]$ and if
\[
\inf [F_{(r)}^+, F_{(r)}^-] > A > -\infty,
\]
then $F_{(r-1)}$ is continuous.

(b) If $F_{(r)}$ is continuous in $[a,b]$ then $F_{(r)}^{(r)} = F_{(r)}$.

(c) If $F_{(r)}$ exists at all points of $[a,b]$ then $F_{(r)}$, possesses both the Darboux property and the mean-value property.

The definitions of the terms used in (c) can be found in [14].

3. $n$-convex functions. A real-valued function $F$ defined on the closed bounded interval $[a, b]$ is said to be $n$-convex on $[a, b]$ if and only if for all choices of $(n + 1)$ distinct points, $x_0, \ldots, x_n$ in $[a, b]$, $V_n(F; x) \geq 0$, [4, 7, 15]. If $-F$ is $n$-convex then $F$ is said to be $n$-concave. The only functions that are both $n$-convex and $n$-concave are polynomials of degree at most $n - 1$, (Lemma 1).

If $n = 1$ this is just the class of monotonic increasing functions and $n = 2$ is the class of convex functions; (the class $n = 0$ is just the class of nonnegative functions, but we will usually only be interested in $n \geq 1$).

THEOREM 5. Let
\[
P_k = (x_k, y_k), 1 \leq k \leq n, n \geq 2, a \leq x_1 < \cdots < x_n \leq b,
\]
be any $n$ distinct points on the graph of the function $F$. Then $F$ is $n$-convex if and only if for all such sets of $n$ distinct points, the graph lies alternately above and below the curve $y = \pi_n(F; x; P_k)$, lying below if $x_{n-j} \leq x \leq x_n$. Further $\pi_{n-1}(x; P_k) \leq F(x), x_n \leq x \leq b$; and $\pi_{n-1}(x; P_k) \leq F(x)(\geq F(x))$ if $a \leq x < x_1$, $n$ being even (odd).

Proof. Let $x_0 \neq x_k, 1 \leq k \leq n, x_1 < x_0 < x_n$ and suppose in fact $x_j < x_0 < x_{j+1}$. If $F$ is $n$-convex then $V_n(F; x_0, \cdots, x_n) \geq 0$; i.e.,
\[
\sum_{k=1}^{n} \frac{F(x_k)}{w_n(x_k)} \geq -\frac{F(x_0)}{w_n(x_0)},
\]
or $F(x_0) \geq -\sum_{k=1}^{n} F(x_k)[w_n(x_k)/w_n(x_k)] = \pi_{n-1}(x_0, P_k)$, if $(n - j)$ is even, but $F(x_0) \leq \pi_{n-1}(x_0, P_k)$ if $(n - j)$ is odd. This proves the necessity; the sufficiently is immediate by reversing the argument. The last remark follows in a similar way by considering $x_n < x_0 < b$, and $a \leq x_0 < x_1$.

This theorem generalizes the property that a convex function always lies below its chord.
THEOREM 6. If $F$ is an $n$-convex function on $[a, b]$ and
\[ a \leq x_1 < \cdots < x_n \leq b, a \leq z_1 < \cdots < z_n \leq b, z_k \leq x_k, 1 \leq k \leq n, \]
then $V_{n-1}(F; z_k) \leq V_{n-1}(F; x_k)$.

Proof. The following particular case suffices to prove this result.
\[ x_k = z_k, k \neq j + 1, x_j < z_{j+1} < x_{j+1}. \]
Then, as in Theorem 5,
\[ \text{sign} [F(z_{j+1}) - \pi_{n-1}(z_{j+1}; x_k)] = (-1)^{n-j}. \]
Hence, with this $\pi_{n-1},$
\[ V_{n-1}(F; z_k) - V_{n-1}(\pi_{n-1}; z_k) = \frac{F(z_{j+1}) - \pi_{n-1}(z_{j+1}; x_k)}{\prod_{k \neq j+1} (z_{j+1} - x_k)} \leq 0. \]
That is
\[ V_{n-1}(F; z_k) \leq V_{n-1}(\pi_{n-1}; z_k) = V_{n-1}(F; x_k), \text{ by (13).} \]

THEOREM 7. If $F$ is $n$-convex in $[a, b]$ then
(a) $F^{(r)}$ exists and is continuous in $[a, b], 1 \leq r \leq n - 2,$
(b) both $F^{(n-1), -}, F^{(n-1), +}$ are monotonic increasing and if
\[ a \leq x_1 < \cdots < x_n \leq x \leq y_1 < \cdots < y_n \leq b \]
then
\[ (n - 1)! \quad V_{n-1}(F; x_k) \leq F^{(n-1), -}(x) \leq F^{(n-1), +}(x) \leq (n - 1)! \quad V_{n-1}(F; y_k), \]
(c) $F^{(n-1), +} = (F^{(n-2)})^{+}, F^{(n-1), -} = (F^{(n-2)})^{-},$
(d) $F^{(n-1)}$ exists at all except a countable set of points.

Proof. Using Theorem 2, it is an immediate consequence of Theorem 6 that $F^{(r), +}$ exists in $[a, b], F^{(r), -}$ exists in $[a, b], 1 \leq r \leq n - 1$ and that (b) holds.
From (b) we get that both $F^{(n-1), +}, F^{(n-1), -}$ are continuous except on a countable set. Then, again from (b), we have that $F^{(n-1), +} = F^{(n-1), -}$ except on a countable set.
Then if we prove (a) and (c), (d) is immediate.
Suppose $a \leq x_1 < \cdots < x_n \leq b$ then repeated application of (10) gives
Now let $x \to x_2$, then by Theorem 6 the left-hand side of this expression tends to a finite limit, $K_1$, say: i.e.,

$$K_1(x_2, \cdots, x_n) = \frac{D^iF(x_2) - V_1(F; x_2, x_3) - V_2(F; x_2, x_3, x_4)}{(x_2 - x_3)} \cdots \frac{(x_2 - x_n)}{(x_2 - x_n)}.$$

If now $x_3 \to x_2$ we get a finite limit on l.h.s. of this last expression: hence $D^i_F(x_2) = D^i_F(x_2)$; that is $DF(x_2)$ exists. A similar argument shows $DF$ is continuous in $]a, b[$.

In a similar way, expressing $V_{n-1}$ in terms of $V_2, V_3, \cdots$ we show that $D^r_F(x_0) = D^r_F(x_3)$ and so by Corollary 3(b), $D^r_F(x_3)$ exists then as above $D^nF$ exists and is continuous in $]a, b[$.

In this way we show $D^rF$ exists and is continuous in $]a, b[, 1 \leq r \leq n - 2$. Hence, by Theorem 2, $F'_r$ exists and is continuous in $]a, b[, 1 \leq r \leq n - 2$ and so finally, by Theorem 4(b), the same is true of $F^{(n)}$. This proves (a).

For the proof of (c) let $x_0 < \cdots < x_{2n-3}$ then repeated application of (10) gives

$$\sum_{k=0}^{n-2} (x_k - x_{k+n-1}) V_{n-1}(F; x_k, \cdots, x_{k+n-1}) = V_{n-2}(F; x_0, \cdots, x_{n-2}) - V_{n-2}(F; x_{n-2}, \cdots, x_{2n-3}).$$

Let $x_k \to x_0$, $1 \leq k \leq n - 2$, $x_k \to x_{n-1}$, $n \leq k \leq 2n - 3$ then by Theorem 6 the limit on the left hand side exists, and the value limit on the right hand side follows from (a). Thus we get an expression of the form

$$(n - 1)(x_0 - x_{n-1}) K(x_0, x_{n-1}) = \frac{1}{(n - 2)!} \{F_{(x_0)}^{(n-2)} - F_{(x_{n-1})}^{(n-2)}\}.$$

Now dividing and letting $x_{n-1} \to x_0$ we get

$$(n - 1)! \lim_{x_{n-1} \to x_0^+} K(x_0, x_{n-1}) = (F^{(n-2)})'_+(x_0);$$

a simple application of (11) shows that the left hand side of this last expression is equal to $F_{(n-1),+}(x_0)$. This completes the proof of the first
part of (c), the rest follows using a similar argument.

Formula (18) is due to James [7, Lemma 10.4], who however assumes the existence of $F_{(n-1)}$ in $[a, b]$.

**COROLLARY 8.**

(a) $F$ is $n$-convex on $[a, b]$ if and only if $F$ differs by a polynomial of degree at most $(n - 1)$ from $\int_a^x (x-t)^{n-1} \mu(dt)$, for some Lebesgue-Stieltjes measure $\mu$. In particular if and only if $F$ is the $(n - 1)$st integral of a monotonic function.

(b) If $F$ is $n$-convex in $[a, b]$, $|F| \leq k$, then $|F_{(k)}(x)| \leq AK \sup \{1/(b-x)^k, 1/(x-a)^k\}$, $0 \leq k \leq n - 1$ where $A$ is a constant independent of $k, F$ and $x$, and where if $k = n - 1$ the derivative is to be interpreted as $\sup \{|F_{(n-1),+}(x)|, |F_{(n-1),-}(x)|\}$.

(c) If $F$ is $n$-convex on $[a, b]$, $a \leq x \leq y \leq b$, $a \leq x + h \leq y$, and $x \leq y + k \leq b$ then

$$\gamma_{n-1}(F; x; h) \leq F_{(n-1),-}(y) \quad \text{and} \quad F_{(n-1),+}(x) \leq \gamma_{n-1}(F; y; k).$$

**Proof.** (a) This is immediate from Theorem 7 (b).

(b) From (18) we have that

$$\frac{1}{(n-1)!} \sum_{s=0}^{n-1} \frac{F(x_k)}{w'(x_k)} \leq \sup \{F_{(n-1),+}(x), F_{(n-1),-}(x)\} \leq \frac{1}{(n-1)!} \sum_{s=0}^{n-1} \frac{F(y_s)}{w'(y_s)}$$

from which (b) in the case $k = n - 1$ is easily deduced. The rest follows by integration, using, (a).

(c) Immediate using (18), (11), (6) Theorems 2 and 4.

The definition, (12), of $\pi_r(x; P_k)$ can be extended to cover the case when not all of the $P_k$ are distinct. Thus if only $s$ of these points are distinct then besides giving the values at the $s$ points, a total of $r + 1 - s$ derivatives must also be given—either $r + 1 - s$ derivatives all at one point, or $r + 1 - s$ first derivatives at $r + 1 - s$ distinct points, (when $r + 1 - s \leq s$), etc. Theorem 5 can be extended, using Theorems 6, 7 and taking limits; thus as an example of many possible extensions we state

**THEOREM 9.** Let $P_k = (x_k, y_k), 1 \leq k \leq r, a \leq x_1 < \cdots < x_r \leq b$, be $r$ distinct points on the graph of the function $F$. Suppose that $F_{(s)}(x_k)$ exists, $1 \leq s \leq n - r$. Then Theorem 5 holds if $\pi_{s-n}(x; P_k)$ is taken to have $\pi_{s-n}(x; P_k) = F(x), 1 \leq s \leq r, \pi_{s-n}(x; P_k) = F_{(s)}(x), 1 \leq s \leq n - r$, and if $P_1$ is considered as $n - r + 1$ points at and to the right of $P_1$ but to the left of $P_r$.

**THEOREM 10.** (a) If $F$ is $n$-convex on $[a, b]$ and $P_k = (x_k, y_k), 1 \leq k \leq n$ are $n$ distinct points on the graph of $F, a \leq x_1 < b$, let
As $h \to 0^+$, $\pi_{n+1}(x; P_k)$ converges uniformly to the right tangent polynomial at $x$, 

$$
\tau_{n+1}(F; x; x_i) = \tau_+(x) = F(x_i) + \sum_{k=1}^{n-2} \frac{(x - x_i)^k}{k!} F^{(k)}(x_i) 
+ \frac{(x - x_i)^{n-1}}{(n-1)!} F^{(n-1)}_+(x_i), \quad x_i \leq x \leq b.
$$

Further on the right of $x$, $\tau_+ \leq F$.

(b) A similar result holds for the left tangent polynomial at $x_i$, $\tau_-(x; x_i), a \leq x \leq x_i, a < x_i \leq b$. However in this case if $n$ is even (odd) then on the left of $x_i$, $\tau_- \geq F(\geq F)$.

(c) At all but a countable set of points $x_i$, a similar result holds for the tangent polynomial at $x_i$, $\tau(x; x), a < x < b, a < x_i < b$. However if $n$ is even the graph of $\tau$ lies below that of $F$, whereas if $n$ is odd the graphs cross, $\tau$ being above on the left of $x_i$, and below on the right of $x_i$.

Proof. It suffices to consider (a). But (a) is a simple consequence of Theorems 5, 7, (11), and (14).

**Corollary 11.** (a) If $F$ is $n$-convex in $[a, b]$ then

$$
\inf \{F^{(n)}_+, F^{(n)}_\tau\} \geq 0.
$$

(b) If $F$ is $n$-convex in $[a, b]$ and $F^{(n-1)}_+$ exists in $[a, b]$ then it is continuous.

(c) If $F$ is $n$-convex in $[a, b]$ then $F^{(n-1)}_+$ is upper-semi continuous (u.s.c.), $F^{(n-1)}_-\tau$ is lower semi-continuous (l.s.c.).

Proof. (a) Suppose in Theorem 10, for simplicity, that $x_i = 0$. Then $F'$ lies above the right tangent polynomial at $x = 0$, i.e.,

$$
\frac{F(x) - \tau_+(x)}{x} \geq 0,
$$

in some interval $[0, h]$. Hence $F^{(n)}_+(0) \geq 0$: in a similar way $F^{(n)}_-\tau(0) \geq 0$.

(b) Immediate from (a), Theorem 4(a), Theorem 7(a).

(a) This is just Theorem 3.2 [3], adapted to one sided derivatives. The following theorem generalizes a result well known when $n = 1, [13, Corollary 32.3]$ and $n = 2 [7, Th. 4]$.

**Theorem 12.** If $F$ is $n$-convex on $[a, b], a < \alpha < \beta < b, E_k = \{x; \alpha \leq x \leq \beta \text{ and } F^{(n)}_k(x) \geq k\}$ then

$$
k \cdot m^*(E_k) \leq 2n[F^{(n-1)}_\tau(\beta) - F^{(n-1)}_+(\alpha)].
$$


(where \(m^*\) denotes the outer Lebesgue measure).

**Proof.** For simplicity we will ignore the countable set where \(F_{(n+1)}\) may not exist and suppose that \(k > 0\). Further let \(E_k^+\) be as \(E_k\) but with \(F_{(n+1)}\) instead of \(F_{(n)}\) and suppose \(m^*E_k^+ > 0\); with a similar definition for \(E_k^-\).

If then \(\epsilon > 0, x \in E_k^+\) there is an \(h > 0\) such that
\[
g_n(F; x; h) \geq F_{(n+1)}(x) - \epsilon \geq k - \epsilon .
\]
So, by [20], there is a finite family of nonoverlapping intervals \([x_i, x_i + h_i], i = 1, \ldots, p\) such that \(x_p + h_p \leq \beta\),
\[
g_n(F; x_i, h_i) \geq k - \epsilon, i = 1, \ldots, p,
\]
and
\[
\sum_{i=1}^p h_i \leq m^*E_k^- - \epsilon .
\]
Thus
\[
\sum_{i=1}^p h_i g_n(F; x_i, h_i) \geq (k - \epsilon)(m^*E_k^- - \epsilon) ;
\]
but since
\[
h g_n(F; x, h) = n\{g_{n-1}(F; x, h) - F_{(n+1)}(x)\}
\]
we have that
\[
\sum_{i=1}^p \{g_{n-1}(F; x_i, h_i) - F_{(n+1)}(x_i)\} \geq \frac{k - \epsilon}{n}(m^*E_k^- - \epsilon) .
\]
However by Corollary 8(c)
\[
\sum_{i=1}^{p-1} \{F_{(n+1)}(x_{i+1}) - g_{n-1}(F; x_i, h_i)\} \geq 0 ,
\]
\[
F_{(n-1)}(x_i) - F_{(n-1)}(\alpha) \geq 0 ,
\]
\[
F_{(n-1)}(\beta) - g_{n-1}(F; x_p, h_p) \geq 0 .
\]
Adding the last four inequalities we get that
\[
F_{(n-1)}(\beta) - F_{(n-1)}(\alpha) \geq \frac{k - \epsilon}{n}(m^*E_k^- - \epsilon) .
\]
This together with a similar inequality for \(E_k^-\), implies (20).

A function that is the difference of two \(n\)-convex functions will be called \(\delta\)-\(n\)-convex; as in the cases \(n = 1\) and \(n = 2\), [16], such
functions can be characterized by their variational properties.

If \( F \) is defined on \([a, b]\) as well as \( F_{(k)}, 1 \leq k \leq n - 1 \), let us write
\[
\omega_n(F; a, b) = \omega_n(a, b) = \max \left\{ \sup_{a < x < b} |(x - a)\gamma_n(F; a; x - a)|, \right. \\
\left. \sup_{a < x < b} |(b - x)\gamma_n(F; a; b - x)| \right\}.
\]
this quantity was introduced by Sargent [19].

**Theorem 13.** A function \( F \) defined on \([a, b]\) is \( \delta \)-\( n \)-convex if and only if either of the following conditions is satisfied.

(a) \( \sum_{k=1}^{m} \omega_n(F; a_k, b_k) < K \) for all finite sets of nonoverlapping intervals, \([a_k, b_k], 1 \leq k \leq m\).

(b) \( \sum_{k=0}^{n} |(x_k - x_{k+n}) V_n(F; x_k, \cdots, x_{k+n})| < K \) for all finite sets of distinct points \( x_0, \cdots, x_{m+n} \).

**Proof.** The discussion of (b) is similar to the case \( n = 2 \) in [16] but using Corollary 8(a).

If (a) is satisfied then \( F_{(n-1)} \) is of bounded-variation [19, Lemma 1], and so by Corollary 8(a) \( F \) is \( \delta \)-\( n \)-convex.

If \( F \) is \( n \)-convex then by (21) and Corollary 8(c),
\[
(x - a)\gamma_n(F; a; x - a) = n(\gamma_{n-1}(F; a; x - a) - F_{(n-2)}(a)) \geq 0
\]
and so by Corollary 8(c)
\[
\omega_n(F; a, b) \leq n(F_{n-1}(b) - F_{(n-1)}(a)).
\]

From this it easily follows that if \( F \) is \( \delta \)-\( n \)-convex then (a) holds.

4. Sufficient conditions for \( n \)-convexity. In this section we obtain some sufficient conditions for a function to be \( n \)-convex. First we prove the following generalization of a well-known property of convex functions.

**Theorem 14.** (a) If \( F \) is \( n \)-convex in \([a, b]\) then \( F^{(n-2)} \) has no proper maximum in \([a, b]\).

(b) A function \( F \) with continuous derivative of order \((n - 2)\) is \( n \)-convex if and only if no function of the form \( F(x) + \sum_{k=0}^{n-2} a_k x^k \) has its derivative of order \((n - 2)\) attaining a maximum in \([a, b]\).

**Proof.** (a) Suppose \( F^{(n-2)} \) has a proper maximum at \( x_0 \), then consider \( G(x) = F(x) - \pi_{n-2}(x; P_0) \), where the polynomial \( \pi_{n-2} \) is determined uniquely by the conditions
\[ G(x_0) = G'(x_0) = \cdots = G^{(n-2)}(x_0) = 0. \]

Now consider \( \pi_{n-2}(x; Q_k) \) where \( Q_k = (x_k, G(x_k)), 0 \leq k \leq n - 2, \) \( x_0 < \cdots < x_{n-2}. \) Then by Theorem III [4], (13), and Lemma 1(b), the coefficient of \( x^{n-2} \) in \( \pi_{n-2}(x; Q_k) \) is \( G^{(n-2)}(x_0 + \delta), x_0 + \delta \) being some point in \( [x_0, x_{n-1}] \). Hence, using Theorem 7(a), since \( x_0 \) is a proper maximum of \( G^{(n-2)} \) and \( G^{(n-2)}(x_0) = 0, \) if \( x_0, \cdots, x_{n-2} \) are close enough together this coefficient is not positive.

Let \( x_k \to x_0, \) \( 1 \leq k \leq n - 3 \) then \( \pi_{n-2}(x; Q_k) \) becomes a polynomial of degree \( n - 2 \) with its value and that of its first \( (n-3) \) derivatives at \( x_0 \) being zero; it’s \( (n-2) \)nd derivative is nonpositive. Hence, by Theorem 9, \( G \leq 0 \) in \( [x_0, x_{n-2}] \).

In a similar way \( G \geq 0(\leq 0) \) in some interval to the left of \( x_0 \) when \( n \) is odd (even). Further in every such interval around \( x_0 \) there are points where these inequalities are strict.

Now consider the \( (n+1) \) points \( z_0, \cdots, z_n \) where

\[ z_0 < z_1 < \cdots < z_{\lfloor n/2 \rfloor} = x_0 < \cdots < z_n. \]

Then

\[ V_n(F; z_k) = V_n(G; z_k) = \frac{G(z)}{w'_n(z)} + F + \frac{G(z)}{w'_n(z)} \geq 0. \]

If then \( z_1, \cdots, z_{n-1} \) tend to \( x_0 \) then \( K \to 0 \) and we get

\[ \frac{G(z_0)}{(z_0 - x_0)^{n-1}(z_0 - z_n)} + \frac{G(z_n)}{(z_n - x_0)^{n-1}(z_n - z_0)} \geq 0. \]

But whether \( n \) is even, or odd both terms on the l.h.s. of this expression can be chosen to be negative-which contradiction completes the proof of (a).

(b) The necessity is evident. Suppose then that \( F \) is not \( n \)-convex. Then by Theorem 5 there exists a polynomial \( \pi_{n-1}(x; P_k) \) such that the two curves \( y = F(x), y = \pi_{n-1}(x; P_k) \) do not intertwine correctly.

Consider \( G(x) = F(x) - \pi_{n-1}(x; P_k); \) then \( G(x_0) = \cdots = G(x_n) = 0 \) and \( G \) changes sign at most \( (n - 2) \) times. Hence \( G^{(n-2)} \) has three zeros and so has a local maximum. This completes the proof.

**Corollary 15.** (a) If \( F \) is \( n \)-convex then \( F^{(r)} \) is \( (n - r) \)-convex, \( 1 \leq r \leq n - 2. \)

(b) If \( F \) is \( n \)-convex then \( F^{(n)} \) exist a.e.

**Proof.** (a) The case \( r = n - 2 \) is just Theorem 14(b). In general \( F^{(k)}, 1 \leq k \leq n - 3, \) has a continuous derivative of order \( n - k - 2 \) satisfying the hypotheses of Theorem 14(b), and hence \( F^{(k)} \) is \( (n - k) \)-convex.
A CRITERION FOR $n$-CONVEXITY

(b) Since $F^{(n-2)}$ is convex this follows immediately from well-known properties of convex functions.

Note that the case $r = n - 1$ of Corollary 15(a) is just the last part of Theorem 7(b).

We now wish to prove a converse of Corollary 11(a). Because of applications to symmetric Perron integral, [7], this converse will be obtained in terms of de la Vallée Poussin derivatives and the results in terms of Peano derivatives will be simple corollaries. A direct proof could be constructed from the proof of the more general results.

**Theorem 16.** If $F$ satisfies $C_{2m}$, $m \geq 1$, in $]a, b[$ and

(a) $D_{2m}F(x) \geq 0$, $x \in ]a, b[ \sim E$, $|E| = 0$,

(b) $D_{2m}F(x) > -\infty$, $x \in ]a, b[ \sim S$, $S$ a scattered set,

(c) $\lim_{h \to 0} h \theta_{2m}(F; x; h) \geq 0 \geq \lim_{h \to 0} h \theta_{2m}(F; x; h)$, $x \in S$ then $F$ is $2m$-convex. (A set is said to be scattered if it contains no subsets that are dense in themselves.)

**Proof.** If $E = S$ then by Theorem 6.1, [9], (a), (b), (c) imply $D_{2m}F \geq 0$ in $]a, b[$ and so the result follows from Theorem 4.2, [8].

Given $\varepsilon > 0$, $T$, $|T| = 0$, $T \in G$, $T \neq \emptyset$ let $\chi_{t, r} = \chi$ be a function on $]a, b[$ such that

(i) $\chi$ is absolutely continuous,

(ii) $\chi$ is differentiable,

(iii) $\chi'(x) = \infty$, $x \in T$,

(iv) $0 \leq \chi'(x) < \infty$, $x \in T$,

(v) $\chi(a) = 0$, $0 \leq \chi(b) \leq \varepsilon/(b - a)^{2m-1}$. That such a function exists is well known, [21]. Then define

$$\Psi_{t, r, 2m}(x) = \mathcal{V}(x) = \frac{1}{(2m - 2)!} \int_{a}^{x} (x - t)^{2m-2} \chi(t) dt,$$

the $(2m - 1)st$ integral of $\chi$. Then $\mathcal{V}^{(2m-1)}(x) = \chi(x)$ and, using (2), we have on integrating by parts that

$$D_{2m}F(x) \geq m\chi'(x) \geq 0.$$

If now $E \subset T$ then we easily see that (i) $\mathcal{V}$ is $C_{2m}$, and $2m$-convex, (ii)
If \( n \geq 2 \), and (i) \( F_{(n-1)} \) exists in \([a, b] \), (ii) \( F_{(n-1)}(x) \geq 0 \), \( x \in [a, b] \), \( x \sim E \), \( |E| = 0 \), (iii) \( F_{(n-1)}(x) \geq 0 \), \( x \in [a, b] \), \( x \sim E \), \( |E| = 0 \), (iv) \( F_{(n-1)}(x) \sim C \), \( C \) countable, then \( F \) is \( n \)-convex.
Proof. As in the proof of Theorem 16 we can assume that \( E = C \)
and so suppose \( \bar{F}(x) \geq 0 \) except when \( x = x_0, x_1, \ldots \). We may assume that for all \( k \in N, x_k \neq b \).

Adopting a procedure due to Bosanquet [1] and Sargent [18] we exhibit for each \( k \in N \) a monotonic \( n \)-convex function \( Z_k \) with the following properties

(i) \( Z_k(a) = 0, Z_k(b) \leq [(b - a)^{n-r+1}/(n - r - 1)!]2^{-(k+1)}\varepsilon, 0 \leq r \leq n - 1, \)

(ii) \( (F + Z_k)_n(x) \geq 0, \)

(iii) \( V_n(Z_k; y_r) \leq K2^{-(k+1)}\varepsilon, \) for all \( (n + 1) \) distinct points \( y_0, \ldots, y_n \).

Then if we define \( G(x) = F(x) + \sum_{k \in N} Z_k(x), G_n(x) \geq 0 \) everywhere and so is \( n \)-convex, by usual arguments; but

\[
V_n(G; y_r) = V_n(F; y_r) + \sum_{k \in N} V_n(Z_k; y_r)
\]

and so \( V_n(F; y_r) \geq -K\varepsilon \), which implies \( F \) is \( n \)-convex.

It remains to define the function \( Z_k \). Since \( C_n \) is satisfied, we have, by (4) and (6), \( \lim_{n \to \infty} h_n(x_k; x_k; h) = 0 \) so we can find a sequence \( y_1, y_2, \ldots \) in \( [x_k, b] \) such that \( 0 < y_{s+1} - x_k = h_{s+1} < \frac{1}{18}(y_s - x_k) = h_s/2, \) and \( h_s \gamma_n(F; x_k; h_s) > -\varepsilon \cdot 2^{-(k+s)} \). Now define the function \( z_k \) in such a way as to be continuous and

\[
z_k(x) = 0, a \leq x \leq x_k,
\]

\[
= 2^{-(k+1)}\varepsilon, y_1 < x \leq b,
\]

\[
= 2^{-(k+s)}\varepsilon, x = y_s, s = 1, 2, \ldots,
\]

\[
= \text{linear in } [y_{s+1}, y_s], s = 1, 2, \ldots.
\]

Then \( z_k \) is continuous, increasing on \( [a, b] \), \( z_k(a) = 0, z_k(b) = 2^{-(k+1)}\varepsilon, z_k(x_k) = 0, z_k(x)/x - x_k \) decreases in \( [x_k, b] \). It is then easily checked that

\[
\int_0^{h_s} (h_s - t)^{n-2}z_k(x_k + t)dt \geq \frac{z_k(y_s)h_s^{n-1}}{n(n - 1)} = \frac{2^{-(k+s)}h_s^{n-1}\varepsilon}{n(n - 1)}.
\]

Define then,

\[
Z_k(x) = \frac{1}{(n - 2)!} \int_a^x (x - t)^{n-2}z_k(t)dt,
\]

the \((n - 1)\)st integral of \( z_k \). Then \( Z_k^{(n-1)} = z_k \) and using Theorem 7, and Corollary 8, \( Z_k \) clearly has all properties wanted except possibly (ii). This we now check. First note that by (21)

\[
h_s \gamma_n(Z_k; x_k, h_s) = n\gamma_n(Z_k; x_k, h_s).
\]

So as in the proof of (23),
\[ h_n^r(Z_k; x_k, h_s) = n \frac{(n - 1)}{h_n^{n-1}} \int_0^{h_s} (h_s - t)^{n-2} z_k(x_k + t) dt = 2^{-(k+1)} \varepsilon. \]

Hence,
\[ h_n^r(Z_k + F; x_k, h_s) \geq 0 \]
which completes the proof.

A theorem of a slightly different form can be obtained using the symmetric Riemann derivatives.

Let us say a real valued function \( F \) on \([a, b]\) is of type \( D_r \) if for all sets of \( (r + 1) \) distinct points \( x_0, \cdots, x_r \) in \([a, b]\)
\[(26) \inf_{a < x < b} \bar{D}_s F(x) \leq r! V_r(F; x_k) \leq \sup_{a < x < b} D_r F(x). \]

The following simple lemmas will be useful.

**Lemma 20.** If \( F^{(r-2)} \) exists and is continuous in \([a, b]\) then for sets of \( (r + 1) \) distinct points \( x_0, \cdots, x_r \) in \([a, b]\)
\[ \inf_{a < x < b} \bar{D}_s F^{(r-2)}(x) \leq r! V_r(F; x_k) \leq \sup_{a < x < b} D_r F^{(r-2)}(x). \]
In particular if \( F^{(r)} \) exists in \([a, b]\) then \( F \) is of type \( D_r \).

*Proof.* Let \( G(x) = F(x) - \pi_{r-1}(F; x_0, \cdots, x_{r-1}) - \lambda P(x) \) where \( P \) is a polynomial of degree \( r \), \( \lambda \) a constant determined by requiring that \( G(x_k) = 0 \), \( 0 \leq k \leq r \) and \( V_r(F; x_k) = \lambda. \)

Then since \( G \) has at least \( (r + 1) \) zeros \( G^{(r-2)} \) has at least 3 zeros and so has a nonnegative maximum; that is for some \( y \) \( V_r(G^{(r-2)}; y_1, y_2, y_3) \leq 0 \) for all \( y_1, y_2, y_3 \) near enough to \( y \); that is
\[ 2 \cdot V_2(G^{(r-2)}; y_1, y_2) = 2 V_2(F^{(r-2)}; y_1, y_2) > r! \lambda \leq 0. \]
The proof now follows that in [6].

**Lemma 21.** If \( F \) is of type \( D_n \) then
\[ \inf_{a < x < b} \bar{D}_s F(x) = \inf_{a < x < b} D_r F(x), \sup_{a < x < b} \bar{D}_s F(x) = \sup_{a < x < b} D_r F(x). \]

*Proof.* The case \( n = 2 \) and more is proved in [6, p. 9]. The proof of the general case is the same.

**Theorem 22.** If \( F \) is of type \( D_n \) and (a) \( D_r F(x) \geq 0 \), \( x \in ]a, b[ \sim E, |E| = 0 \), (b) \( D_r F > -\infty \), then \( F \) is \( n \)-convex.

*Proof.* Since the \( 2m \)-convex function \( \Psi \) of Theorem 16 is, using
Lemma 20, of type $D_{2m}$ we can, as in Theorem 16, assume $E = \emptyset$. The result is then a trivial consequence of (26).

**Corollary 23.** If $F, G$ are such that (a) $F - G$ is of type $D_{n}$, (b) $\bar{D}^*_r(F - G)(x) \geq 0 \geq F_r(F - G)(x)$, $x \in [a, b]$, $|E| = 0$, (c) $D_r(F - G) > -\infty$, $D_r(F - G) < \infty$, then (24) holds.

It would be of interest to produce some reasonable conditions on $F$ that ensure it is of type $D_r$. It is known, [15], that if $F$ is continuous then $F$ is of type $D_2$, but Kassimatis, [10], has pointed out that if $r > 2$ this is false. One would expect the existence and continuity of $F^{(r-2)}$ to imply $F$ is of type $D_r$ but this has not been proved. Let us say $F$ is of type $d_r$ when

$$\inf_{a < x < b} D_r F(x) \leq r! V_r(F; x_b) \leq \sup_{a < x < b} \bar{D}_r F(x).$$

If in Theorem 22 and Corollary 23 we weaken our hypothesis to $F$ being of type $d_r$, obvious modifications of the other conditions will produce analogous theorems. It has been proved in [2] that if $F^{(r-2)}$ exists and is continuous, $r = 2, 3, 4$, then $F$ is $d_r$; unfortunately the method fails if $r > 4$.

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Norman Larrabee Alling, *Analytic and harmonic obstruction on nonorientable Klein surfaces* ........................................................... 1
Shimshon A. Amitsur, *Embeddings in matrix rings* .................................................. 21
William Louis Armacost, *The Frobenius reciprocity theorem and essentially bounded induced representations* .................................................. 31
Kenneth Paul Baclawski and Kenneth Kapp, *Topisms and induced non-associative systems* ............................................................... 45
George M. Bergman, *The index of a group in a semigroup* ................................... 55
Simeon M. Berman, *Excursions above high levels for stationary Gaussian processes* .............................................................. 63
Peter Southcott Bullen, *A criterion for n-convexity* ............................................. 81
W. Homer Carlisle, III, *Residual finiteness of finitely generated commutative semigroups* ................................................................. 99
Roger Clement Crocker, *On the sum of a prime and of two powers of two* .......... 103
David Eisenbud and Phillip Alan Griffith, *The structure of serial rings* .............. 109
Timothy V. Fossum, *Characters and orthogonality in Frobenius algebras* ........ 123
Hugh Gordon, *Rings of functions determined by zero-sets* ................................ 133
William Ray Hare, Jr. and John Willis Kenelly, *Characterizations of Radon partitions* ........................................................................................................... 159
Philip Hartman, *On third order, nonlinear, singular boundary value problems* ................................................................................................. 165
David Michael Henry, *Conditions for countable bases in spaces of countable and point-countable type* .................................................. 181
James R. Holub, *Hilbertian operators and reflexive tensor products* ............... 185
Robert P. Kaufman, *Lacunary series and probability* ........................................ 195
Erwin Kreyszig, *On Bergman operators for partial differential equations in two variables* .......................................................................................... 201
Chin-pi Lu, *Local rings with noetherian filtrations* ............................................... 209
Louis Edward Narens, *A nonstandard proof of the Jordan curve theorem* ........ 219
Joseph Earl Valentine and Stanley G. Wayment, *Wilson angles in linear normed spaces* .............................................................................. 239
Hoyt D. Warner, *Finite primes in simple algebras* ............................................... 245
Horst Günter Zimmer, *An elementary proof of the Riemann hypothesis for an elliptic curve over a finite field* ........................................... 267