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A serial ring (generalized uniserial in the terminology of Nakayama) is one whose left and right free modules are direct sums of modules with unique finite composition series (uniserial modules.) This paper presents a module-theoretic discussion of the structure of serial rings, and some one-sided characterizations of certain kinds of serial rings. As an application of the structure theory, an easy proof is given of A. W. Goldie's characterization of serial rings with trivial singular ideal.

In an earlier paper [8], we considered some occurrences and applications of serial rings in other areas of ring theory. In addition, we gave a short conceptual proof of Nakayama's Theorem [19, Th. 17], which states that any module over a serial ring is a direct sum of uniserial modules. This generalizes the theorem that every module over an artinian principal ideal ring is a direct sum of uniserial modules [13, § 15] as well as the corresponding theorem for semi-simple artinian rings.

Of course, semi-simple artinian rings may be characterized by saying only that each of their left modules is a direct sum of simple modules. Fuller [10, 5.4] has supplied the analogous result for serial rings. In § 1 of this paper, we provide a method (Theorem 1.2) for deducing that a left artinian ring, under rather strong assumptions on its category of left modules, is also right artinian. Applying this method, we easily obtain Fuller's one-sided characterization of serial rings in a style quite different from his.

Section 2 is concerned with various characterizations of artinian principal ideal rings. Among these, a one-sided characterization is given (Theorem 2.1 (4)). An interesting sidelight of this section is a module-theoretic proof of the classical structure theorem for principal ideal rings (Corollary 2.2. For a classical proof, see [13, § 15]).

In § 3, we present a structure theorem for serial rings (Theorem 3.1). A considerable part of this theorem is a restatement of results Murase [16, 17, 18] and of Amdal-Ringdal [1, 2]. In part, our theorem shows that a serial ring A has a ring decomposition $A = A_0 \times A_1 \times A_2 \times A_3$ such that

(0) A_0 is semi-simple, and A_i has no semi-simple factor for $i > 0$.

(1) A_1 is an artinian principal ideal ring.

(2) A_2 is a product of rings each of which is Morita equivalent to a factor ring of a full ring of upper triangular matrix rings over a division ring.

(3) $A_3/(\operatorname{Rad} A_3)^2$ is QF and A_3 has no homogeneous projective modules (see the definition of homogeneous in § 2).

As an application of the structure theory, we give in § 4 a new proof of a theorem of Goldie [11, Th. 8.11].

Throughout this paper, rings have identities and all modules will be unital. If A is a ring, $\operatorname{Rad} A$ is its Jacobson radical. A principal indecomposable A-module is an indecomposable direct summand of A. If M is an A-module, then soc M denotes the sum of all simple submodules of M and $\lambda(M)$ denotes the length of a composition series of M (if one exists). We observe that, if A is left artinian and if M is a uniserial left A-module, with $N = \operatorname{Rad} A$, then $M \supset NM \supset N^2M \supset \cdots \supset N^kM = 0$ is the form of a unique composition series for M. We recall that a QF (Quasi-Frobenius) ring is an artinian ring which is right and left self injective. A ring A is said to be semiprimary if $\operatorname{Rad} A$ is nilpotent and $A/\operatorname{Rad} A$ is semisimple artinian. Finally, our homological notation follows that of MacLane [15].

1. Going from left to right. In this section we set forth conditions on the category of left modules over a left artinian ring which force the ring to be right artinian. The tool we will use for this purpose is the stable duality functor of Auslander and Bridger [4], whose definition we sketch below. We are grateful to M. Auslander for his suggestion that this functor was the appropriate tool. Its use has resulted in a considerable simplification of our original proofs.

Let M be a finitely presented module over any ring A, and let $P \xrightarrow{\varphi} Q \twoheadrightarrow M$ be exact, with P and Q finitely generated projectives. Let $D(M) = \operatorname{Coker}(Q^* \xrightarrow{\varphi^*} P^*)$ where $-^* = \operatorname{Hom}_A(-, A)$. If $P_1 \xrightarrow{\varphi_1} Q_1 \twoheadrightarrow M$ is another such sequence, then there are finitely generated projective modules F and G such that $F \oplus D(M) \cong G \oplus D_1(M)$, where $D_1(M) = \operatorname{Coker}(\varphi_1^*)$. Thus $D(M)$ is unique up to stable equivalence. Note that if M is a left module, $D(M)$ is a right module. Thus $D(D(M))$ is a left module, and M is stably isomorphic to $D(D(M))$.

LEMMA 1.1. *Let A be a semiprimary ring, X an indecomposable right direct summand of A. Let T be a submodule of X, and suppose T has finite length. Then* $\operatorname{End}_A(X/T)$ *is a local ring.*

Proof. Note that the conclusion is easy for $T = 0$.

Let $N = \operatorname{Rad} A$. Since XN is the unique maximal submodule of X, we may assume $T \subset XN$. Since X is a principal right ideal,

$$\{\varepsilon \in \operatorname{End}_A(X/T) \mid \operatorname{Im}(\varepsilon) \subseteqq XN/T\}$$

is an ideal of $\operatorname{End}_A(X/T)$. We will show that this is the unique maximal ideal, by showing that if $\varepsilon: X/T \to X/T$ is such that $\operatorname{Im} \varepsilon \nsubseteqq XN/T$, then ε is an isomorphism.

Such an ε is an epimorphism because XN is the unique maximal submodule of X. Let $\pi: X \to X/T$ be the canonical projection. Then both π and $\varepsilon\pi$ are epimorphisms onto X/T, so by Schanuel's Lemma,

$$T \oplus X \cong \operatorname{Ker} \varepsilon\pi \oplus X .$$

$\operatorname{End}_A(X)$ is local, so by [21, Th. 2.6], $T \cong \operatorname{Ker} \varepsilon\pi$. Thus $\operatorname{Ker} \varepsilon\pi$ has the same (finite) length as T. But $T \subseteqq \operatorname{Ker} \varepsilon\pi$ as submodules of X: hence $T = \operatorname{Ker} \varepsilon\pi$. Thus ε is a monomorphism.

THEOREM 1.2. *Suppose A is a left artinian ring with only finitely many nonisomorphic finitely generated indecomposable left modules. Then the same statements hold when "left" is replaced by "right".*

Proof. We first show that A is right artinian. A is in any case semiprimary, so by a theorem of Eilenberg [7], every projective right A-module is a direct sum of principal indecomposable right A-modules. If A is not right artinian, we may find a principal indecomposable right A-module X and an infinite increasing sequence of submodules

$$S_1 \subsetneqq S_2 \subsetneqq \cdots \subsetneqq X ,$$

such that each S_i has finite length. Since X has a local endomorphism ring, Schanuel's lemma and [21, Th. 2.6] show that $X/S_i \ncong X/S_j$ for $i \neq j$.

Let $U_1, \cdots, U_n$ be representatives of the finitely generated indecomposable nonprojective left A-modules. For each i we may write $D(X/S_i) = V_i \oplus \text{Projective}$, where V_i is a direct sum of certain U_j's. $D(V_i)$ is of course a direct sum of $D(U_j)$'s, and we have

$$(*) \qquad X/S_i \oplus \text{Projective} \cong D(V_i) \oplus \text{Projective} .$$

By Lemma 1.1, the left hand side of (*) is a direct sum of indecomposable modules with local endomorphism rings. Using [21, Lemma 2.9 and Th. 2.6] repeatedly, we see that there is an index $j = j(i)$ such that U_j is a summand of V_i, and X/S_i is a summand of $D(U_j)$.

By [21, Th. 2.6], the complement of X/S_i in the right hand side of (*) is projective, so we may write

$$X/S_i \oplus \text{Projective} \cong D(U_{j(i)}) \oplus \text{Projective} .$$

Since there are finitely many U_j's and infinitely many S_i's, there are indices i, i' such that $i \neq i'$ but $j(i) = j(i')$. For these indices we have

$$X/S_i \oplus \text{Projective} \cong X/S_{i'} \oplus \text{Projective} .$$

Both sides of this equation are sums of indecomposable modules with local endomorphism rings, so by the Krull-Schmidt Theorem, $X/S_i \cong X/S_{i'}$, a contradiction. Hence A is right artinian, and in particular, the Krull-Schmidt Theorem holds in the category of finitely generated right A-modules.

It is now easily seen that for any finitely generated right or left A-module M, M and $D(M)$ have the same number of nonprojective indecomposable summands. Also, two finitely generated A-modules without projective summands are isomorphic if and only if they are stably isomorphic. The conclusion of Theorem 1.2 now follows easily (A has the same number of finitely generated indecomposable modules on the right as on the left).

As mentioned in the introduction, the following result was first proved by Fuller [10, 5.4].

THEOREM 1.3. *Let A be a ring each of whose finitely generated left modules is a direct sum of uniserial modules. Then A is serial, and thus every left or right A-module is a direct sum of uniserial modules.*

REMARK. This theorem shows that the generalized left uniserial rings of Griffith [12] are the same as serial rings.

Proof of 1.3. A must be left artinian, so Theorem 1.2 shows that A is right artinian. If M is any finitely generated right module, then $D(M)$ is a direct sum of uniserial left modules. On the other hand, if U is a uniserial left module, then there is an exact sequence $X \rightarrow Y \twoheadrightarrow U$ with X and Y principal indecomposable A-modules. Thus $D(U) = \operatorname{Coker}(Y^* \rightarrow X^*)$ is a homomorphic image of X^*, which is a principal indecomposable right A-module. Thus $D(D(M))$ is a direct sum of homomorphic images of principal indecomposable right A-modules, and it follows by the Krull-Schmidt Theorem that the same is true of M. By Nakayama [20, Th. 3], A is a serial ring.

2. Artinian principal ideal rings. We will say that a module of finite length is *homogeneous* if its composition factors are all isomorphic to one another.

THEOREM 2.1. *Let A be a ring with radical N. Then the following are equivalent.*

(1) *A is an artinian principal ideal ring.*

(2) *A is serial and every ideal of A is a principal left ideal.*

(3) *Every left A-module is a direct sum of homogeneous uniserial modules.*

(4) *A is serial and, as a left A-module, is the direct sum of homogeneous uniserial modules.*

REMARK. Since (1) is left-right symmetric, each of (2), (3), and (4) is equivalent to the corresponding statement about right modules.

Proof. (1) $\Rightarrow$ (2): It suffices to prove A serial. For this it is enough to show that if X is an indecomposable summand of A, then $\operatorname{soc}(X/N^k X)$ is simple for every k, or equivalently, that $\lambda(\operatorname{soc}(A/N^k)) \leqq \lambda(A/N)$. Of course, the preimage in A of $\operatorname{soc}(A/N^k)$ is principal, so $\operatorname{soc}(A/N^k)$ is cyclic as an A/N-module. The required inequality follows readily.

(2) $\Rightarrow$ (3): Let $A \cong \coprod X_i$ as left A-modules, where each X_i is indecomposable. To prove that every indecomposable left A-module is homogeneous, it now suffices to prove that the modules X_i are all homogeneous. By assumption, $\operatorname{soc}(A) = \coprod \operatorname{soc} X_i$ is principal. Since it has the same length as A/N, we must have $A/N \cong \operatorname{soc} A$. Choose a simple A-module S and set $J = \{j \mid \operatorname{soc} X_j \cong S\}$, $T = \coprod_{j \in J} \operatorname{soc} X_j$. T is an ideal of A, and A/T is also an artinian principal ideal ring, so again we have that $(A/T)/\operatorname{Rad}(A/T) \cong \operatorname{soc}(A/T)$. Counting the number of simple summands on each side of this equation which are isomorphic to S, we see that, for each $j \in J$ such that $\lambda(X_j) \geqq 2$, $\operatorname{soc}(X_j/\operatorname{soc} X_j) \cong S \cong \operatorname{soc} X_j$. Continuing in this manner (or using the periodicity theorem [8, 2.3]), we see that each X_j is homogeneous. Since S was arbitrary, it follows that every X_i is homogeneous.

(3) $\Rightarrow$ (4): This follows from Theorem 1.3.

(4) $\Rightarrow$ (1): If X and Y are homogeneous uniserial projective modules, then $\operatorname{Hom}_A(X, Y) \neq 0$ if and only if $X \cong Y$. Of course, we may assume that A is indecomposable, and we see at once that A is a direct sum of isomorphic indecomposable left modules; $A = \coprod_{i=1}^n X_i$. Hence A is isomorphic to the ring of $n \times n$ matrices over the local serial ring $\operatorname{End}_A(X_i)$ (See [8, Corollary 2.2]). Thus the category of A-modules is equivalent to the category of $\operatorname{End}_A(X_i)$-

modules, so every indecomposable right or left A-module is homogeneous.

By symmetry, it suffices to prove that any left ideal L is principal. This is equivalent to showing that L/NL is a cyclic A/N-module. But L/NL is a direct summand of $\operatorname{soc}(A/NL)$, so it is enough to show that $\operatorname{soc}(A/NL)$ is a cyclic A/N-module. But A/NL is a direct sum of homogeneous uniserial modules. Hence, $\operatorname{soc}(A/NL) \cong A/N \otimes_A A/NL$ which is cyclic as an A/N-module, since A/NL is cyclic as an A-module.

We showed at the beginning of the proof of $(4) \Rightarrow (1)$ that an indecomposable serial ring, each of whose indecomposable modules is homogeneous, is a full matrix ring over a local serial ring. Moreover, a local serial ring is trivially seen to be a principal ideal ring. Combining these remarks with Theorem 2.1, we obtain the classical result [13, § 15].

COROLLARY 2.2. *A ring is an artinian principal ideal ring if and only if it is a product of full matrix rings over local artinian principal ideal rings.*

In [9, Remark, p. 249], Fuller has proven that a ring A is an artinian principal ideal ring if and only if it and each of its factor rings are QF. Our proof of Proposition 2.3 is very similar to the proof of Fuller's Theorem, except that we use the periodicity theorem [8, 2.3] to complete the argument.

PROPOSITION 2.3. *A ring A is an artinian principal ideal ring if and only if A is serial and* QF, *and for every minimal ideal T of A, A/T is* QF.

Proof. $\Rightarrow$: This is immediate from Corollary 2.2.

$\Leftarrow$: By Theorem 2.1, it suffices to show that, if $A = \coprod X_i$ is a direct sum decomposition of A into indecomposable modules, then the X_i are homogeneous. Recall that a QF ring contains a simple module of each isomorphism type. If A is semi-simple, there is nothing to prove. Otherwise, let S be a simple A-module which is not injective, and set $Y = \coprod_{\operatorname{soc} X_i \cong S} X_i$. Note that $\operatorname{soc} Y = T$ is a minimal two sided ideal contained in $\operatorname{Rad} A$. Thus S is an A/T-module. Since A/T is QF, S must appear in $\operatorname{soc}(A/T) \cong \operatorname{soc}(Y/T) \oplus \operatorname{soc}(A/Y)$. Since by construction S is not a submodule of A/Y, S must be a submodule of Y/T. A is QF, so $\operatorname{soc} X_i \cong \operatorname{soc} X_j$ implies that $X_i \cong X_j$. Hence $\operatorname{soc}(Y/T)$ is homogeneous and thus is a direct sum of copies of S. However,

$$\mathrm{soc}\,(Y/T) \cong \coprod_{X_i \subseteq Y} \mathrm{soc}\,(X_i/\mathrm{soc}\,X_i)\,.$$

Using the periodicity theorem [8, 2.3] we see that each $X_i \subseteq Y$ is homogeneous. Since S was an arbitrary noninjective simple module, every X_i is homogeneous.

3. The structure of serial rings. In this section we present a description, in module-theoretic terms, of the structure of serial rings. The parts of the theorem dealing with artinian principal ideal rings were established in § 2. The remainder of the theorem is equivalent to various results of Murase [16, 17, 18] and Chase [5, 6]. Since Murase did not formulate his work in module-theoretic terms, it has seemed worthwhile to include some restatements. We have also included a sketch of a module-theoretic proof of one of his theorems on triangular matrix rings (Theorem 3.2).

THEOREM 3.1. *Let A be a serial ring. Then $A = A_0 \times A_1 \times A_2 \times A_3$ (ring direct sum) such that*

(0) *A_0 is semi-simple, and A_i has no semi-simple factor for $i > 0$.*

(1) *Every indecomposable A_1-module is homogeneous; equivalently,*

(1′) *A_1 is* QF, *and if I is a minimal ideal of A, then A/I is* QF; *equivalently,*

(1″) *A_1 is an artinian principal ideal ring.*

(2) gl. dim $A_2/(\mathrm{Rad}\,A_2)^2 < \infty$; *equivalently,*

(2′) *Every homomorphic image of A_2 has finite global dimension; equivalently,*

(2″) *A_2 is a product of rings each of which is Morita-equivalent to a factor ring of a full ring of upper triangular matrix rings over a division ring.*

(3) *$A_3/(\mathrm{Rad}\,A_3)^2$ is* QF *and A_3 has no homogeneous projective modules.*

Proof. The equivalences (1) $\Leftrightarrow$ (1′) $\Leftrightarrow$ (1″) were established in § 2. The equivalences (2) $\Leftrightarrow$ (2′) is due to Chase [5] (Chase's theorem appears in this paper as Theorem 3.4). The equivalence (2′) $\Leftrightarrow$ (2″) is due to Murase [17, Ths. 17 and 18]; we sketch a new proof of it in Theorem 3.2.

The splitting of a serial ring which we will now indicate is essentially that given by Murase in [16, 17]. Specifically, his type one includes A_0 and A_2, A_1 is of the first kind in the second category, and A_3 is of the second kind in the second category.

Murase's proof that an indecomposable serial ring is one of the above kinds relies on a study of Kuppisch series (see [14]) for serial rings. We will give an alternate description.

Let $\operatorname{Rad} A = N$, and let $A = \coprod_{i \in I} X_i$ be a decomposition of A into a direct sum of indecomposable projective left modules. Set:

$I_0 = \{i \in I \mid X_i$ is simple and injective$\}$

$I_1 = \{i \in I \mid X_i$ is homogeneous but not simple$\}$

$I_2 = \{i \in I \mid i \notin I_0$ and $hd_{A/N^2}(X_i/NX_i) < \infty\}$

$I_3 = \{i \in I \mid i \notin I_0 \cup I_1 \cup I_2\}$.

It is easy to verify that I is the disjoint union of I_0, I_1, I_2, and I_3. Set $A_j = \coprod_{I_j} X_i$. The A_j may be seen to be ideals of A, and are thus ring direct summands of A.

It is immediate that A_0 is semi-simple and that every indecomposable A_1-module is homogeneous. If $\operatorname{Rad} A_2 = N_2$, then each of the finitely many simple A_2/N_2^2-modules has finite homological dimension. By a theorem of Auslander [3, Proposition 10], gl. dim. (A/N^2) is the supremum of these homological dimensions and is therefore finite.

Finally, A_3/N_3^2 has no simple projectives; hence every indecomposable summand of A_3/N_3^2 has the same length and is therefore injective. Thus A_3/N_3^2 is QF. This completes the proof of (1), (2), and (3).

We will now present a new proof of the implication (2) $\Rightarrow$ (2″).

It is easy to see that if A is a serial ring with radical N such that gl. dim. $(A/N^2) < \infty$, then A must have a simple projective. It then follows that if A is indecomposable, A is, in the terminology of Murase [16], of the first category. Murase proves [17, Ths. 17, 18]:

THEOREM 3.2. *Let A be a serial ring with radical N. Then* gl. dim. $(A/N^2) < \infty$ *if and only if A is the direct product of rings, each of which is Morita equivalent to a factor ring of a full ring of upper triangular matrices over a division ring.*

We will sketch a module-theoretic proof of this result.

First we state a familiar lemma describing the homomorphic images of upper triangular matrix rings. Let $\Omega_n = \{(i, j) \mid 1 \leqq i, j \leqq n\}$. The points of Ω_n are to be thought of as the elements of an $n \times n$ matrix, (i, j) being the intersection of the i-th row and the j-th column. A subset $\Lambda \subset \Omega_n$ will be called a staircase set of rank n if

(1) $(i, j) \in \Lambda$ implies $j \geqq i + 2$ (i.e., Λ contains no elements on or below the diagonal just above the main diagonal).

and

(2) $(i, j) \in \Lambda$ implies $(k, l) \in \Lambda$ whenever $k \leqq i$ and $l \geqq j$.

For example, the entries which lie in the shaded region of the matrix in Figure 1 form a staircase set:

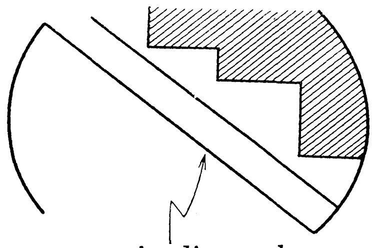

FIGURE 1.

LEMMA 3.3. *Let $T_n(D)$ be the ring of $n \times n$ upper triangular matrices with entries in the division ring D. There is a one-to-one correspondence between staircase sets of rank n and ideals I of $T_n(D)$ such that $T_n(D)/I$ is indecomposable. This correspondence is given by*

$$\Lambda \longleftrightarrow \{(a_{ij}) \in T_n(D) \mid a_{ij} = 0 \text{ for } (i, j) \notin \Lambda\} = I .$$

Thus any indecomposable homomorphic image of $T_n(D)$ may be regarded as the set of upper triangular matrices with zeros in all the entries in some staircase set, multiplication being the usual matrix multiplication, followed by a replacement of any nonzero entries within the staircase set by zeros. Note in particular that if A is an indecomposable homomorphic image of $T_n(D)$, and e_n is the matrix with 1 in the place (n, n) and zeros elsewhere, then e_nA is the only simple indecomposable right summand of A.

In proving Theorem 3.2, we will make use of the following theorem of Chase [5, Th. 4.1].

THEOREM 3.4. *Let A be a semiprimary ring with radical N. The following are equivalent:*

(1) gl. dim. $(A/N^2) < \infty$.

(2) gl. dim. $(A/I) < \infty$ *for all ideals I of A.*

(3) *There exists a full set of primitive idempotents $e_1, \cdots, e_k$ of A such that $e_iNe_j = 0$ for all $j \leqq i$.*

A ring of the sort described in Chase's Theorem is called a generalized triangular matrix ring.

To prove Theorem 3.2, it thus suffices to show that an indecomposable serial generalized triangular matrix ring A is Morita equivalent to a homomorphic image of a full ring of triangular matrices over a division ring. (The converse of this is an easy verification.) By passing to a Morita equivalent, we may assume that if $A = \coprod_{i=1}^{n} Ae_i$, where the e_i are primitive idempotents, then $Ae_i \not\cong Ae_j$ for $i \neq j$. By Theorem 3.4 we have, for $N = \operatorname{Rad} A$, the equations

$$e_iAe_j = 0 \quad \text{for every} \quad j < i$$
$$e_iNe_i = 0 \quad \text{for every} \quad i\ .$$

We will do an induction on the number of primitive orthogonal idempotents of A. For $n = 1$, A is a division ring, and the theorem is obvious (A is a 1×1 triangular matrix ring).

Suppose $n > 1$. Set $e = e_n$ and $f = 1 - e$. By Chase's Theorem, we have

$$A = \left\{\begin{pmatrix} r & m \\ 0 & d \end{pmatrix} \middle|\, r \in fAf,\ m \in fAe,\ d \in eAe \right\}$$

with the "usual" matrix multiplication. Of course eAe is a division ring. Since A is assumed indecomposable, $M = fAe \neq 0$. Set $D = eAe$, $S = fAf$, $J = \operatorname{Rad} S$. Then

$$\operatorname{Rad} A = N = \left\{\begin{pmatrix} j & m \\ 0 & 0 \end{pmatrix} \middle|\, j \in J,\ m \in M \right\}$$
$$N^2 = \left\{\begin{pmatrix} j & m \\ 0 & 0 \end{pmatrix} \middle|\, j \in J^2,\ m \in JM \right\}.$$

Thus $N/N^2 = J/J^2 \oplus M/JM$ as A-A-bimodules. Since A is serial and has only one more idempotent than S, M/JM must be simple as a left S and as a right eAe-module. In particular, M is indecomposable as a left S-module. Since A is indecomposable, this shows that S is indecomposable as a ring. But S is a factor ring of A. Thus S is an indecomposable serial generalized triangular matrix ring with only $n - 1$ orthogonal primitive idempotents. Hence by induction, S is a factor ring of an $(n-1) \times (n-1)$ triangular matrix ring over a division ring D'. Since M/JM is a simple left S-module, it is a one dimensional left D'-module. As it is also a one dimensional right D-module, we have $D \cong \operatorname{End}_{D'}(M/JM) \cong D'$ (where endomorphisms are written on the right).

We will next prove that M is a homomorphic image Se_{n-1}, say $M = Se_{n-1}/J^k e_{n-1}$. This will finish the proof; for let Λ be the staircase set of rank $n-1$ corresponding to S, as in the discussion following Lemma 3.3. Take $\Lambda' = \Lambda \cup \{(j, n) \mid j \leqq n - k\}$. It is easily seen that Λ' is a staircase set of rank n, and that A is the homomorphic image of $T_n(D)$ corresponding to Λ'.

Since A and S are serial, we have:

$$\lambda_A(N/N^2) = \lambda_S(J/J^2) + 1$$

$$\lambda_A(N/N^2) = \text{the number of } e_i \text{ such that } \lambda_A(e_iA) \geqq 2$$

$$\lambda_S(J/J^2) = \text{the number of } e_i,\ i \leqq n-1, \text{ such that } \lambda_S(e_iS) \geqq 2\ .$$

Now $\lambda_A(e_nA) = 1$ trivially, so there is $i \leqq n-1$ such that $\lambda_A(e_iA) \geqq 2$ and $\lambda_S(e_iS) = 1$.

By the remark following Lemma 2.6, $i = n-1$. Hence $e_{n-1}N = e_{n-1}M \neq 0$. This shows $e_{n-1}Ne_n \neq 0$, so there is a homomorphism of left S-modules $0 \neq \varphi\colon Se_{n-1} \to M$. But M is indecomposable, and hence is uniserial as an S-module, so M is the homomorphic image of some Se_k. Clearly Se_{n-1}/Je_{n-1} is isomorphic to a composition factor of M, and is not isomorphic to any composition factor of Se_k for $k \neq n-1$. Hence M is a homomorphic image of Se_{n-1}. This concludes the proof of Theorem 3.2.

4. Serial rings with zero singular ideal.

Let A be a ring. In [11], Goldie defines $Z(A)$, the singular ideal of A, as the set of elements of A whose left annihilators are essential left ideals. He proves [11, Th. 8.11].

THEOREM 4.1. *Let A be an indecomposable serial ring. Then $Z(A) = 0$ if and only if A is Morita equivalent to a full ring of upper triangular matrices over a division ring.*

As an application of the structure theorem, we will give a new proof of this result. But first we recall some immediate consequences of the definition of $Z(A)$.

If A is artinian, then a left ideal of A is essential if and only if it contains the left socle, say $\operatorname{soc}(A) = S$. Thus $Z(A) =$ right ann. $(S) \cong \operatorname{Hom}_A(A/S, A)$. If $A = \coprod X_i$, where the X_i are indecomposable left A-modules, then we see that $Z(A) = 0$ if and only if $\operatorname{Hom}_A(X_i/\operatorname{soc}(X_i), X_j) = 0$ for every i and j. In particular, the vanishing of the singular ideal of an artinian ring is invariant under Morita equivalence.

Proof of Theorem 4.1. Suppose that A is indecomposable and serial, and $Z(A) = 0$. We will show, in the language of Theorem 3.1 and Lemma 3.3, that either $A = A_0$, or $A = A_2$ and the associated staircase set is empty. This will clearly suffice to prove 4.1.

If $A = A_1$ then $A/\operatorname{soc}(A) \cong \operatorname{Rad} A$; so $Z(A) \neq 0$.

Suppose $A = A_3$. Then no indecomposable summand of A is simple, and by Kuppisch [14, Satz 5] we may arrange a decomposition $A = \coprod_{i=1}^{n} X_i$ of A into indecomposable modules in such a way that there are epimorphisms $\varphi_i: X_i \twoheadrightarrow (\mathrm{Rad}\, A)X_{i+1}$ for $i > n$, and $\varphi_n: X_n \twoheadrightarrow (\mathrm{Rad}\, A)X_1$. By a length argument, not all of these epimorphisms can be monomorphisms; hence for some i, $\mathrm{soc}\,(X_i) \subseteqq \mathrm{Ker}\, \varphi_i$. Thus $Z(A) \neq 0$.

To prove Goldie's Theorem, we only need show that if $A = A_2 = T_n(D)/I$ and $Z(A) = 0$, then $I = 0$; that is, the staircase set Λ associated to I is empty (see § 3). Suppose this were not so, and choose $(i, j) \in \Lambda$ with minimal j. Let e_k be the idempotent of A represented by the matrix with a 1 in the place (k, k) and zeros elsewhere. Then $\lambda(Ae_{j-1}) = j - 1$. Of course, there is an epimorphism $\varphi: Ae_{j-1} \twoheadrightarrow \mathrm{Rad}\,(A)e_j$. But $0 \neq \lambda(\mathrm{Rad}\,(A)e_j) \leqq j - i - 1$. Thus $\mathrm{soc}\,(Ae_{j-1}) \subseteqq \mathrm{Ker}\, \varphi$, so $\mathrm{Hom}_A\,(Ae_{j-1}/\mathrm{soc}\,(Ae_{j-1}), Ae_j) \neq 0$.

Added in proof. In [22], Skorjakov defines a module to be *chained* if its submodules are linearly ordered, and proves a theorem stronger than our Theorem 1.3: if A is a ring such that every left A-module is a direct sum of chained modules, then A is serial. His methods are quite different from those of section one of this paper.

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