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Let $P_{\alpha\beta\lambda}$ be the singular boundary value problem on $0 \leqq t < \infty$ consisting of the nonlinear ordinary differential equation $y''' + y'' + \lambda(1 - y'^2) = 0$, the boundary conditions $y(0) = \alpha$, $y'(0) = \beta$ and $y'(\infty) = 1$, and the condition $\beta < y'(t) < 1$ for $t > 0$. The problems $P_{\alpha\beta\lambda}$ arise in boundary layer theory and questions of existence and uniqueness have been settled for parameters on the range: (λ, α) arbitrary, $0 \leqq \beta < 1$. If $\lambda = 0$, a "discontinuity" occurs in the existence theory at $\beta = 0$ in the following sense: if $0 < \beta < 1$, then $P_{\alpha\beta0}$ has a solution for all α, but if $\beta = 0$, then there is a number A_0 with the property that $P_{\alpha00}$ has a solution if and only if $\alpha \geqq A_0$. In this paper, it is shown that, if $\lambda > 0$, a similar "discontinuity" occurs at $\beta = -1$; namely, if $\lambda > 0$ and $-1 < \beta < 1$, then $P_{\alpha\beta\lambda}$ has a solution for arbitrary α, while if $\beta = -1$, then there exists a number A_λ such that $P_{\alpha,-1,\lambda}$ has a solution if $\alpha > A_\lambda$ but no solution if $\alpha < A_\lambda$.

1. Questions of existence and nonexistence, uniqueness and nonuniqueness for the problem (1.1) – (1.4):

$$y''' + yy'' + \lambda(1 - y'^2) = 0 , \tag{1.1}$$

$$y'(t) \to 1 \quad \text{as} \quad t \to \infty , \tag{1.2}$$

$$y(0) = \alpha, \; y'(0) = \beta , \tag{1.3}$$

$$\beta < y'(t) < 1 \quad \text{for} \quad 0 < t < \infty , \tag{1.4}$$

have been settled, in the case $\beta = 0$, by Iglisch, Grohne, and Kemnitz; cf. [2],]6], [7], [8]. These papers include results of Weyl [10] for $\lambda \geqq 0$, but not those of Coppel [1] for $\lambda \geqq 0$, $\alpha \geqq 0$ and $0 \leqq \beta < 1$. See [2] and [4] for the cases $\lambda = 0$, $0 \leqq \beta < 1$. Asymptotic behavior of solutions is discussed in [1] and [3]. A complete summary of this theory for arbitrary λ, α, and $0 \leqq \beta < 1$ is given in [4, pp. 519–537].

Although the cases $\beta < 0$ do not occur in fluid mechanics, there is no mathematical reason for ignoring them. We shall prove the following assertion which exhibits the "discontinuity" in the existence theory at $\lambda > 0$, $\beta = -1$.

THEOREM 1.1. *Let $\lambda > 0$. If $-1 < \beta < 1$, then* (1.1)–(1.4) *has a* *solution for all* α. *If* $\beta = -1$, *then there exists a number* A_λ *such that* (1.1) – (1.4) *has a solution if* $\alpha > A_\lambda$, *but has no solution if* $\alpha < A_\lambda$.

The first statement, that, concerning $-1 < \beta < 1$, can easily be obtained by known arguments; cf., e.g., Coppel [1]. In order to obtain additional information about the solutions for use in the proofs of the statements concerning $\beta = -1$, a proof indicated in [4, Exercise 6.4, pp. 524–525] will be employed. It is known (Iglisch [7]) that the solutions in Theorem 1.1 are unique if $0 \leqq \beta < 1$; cf. [4, p. 523] (or Theorem 6.1 below). Questions of uniqueness will remain undecided for $-1 \leqq \beta < 0$, except in the case $\lambda = 1/2$, where they will be decided affirmatively (Theorem 6.1). The related question of existence when $\alpha = A_\lambda$ will remain open.

It can be remarked that (1.4) is equivalent to

$$y''(t) > 0 \quad \text{for} \quad 0 \leqq t < \infty \tag{1.5}$$

if $-1 \leqq \beta < 1$ and $\lambda > 0$. For, on the one hand, (1.1) has the trivial solutions $y = \alpha \pm t$, $y' = \pm 1$, $y'' = 0$. On the other hand, if $\lambda > 0$ and $y(t)$ is any nontrivial solution of (1.1) satisfying $y''(t_0) = 0$, then $y'''(t_0) < 0$ or $y'''(t_0) > 0$ according as $|y'(t_0)| < 1$ or $|y'(t_0)| > 1$ and, correspondingly, $y'(t)$ has a strict maximum or a minimum at $t = t_0$.

2. Existence statements ($\lambda > 0$, $|\beta| < 1$).

The corresponding part of Theorem 1.1 has the following refinements:

THEOREM 2.1. *For* α, β, λ *on the set*

$$\{(\alpha, \beta, \lambda) : -\infty < \alpha < \infty, \ -1 < \beta < 1, \ \lambda > 0\}, \tag{2.0}$$

there exist solutions $y = y_*(t; \alpha, \beta, \lambda)$, $y^*(t; \alpha, \beta, \lambda)$ *of* (1.1) – (1.4), *which may coincide, with the following two properties*: (i) *the functions*

$$y_*, y^* \text{ and } -(1-y_*')/(1-\beta), \ -(1-y^{*\prime})/(1-\beta), \text{ hence } y_*', y^{*\prime}, \tag{2.1}$$

and

$$y_*''/(1-y_*') \quad \text{and} \quad y^{*\prime\prime}/(1-y^{*\prime}) \tag{2.2}$$

are increasing functions of α, β, *and* λ *for fixed* $t \geqq 0$ *and* $t > 0$, *respectively*; (ii) *if* $y(t)$ *is any solution of* (1.1) – (1.4), *then*

$$\begin{gathered} y_* \leqq y \leqq y^*, \qquad y_*' \leqq y' \leqq y^{*\prime}, \\ y_*''/(1-y_*') \leqq y''/(1-y') \leqq y^{*\prime\prime}/(1-y^{*\prime}). \end{gathered} \tag{2.3}$$

The proof of Theorem 2.1 will be given in § 3. It depends on some essentially known results (Lemmas 1A, 2A, and 3A) on linear, second order differential equations stated in the Appendix below for easy reference in the forms to be used here.

3. Proof of Theorem 2.1.

(a) *Existence.* Assume that

$$\lambda > 0 , \quad -1 < \beta < 1 , \quad -\infty < \alpha < \infty . \tag{3.0}$$

Introduce the function

$$h = 1 - y' , \tag{3.1}$$

so that (1.1) becomes a linear second order equation for h,

$$h'' + yh' - \lambda(1 + y')\, h = 0 . \tag{3.2}$$

Define a sequence of successive approximations $\{y_n(t)\}$ as follows:

$$h_0 \equiv 1 - \beta , \quad y_0 \equiv \alpha + \beta t , \quad y_0' \equiv \beta . \tag{3.3}$$

If h_k, y_k, y_k' have been defined for $k = 1, \cdots, n$ satisfying

$$h_n(0) = 1 - \beta; \ h_n(t) > 0 \ \text{and} \ h_n'(t) < 0 \qquad \text{for } t \geqq 0 , \tag{3.4$_n$}$$

$$y_n' = 1 - h_n \ \text{ and } \ y_n(t) = \alpha + \int_0^t y_n'(s)\, ds , \tag{3.5$_n$}$$

let $h = h_{n+1}(t)$ be a solution of

$$h'' + y_n(t)h' - \lambda(1 + y_n'(t))\, h = 0 , \tag{3.6$_n$}$$

satisfying (3.4_{n+1}). Actually, such a solution exists, is unique, and satisfies the analogue of

$$h_n(t) \longrightarrow 0 , \quad \text{i.e., } y_n'(t) \longrightarrow 1 , \ \text{ as } t \longrightarrow \infty \tag{3.7}$$

(when $n \geqq 1$). This follows from the Lemma 1A applied to (3.6_n). The conditions (5) and (7) of the Lemma 1A can be verified directly for $n = 0$, and are a consequence of the Remark following Lemma 1A if $n \geqq 1$, since $1 + y_n' = 2 - h_n(t) \geqq 1 + \beta > 0$ for $\beta > -1$.

The sequences so defined satisfy

$$h_n > h_{n+1} \ \text{ for } t > 0; \ h_n'/h_n > h_{n+1}'/h_{n+1} \qquad \text{for } t \geqq 0 ; \tag{3.8$_n$}$$

$$y_n' < y_{n+1}' \ \text{ and } \ y_n < y_{n+1} \qquad \text{for } t > 0 . \tag{3.9$_n$}$$

The relations (3.8_0) hold, since $h_0 \equiv 1 - \beta$ and $h_1(0) = 1 - \beta$, $h_1'(t) < 0$ for $t \geqq 0$. Hence (3.9_0) holds. Assume (3.9_n). Then (3.8_{n+1}) follows from Lemma 2A (and its proof) if (3.5_n) has a unique solution $h = h_{n+1}$ satisfying (3.4_{n+1}). But this uniqueness has already been noted. This proves $(3.8_n)-(3.9_n)$ for $n=0, 1, \cdots$.

Define another sequence of successive approximations $\{Y_n(t)\}$ as follows:

$$H_0 \equiv 0, \quad Y_0(t) = \alpha + t, \quad Y_0' \equiv 1 . \tag{3.10}$$

If H_k, Y_k, Y_k' have been defined for $k = 1, \cdots, n$ satisfying

$$H_n(0) = 1 - \beta,\ H_n(t) > 0 \quad \text{and} \quad H_n'(t) < 0 \qquad \text{for } t \geqq 0 , \tag{3.11_n}$$

$$Y_n' \equiv 1 - H_n \quad \text{and} \quad Y_n(t) = \alpha + \int_0^t Y_n'(s)ds , \tag{3.12_n}$$

let $H = H_{n+1}(t)$ be a solution of

$$H'' + Y_n(t)H' - \lambda(1 + Y_n'(t))H = 0 \tag{3.13_n}$$

satisfying (3.11_{n+1}). This solution exists, is unique, and satisfies the analogue of

$$H_n(t) \longrightarrow 0, \text{ i.e., } Y_n'(t) \longrightarrow 1, \text{ as } t \longrightarrow \infty , \tag{3.14_n}$$

by Lemma 1A and the Remark following it.

As above, an induction, based on Lemma 2A, shows that, for $n = 0, 1, \cdots$,

$$H_n < H_{n+1} \quad \text{for} \quad t > 0; \ H_n'/H_n < H_{n+1}'/H_{n+1} \qquad \text{for } t \geqq 0 ; \tag{3.15_n}$$

$$Y_n' > Y_{n+1}' \quad \text{and} \quad Y_n > Y_{n+1} \qquad \text{for } t > 0 . \tag{3.16_n}$$

Similarly, we can obtain

$$H_m(t) < h_n(t) \qquad \text{for } t > 0 \tag{3.17_{mn}}$$

and $m, n = 0, 1, \cdots$; that is,

$$Y_m' > y_n' \quad \text{and} \quad Y_m > Y_n \qquad \text{for } t > 0 . \tag{3.18_{mn}}$$

For (3.17_{0n}), (3.18_{0n}) are trivial for $n = 0, 1, \cdots$. Thus Lemmas 1A and 2A give $(3.17_{1,n+1})$ for $n = 0, 1, \cdots$, while (3.18_{10}) is trivial. This argument which goes from $m = 0$ to $m = 1$ can be repeated for any m to give (3.17_{mn}), (3.18_{mn}). Thus

$$1 \equiv Y_0' \geqq Y_1' \geqq \cdots \geqq y_1' \geqq y_0' \equiv \beta , \tag{3.19}$$

$$\alpha + t = Y_0 \geqq Y_1 \geqq \cdots \geqq y_1 \geqq y_0 = \alpha + \beta t , \tag{3.20}$$

for $t \geqq 0$.

Consequently, the limits

$$y^*(t) = \lim_{n\to\infty} Y_n(t),\ y_*(t) = \lim_{n\to\infty} y_n(t)\ , \tag{3.21}$$

$$y^{*\prime}(t) = \lim_{n\to\infty} Y_n'(t),\ y_*'(t) = \lim_{n\to\infty} y_n'(t)\ , \tag{3.22}$$

exist for $t \geqq 0$. It is clear that the sequences $\{y_n'\}$, $\{y_n''\}$, $\{Y_n'\}$, $\{Y_n''\}$ are uniformly bounded on t-intervals. Hence (3.21), (3.22) hold uniformly on such intervals. If $h_*(t) = 1 - y_*'(t)$ and $h^* = 1 - y^{*\prime}(t)$, then these functions are the unique solutions of

$$h_*'' + y_*(t)h_*' - \lambda(1 + y_*'(t))h_* = 0\ , \tag{3.23}$$

$$h_*'' + y^*(t)h^{*\prime} - \lambda(1 + y^{*\prime}(t))h^* = 0\ , \tag{3.24}$$

satisfying $h_* > 0$, $h_*' < 0$ and $h^* > 0$, $h^{*\prime} < 0$ for $t \geqq 0$; cf. Lemma 3A. By Lemma 1A, we have $h_*(t)$, $h^*(t) \to 0$ as $t \to \infty$. It follows that $y = y_*(t)$, $y^*(t)$ are solutions of (1.1) – (1.4).

(b) *On monotony* (i). We now verify the statements concerning the monotony of (2.1) – (2.2). Let the functions h_n, y_n, H_n, Y_n be written as $h_n(t;\alpha,\beta,\gamma)$, $y_n(t;\alpha,\beta,\lambda)$, $\cdots$. For

$$\alpha_1 \leqq \alpha_2,\ -1 < \beta_1 \leqq \beta_2 < 1,\ 0 < \lambda_1 \leqq \lambda_2\ , \tag{3.25}$$

let $h_{nj}(t) = h_n(t;\alpha_j,\beta_j,\lambda_j)$ for $j = 1, 2$. Then

$$h_{n2}/(1-\beta_2) \leqq h_{n1}/(1-\beta_1) \qquad \text{for } t \geqq 0\ ,$$

$$h'_{n2}/h_{n2} \leqq h'_{n1}/h_{n1} \qquad \text{for } t \geqq 0\ ,$$

$$y'_{n2} \geqq y'_{n1} \text{ and } y_{n2} \geqq y_{n1} \qquad \text{for } t \geqq 0\ ,$$

hold for $n = 0$. An induction shows that these inequalities hold for $n = 0, 1, \cdots$. (Note that the first of these inequalities gives

$$(1-y_{n2}')/(1-\beta_2) \leqq (1-y_{n1}')/(1-\beta_1) \leqq (1-y_{n1}')/(1-\beta_2)\ ,$$

which implies the last two inequalities.) A limit process, as $n \to \infty$, shows that the functions

$$y_*,\ -(1-y_*')/(1-\beta),\ y^{*\prime\prime}/(1-y_*') \tag{3.26}$$

are nondecreasing with respect to α, β, and λ. By considering the equation (3.23) for $h_*(t) = h_*(t;\alpha,\beta,\lambda)$, one obtains the assertions concerning the strict monotony of (3.26) in Theorem 2.1. The monotony statements concerning y^*, $y^{*\prime}$, $y^{*\prime\prime}$ are proved similarly.

(c) *On part* (ii). Let $y(t)$ be a solution of (1.1) – (1.4) and $h(t) = 1 - y'(t)$, so that $h > 0$, $h' < 0$. Then $y_0(t) \leqq y(t) \leqq Y_0(t)$ and $y_0'(t) \leqq y'(t) \leqq Y_0'(t)$. Thus the differential equations (3.2) and (3.6$_0$), (3.13$_0$) imply the case $n = 1$ of

$$h_n \geqq h \geqq H_n \quad \text{and} \quad h_n'/h_n \geqq h'/h \geqq H_n'/H_n\,,$$

(3.27)

$$y_n \leqq y \leqq Y_n \quad \text{and} \quad y_n' \leqq y' \leqq Y_n'\,,$$

by Lemma 2A. A simple induction argument shows the validity of these inequalities for $n = 1, 2, \cdots$ and $t \geqq 0$. Hence (2.3) follows by letting $n \to \infty$.

4. A preliminary result ($\lambda > 0$, $\beta = -1$). Define the functions $\gamma^*(\alpha, \beta, \lambda)$ $\gamma_*(\alpha, \beta, \lambda)$ on the set (2.0) by

$$\gamma^*(\alpha, \beta, \lambda) = y^{*\prime\prime}(0; \alpha, \beta, \lambda), \quad \gamma_*(\alpha, \beta, \lambda) = y_*''(0; \alpha, \beta, \lambda)\,; \tag{4.0}$$

so that, for example, y^* is the solution of (1.1) satisfying the initial condition $y^* = \alpha$, $y^{*\prime} = \beta$, $y^{*\prime\prime} = \gamma^*$ at $t = 0$. By Theorem 2.1, the positive function $\gamma^*(\alpha, \beta, \lambda)/(1-\beta)$ is an increasing function of each of its arguments on the set (2.0). Define $\gamma^*(\alpha, -1, \lambda)$ by

$$\gamma^*(\alpha, -1, \lambda) = 2 \lim_{\beta \to 1} \gamma^*(\alpha, \beta, \lambda)/(1-\beta) = \lim_{\beta \to 1} \gamma^*(\alpha, \beta, \lambda)\,. \tag{4.1}$$

Thus $\gamma^*(\alpha, -1, \lambda)$ is a nondecreasing function of α and of λ, and satisfies

$$0 \leqq \gamma^*(\alpha, -1, \lambda) < 2\,\gamma^*(\alpha, \beta, \lambda)/(1-\beta) \quad \text{for } -1 < \beta < 1\,. \tag{4.2}$$

PROPOSITION 4.1. *The problem* (1.1) – (1.4) *with* $\beta = -1$ *has a solution if and only if*

$$\gamma^*(\alpha, -1, \lambda) > 0\,. \tag{4.3}$$

In particular, if it has a solution for $\beta = -1$ *and some* $\lambda = \lambda_0 > 0$, $\alpha = \alpha_0$, *then it has a solution for* $\beta = -1$, $\alpha \geqq \alpha_0$, $\lambda \geqq \lambda_0$.

Proof. Let $\lambda > 0$ be fixed and denote the solution of (1.1) satisfying the initial conditions $y(0) = \alpha$, $y'(0) = \beta$, $y''(0) = \gamma$ by $y = y(t, \alpha, \beta, \gamma)$.

Suppose that (1.1) – (1.4) has a solution $y(t)$, where $\lambda > 0$, $\beta = -1$. Then (1.4) implies that $-1 < y'(t) < 1$ for $t > 0$. Thus, by Theorem 2.1,

$$y''(t) \leqq \gamma^*(y(t), y'(t), \lambda) \qquad \text{for } t > 0\,.$$

Since $y'(0) = -1 < 0$ implies that $y(t) < \alpha$ for small $t > 0$,

$$y''(t) \leqq \gamma^*(\alpha, y'(t), \lambda) \qquad \text{for small } t > 0\,.$$

Letting $t \to 0$ gives

$$\gamma^*(\alpha, -1, \lambda) \geqq y''(0) > 0 . \tag{4.4}$$

Consequently, (4.3) is a necessary condition for the existence of a solution of (1.1) – (1.4), $\lambda > 0$ and $\beta = -1$.

In order to prove the converse, assume that (4.3) holds. Let $y^*(t) = y^*(t; \alpha, \beta, \lambda)$ be the "maximal" solution of (1.1) – (1.4), $-1 < \beta < 1$, of Theorem 2.1. Thus

$$y^*(t; \alpha, \beta, \lambda) = y(t, \alpha, \beta, \gamma^*(\alpha, \beta, \lambda)) .$$

Let

$$y(t) = y(t, \alpha, -1, \gamma^*(\alpha, -1, \lambda)) = \lim_{\beta \to -1} y^*(t; \alpha, \beta, \lambda) . \tag{4.5}$$

Then $y(t)$ is the solution of (1.1) satisfying the initial conditions

$$y(0) = \alpha , \quad y'(0) = -1 , \quad y''(0) = \gamma^*(\alpha, -1, \lambda) > 0 .$$

In view of (4.4) and the inequalities for $y^* = y^*(t; \alpha, \beta, \lambda)$,

$$\alpha + \beta t \leqq y^* \leqq \alpha + t , \quad -1 < y^* < 1$$

$$0 \leqq y^{*\prime\prime}/(1-y^{*\prime}) \leqq y^{*\prime\prime}(t; \alpha, \beta_0, \lambda)/(1 - y^{*\prime\prime}(t; \alpha, \beta_0, \lambda))$$

for $t \geqq 0$, $-1 < \beta \leqq \beta_0 < 1$, the solution (4.5) exists for $t \geqq 0$ and satisfies

$$-1 \leqq y'(t) \leqq 1 \text{ and } y''(t) \geqq 0 \qquad \text{for } t \geqq 0 .$$

Note that $y''(t) > 0$ for $t \geqq 0$, for if there is a least $t = t_0 > 0$ where $y''(t_0) = 0$, then $y'''(t_0) < 0$ (unless $y'(t_0) = 1$ and $y''(t_0) = 0$ which, by the uniqueness of solutions of (1.1), would imply that $y = \alpha + t$, $y' = 1$, $y'' \equiv 0$). But $y''(t_0) = 0$, $y'''(t_0) < 0$ gives $y''(t) < 0$ for small $t - t_0 > 0$. Hence $y''(t) > 0$ and $-1 < y'(t) < 1$ for $t > 0$. It is easy to show that $y'(t) \to 1$ as $t \to \infty$; e.g., examine the equation (3.2) for $h = 1 - y'$. Thus $y = y(t)$ is a solution of (1.1) – (1.4). This completes the proof of Proposition 4.1.

5. Proof of Theorem 1.1. ($\lambda > 0$, $\beta = -1$). This part of Theorem 1.1 follows from Proposition 4.1 if it is verified that when $\lambda > 0$ and $\beta = -1$, then

(a) there exist α-values for which (1.1) – (1.4) has solutions;

(b) there exist α-values for which (1.1) – (1.4) does not have solutions.

On (a). Let $y(t)$ be the solution of (1.1) – (1.4) for a given $\lambda > 0$, $\beta = 0$, $\alpha = 0$. It will be shown that there exists a (largest) $t_1 < 0$ such that $y(t)$ exists on $t_1 \leqq t < \infty$, $y'(t_1) = -1$, and $y''(t) > 0$ for $t_1 \leqq t < \infty$. In this case, $y(t + t_1)$ is a solution of (1.1) – (1.4) with $\alpha = y(t_1)$, $\beta = -1$, and assertion (a) is proved.

Since $y'''(0) = -\lambda < 0$, it follows that $y''(t) > y''(0) > 0$, $y' < 0$ for small $-t > 0$. Also, $y(t) > y(0) = 0$ as t decreases from 0, as long as $y'(t) < 0$. Thus $y''' = -yy'' - \lambda(1 - y'^2) < 0$ as long as $-1 \leqq y' < 0$. Hence t_1 exists unless there is a $t_0 < 0$ such that $y''(t) \to -\infty$ as $t \to t_0 + 0$, while $y'(t) > -1$ for $t_0 < t \leqq 0$. Since this is impossible, the desired t_1 exists.

The proof of (b) will be obtained in several steps (c) – (g).

(c) It will be shown that for a fixed $\lambda > 0$, there exists a large $-\alpha > 0$ such that $y(t, \alpha, -1, \gamma)$ is not a solution (1.1) – (1.4), $\lambda > 0$ and $\beta = -1$, for any choice of $\gamma > 0$. If $y(t, \alpha, -1, \gamma)$ is a solution, then $0 < y''(0) = \gamma \leqq \gamma^*(\alpha, -1, \lambda) \leqq \gamma^*(0, -1, \lambda)$; cf. (4.4). Hence, it suffices to show that $y(t, \alpha, -1, \gamma)$ is not a solution of (1.1) – (1.4) for γ on the fixed (possibly empty) bounded range $0 < \gamma \leqq \gamma^*(0, -1, \lambda)$.

(d) It will be verified, by an induction, that n differentiations of (1.1) give a differential equation of the form

$$y^{(n+3)} + yy^{(n+2)} + (n - 2\lambda)y'y^{(n+1)} = P_n(y'', \cdots, y^{(n)}) , \tag{5.1}$$

where P_n is an expression of the type

$$P_n = \sum_{j=2}^{n} a_{nj}(\lambda)y^{(j)}y^{(n+2-j)} , \tag{5.2}$$

$$a_{nj}(\lambda) \geqq 0 \qquad \text{for } 2\lambda \geqq n - 1 . \tag{5.3}$$

This statement is correct if $n = 1$, with $P_1 = 0$, for a differentiation of (1.1) gives

$$y^{(4)} + yy''' + (1 - 2\lambda)y'y'' = 0 .$$

Assume (5.1) – (5.3) and differentiate (5.1) to get

$$y^{(n+4)} + yy^{(n+3)} + (n + 1 - 2\lambda)y'y^{(n+2)} = P_{n+1} ,$$

where $P_{n+1} = P_n' + (2\lambda - n)y''y^{(n+1)}$. Hence the analogue of (5.1) – (5.3) holds for $n + 1$.

(e) The solution $y(t) = y(t, \alpha, -1, \gamma)$ satisfies the initial conditions

$$y(0) = \alpha,\ y'(0) = -1,\ y''(0) = \gamma . \tag{5.4}$$

It will be shown, by an induction, that $y^{(k+2)}(0)$ is a polynomial in α, γ of the form

$$y^{(k+2)}(0) = \gamma\Big[(-\alpha)^k + \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} b_{kij}(\lambda)\alpha^j\gamma^i\Big] \tag{5.5$_k$}$$

for $k = 0,\ 1, \cdots$. This is the case for $k = 0$ and $k = 1$, $y''(0) = \gamma$ and $y'''(0) = -\gamma\alpha$. Assume (5.5$_k$) for $0 \leqq k \leqq n$, $n \geqq 1$. Then, by

(5.1),

$$y^{(n+3)}(0) = -\alpha y^{(n+2)}(0) + (n - 2\lambda)y^{(n+1)}(0) + P_n(y''(0), \cdots, y^{(n)}(0)) .$$

Hence (5.5_{n+1}) follows from (5.5_k), $0 \leqq k \leqq n$, and (5.2), (5.3).

Note that (5.5_k) implies, for $k > 0$, that

$$y^{(k+2)}(0) - \tau\alpha y^{(k+1)}(0) = \gamma[(1 + \tau)(-\alpha)^k + \Sigma\Sigma\cdots] , \tag{5.6}$$

for any constant τ, where $\Sigma\Sigma \cdots$ is a sum of terms of the form $C_{kij}(\lambda, \tau)\alpha^j\gamma^i$ for $0 \leqq i, j \leqq k - 1$.

(f) Let $\lambda > 0$ and let $n > 0$ be the integer satisfying

$$n - 1 < 2\lambda \leqq n . \tag{5.7}$$

Let τ denote the number

$$\tau = 2\lambda - n , \quad \text{so that} \quad 0 < 1 + \tau \leqq 1 , \quad \tau \leqq 0 . \tag{5.8}$$

It follows from (5.5_k) and (5.6) that there exists a number $-\alpha > 0$, so large that if $0 < \gamma \leqq \gamma^*(0, -1, \lambda)$, then

$$y^{(k)}(0) > 0 \qquad \text{for } k = 2, \cdots, n + 2 , \tag{5.9}$$

$$y^{(n+2)}(0) - \tau\alpha y^{(n+1)}(0) > 0 . \tag{5.10}$$

(g) It will now be shown that, for such a choice of α, the problem (1.1) – (1.4) with $\beta = -1$ has no solution, To this end, write (5.1) as a second order equation for $v = y^{(n+1)}$,

$$v'' + yv' + (n - 2\lambda)y'v = P_n(y'', \cdots, y^{(n)}) . \tag{5.11}$$

In this equation, make the variation of constants

$$v = w \exp \tau\int_0^t y(r)dr , \tag{5.12}$$

transforming (5.11) into

$$w'' + (2\tau+1)yw' = [(\tau+n-2\lambda)y' - \tau(\tau+1)y^2]\, w + P_n \exp\left(-\tau\int_0^t y(r)dr\right) .$$

Using the definition (5.8) of τ gives

$$w'' + (2\tau+1)yw' = -\tau(\tau+1)y^2 w + P_n \exp\left(-\tau\int_0^t ydr\right) . \tag{5.13}$$

Since $\tau \leqq 0$ and $1 + \tau > 0$, the right side of (5.13) is nonnegative if w and $y'', \cdots, y^{(n)}$ are nonnegative, cf. (5.2), (5.3). Also

$$w = v \exp\left(-\tau \int_0^t ydr\right) , \quad w' = (v' - \tau yv) \exp\left(-\tau\int_0^t ydr\right) , \tag{5.14}$$

where $v = y^{(n+1)}$. In particular, w satisfies the initial conditions

$$w(0) = v(0) = y^{(n+1)}(0) > 0 ,$$

$$w'(0) = v'(0) - \tau\alpha v(0) = y^{(n+2)}(0) - \tau\alpha y^{(n+1)}(0) > 0 .$$

Up to a positive factor, the left side of (5.13) is

$$\left(w' \exp(2\tau+1) \int_0^t y dr\right)'.$$

If w' is positive on an interval $[0, t]$, then the function w, hence $y^{(n+1)}$, and $y'', \cdots, y^{(n)}$ are positive there also, by virtue of (5.9). Consequently, $y'', \cdots, y^{(n+1)}$ and w, w' are positive on their right maximal interval of existence.

If $n > 1$, this implies that $y''' > 0$, hence $y''(t) \geqq \gamma > 0$. If $y(t)$ exists for $t \geqq 0$, then $y'(t) \to \infty$ as $t \to \infty$ and $y(t)$ is not a solution of (1.1) – (1.4).

Consider the case $n = 1$. Suppose, if possible, that $y(t)$ is a solution of (1.1) – (1.4). Then

$$y''' - \tau y y'' = -(1+\tau) y y'' - \lambda(1-y'^2) < 0 \qquad \text{for large } t.$$

This contradicts $w' > 0$; cf. (5.14), where $v = y''$. This completes the proof of (c), hence (b), and of Theorem 1.1.

6. Uniqueness statements. We shall give a new simple proof of uniqueness in the cases $\lambda \geqq 0$, $0 \leqq \beta < 1$ and show that this proof can be modified to obtain uniqueness when $\lambda = 1/2$, $-1 \leqq \beta < 1$.

THEOREM 6.1 *The problem* (1.1) – (1.4) *has at most one solution for* $\lambda \geqq 0$, $0 \leqq \beta < 1$ *and for* $\lambda = 1/2$, $-1 \leqq \beta < 1$.

Proof. ($\lambda \geqq 0$, $0 \leqq \beta < 1$). Along a solution $y = y(t)$ of (1.1) for which $y''(t) > 0$, it is possible to introduce the new independent variable

(6.1) $$x = y'(t) ,$$

so that

(6.2) $$z = y''(t) = dx/dt$$

and

(6.3) $$dy/dx = y'/y'' \quad \text{and} \quad dz/dx = y'''/y'' .$$

Thus (1.1) is equivalent to the (nonautonomous) first order system

$$dy/dx = x/z\,, \qquad dz/dx = -y - \lambda(1-x^2)/z\,, \tag{6.4}$$

in considering solutions for which $y'' > 0$. The partial set of initial conditions (1.3) becomes

$$y = \alpha \text{ at } x = \beta\,. \tag{6.5}$$

If $y(t)$ is a solution of (1.1) – (1.4), $\lambda \geqq 0$, then $y''' \leqq 0$ for large t, so that $y'' \geqq 0$ is nonincreasing and (1.2) implies, therefore, that $y'' \to 0$ as $t \to \infty$. (Actually, the asymptotic behavior of y'' at $t = \infty$ is known; cf. [4, p. 536]). Thus a solution y of (1.1) – (1.4) determines a solution (y, z) of the system (6.4) for $\beta \leqq x < 1$ satisfying (6.5) and

$$(0 <)\, z \longrightarrow 0 \text{ as } x \longrightarrow 1\,. \tag{6.6}$$

It will be shown that (6.4) has at most one solution satisfying (6.5) and (6.6) if $\beta \geqq 0$. In fact, if $0 \leqq \beta \leqq x < 1$, then the right sides of the equations in (6.4) are nonincreasing functions of z and y, respectively. Thus, a theorem of Kamke [9] implies that if (y_1, z_1), (y_2, z_2) are two solutions of (6.4) for $(0 \leqq)\, \beta \leqq x < 1$ such that $y_1 = y_2 = \alpha$ and $0 < z_1 < z_2$ at $x = \beta$, then $y_1 > y_2$ and $z_1 < z_2$ for $\beta < x < 1$. Thus the last equation of (6.4) shows that $d(z_2 - z_1)/dx > 0$ for $\beta \leqq x < 1$. In particular, the limit of $z_2 - z_1$, as $x \to 1$, is positive $(\leqq \infty)$. Hence $z = z_1, z_2$ cannot satisfy (6.6), and Theorem 6.1 is proved for the cases indicated.

Proof. ($\lambda = 1/2$, $-1 \leqq \beta < 1$). The system (6.4) can be reduced to a nonlinear, second order equation by differentiating the second equation of (6.4) with respect to x and using the first equation to obtain

$$d^2z/dx^2 = (2\lambda - 1)x/z + [\lambda(1-x^2)/z^2]dz/dx\,, \tag{6.7}$$

If $2\lambda - 1 = 0$, this reduces to

$$d^2z/dx^2 = \left[\frac{1}{2}(1-x^2)/z^2\right]dz/dx\,. \tag{6.8}$$

This linear, homogeneous equation for dz/dx shows that either $dz/dx \equiv 0$ or $dz/dx \neq 0$ for $\beta \leqq x < 1$. In the case of a solution satisfying (6.6), it follows that

$$dz/dx < 0 \qquad \text{for } \beta \leqq x < 1\,. \tag{6.9}$$

Rewrite (6.8) as a binary, first order system

$$dz/dx = u, \quad du/dx = \frac{1}{2}(1-x^2)u/z^2\,. \tag{6.10}$$

Suppose that (1.1) – (1.4) has two solutions $y_1(t)$, $y_2(t)$, so that $y_1 = y_2 = \alpha$, $y_1' = y_2' = \beta$ and $y_2'' > y_1''$ at $t = 0$. By the last part of (6.4), the corresponding solutions (u_1, z_1), (u_2, z_2) of (5.4) satisfy

$$0 > u_2 \geqq u_1, z_2 > z_1 > 0 \tag{6.11}$$

at $x = \beta$, $-1 \leqq \beta < 1$.

For $-1 \leqq \beta \leqq x < 1$, $u < 0$ and $z > 0$, the right sides of the equations in (6.10) are increasing functions of u, z, respectively. Thus, a theorem of Kamke [9] shows that solutions (u_1, z_1), (u_2, z_2) of (6.10) satisfy (6.11) for $\beta \leqq x < 1$, provided that they satisfy $u < 0$ and $z > 0$ for $(-1 \leqq) \beta < x < 1$ and (6.11) for $t = \beta$.

This leads to the same contradiction obtained in the last proof, and completes the proof of Theorem 6.1.

REMARK. A differentiation of (1.1) with respect to t gives

$$y^{(4)} + yy''' + (1-2\lambda)y'y'' = 0 \,, \tag{6.12}$$

so that if $1 - 2\lambda = 0$,

$$y^{(4)} + yy''' = 0 \,. \tag{6.13}$$

One can give a similar proof of uniqueness ($\lambda = 1/2$, $-1 \leqq \beta < 1$) by writing (6.13) as a first order system for the vector $(y, y', y'', y'''/y'')$, with t as independent variable.

7. Remarks on continuity.

Although we cannot settle the general question of uniqueness for $\lambda > 0$, $-1 \leqq \beta < 0$, the following may be of interest.

PROPOSITION 7.1. *Let $\lambda > 0$, $-1 < \beta < 1$ be fixed and $Z = Z(\beta, \lambda)$ the set of pairs $(\alpha, \gamma) \in R^2$ such that the solution of the initial value problem $y = y(t, \alpha, \beta, \gamma)$ of* (1.1) *and $y = \alpha$, $y' = \beta$, $y'' = \gamma$ at $t = 0$ is a solution* (1.1) – (1.4). *Then Z is homeomorphic to a line.*

Of course, the solution of (1.1) – (1.4) is unique if (α, γ_1), $(\alpha, \gamma_2) \in Z \Rightarrow \gamma_1 = \gamma_2$. In terms of the functions γ_*, γ^* in (4.0), the solution of (1.1) – (1.4) is unique if and only if $\gamma_*(\alpha, \beta, \lambda) = \gamma^*(\alpha, \beta, \lambda)$.

PROPOSITION 7.2. *Let $\lambda > 0$ be fixed. The problem* (1.1) – (1.4) *has a unique solution for (α, β) on the set*

$$-\infty < \alpha < \infty \,, \quad (-1<)\, \beta_0 \leqq \beta < 1 \tag{7.1}$$

if and only if the functions $\gamma_*(\alpha, \beta, \lambda)$, $\gamma^*(\alpha, \beta, \lambda)$ are continuous on (7.1).

It is clear that uniqueness ($\gamma_* = \gamma^*$) implies continuity; cf. the last half of the proof of Proposition 4.1.

Proof of Proposition 7.1. In this proof, we shall only use the uniqueness of the solution of (1.1) – (1.4) for $\lambda > 0$, $\beta = 0$; i.e., the fact that $\gamma_*(\alpha, 0, \lambda) = \gamma^*(\alpha, 0, \lambda)$ is continuous.

For fixed α and $\beta = 0$, the problem (1.1) – (1.4) has a unique solution. Let $y = y(x, \alpha)$, $z = z(x, \alpha)$ be the corresponding solution of the system (6.4), so that $y(0, \alpha) = \alpha$ and $z(0, \alpha) = \gamma_*(\alpha, 0, \lambda) = \gamma^*(\alpha, 0, \lambda)$ and $y(x, \alpha)$, $z(x, \alpha)$ are defined for $0 \leqq x < 1$. Since, by (2.3),

$$0 < \gamma_*(y(x, \alpha), x, \lambda) \leqq z\,(x, \alpha) \leqq \gamma^*(y(x, \alpha), x, \lambda)\ ,$$

it is clear that the solution $(y(x, \alpha), z(x, \alpha))$ of (6.3) exists for $-1 < x < 1$. The continuity of $\gamma^*(\alpha, 0, \lambda)$ implies that $y(x, \alpha)$, $z(x, \alpha)$ are continuous for $-1 < x < 1$, $-\infty < \alpha < \infty$. For fixed β_0, $-1 < \beta_0 < 1$, define the continuous mapping T,

$$T\ \alpha = (\beta_0,\ y(\beta_0, \alpha), z(\beta_0, \alpha))\ ,$$

of the α-line onto a set Z in the plane $x = \beta_0$ of the (x, y, z)-space. Thus $Z = Z(\beta_0, \lambda)$ consists of the set of points (β_0, y_0, z_0) such that the solution of the initial value problem (1.1) and $y(0) = y_0$, $y'(0) = \beta_0$, $y''(0) = z_0$ is a solution of (1.1) – (1.4).

The map T has an inverse obtained as follows: Let $(\beta_0, y_0, z_0) \in Z$ and $y = y(x, y_0, z_0)$, $z = z(x, y_0, z_0)$ the solution of (6.4) satisfying the initial condition $y = y_0$, $z = z_0$ at $x = \beta_0$. Then

$$T^{-1}(\beta_0, y_0, z_0) = y(0, y_0, z_0) = \alpha\ .$$

Clearly, T^{-1} is continuous and onto the α-line. Thus Z is homeomorphic to a line.

Proof of Proposition 7.2. We have to show that the continuity of γ_*, γ^* on (7.1) implies the uniqueness of the solution of (1.1) – (1.4) for $\beta = \beta_0$.

(a) Assume the continuity of $\gamma_*(\alpha, \beta, \lambda)$, $\gamma^*(\alpha, \beta, \lambda)$ on (7.1). We shall show that

$$(7.2)\quad Z = \{(x, y, z) : x = \beta_0, \gamma_*(y, \beta_0, \lambda) \leqq z \leqq \gamma^*(y, \beta_0, \lambda),\ y \text{ arbitrary}\}\ .$$

In other words, if $y(t) = y(t, \alpha, \beta_0, \gamma)$ is the solution of the initial value problem (1.1) and $y = \alpha$, $y' = \beta_0$, $y'' = \gamma$ at $t = 0$, then a necessary and sufficient condition that $y(t)$ be a solution of (1.1) – (1.4) is that

$$(7.3)\qquad \gamma_*(\alpha, \beta, \lambda) \leqq \gamma \leqq \gamma^*(\alpha, \beta, \lambda)\ .$$

The necessity of (7.3) is contained in Theorem 2.1. In order to prove the converse, assume (7.3). It will be shown that $y(t)$ exists for $t \geqq 0$ and satisfies (1.1)–(1.4). This is clear if there is a $t \geqq 0$ where either

$$y''(t) = \gamma_*(y(t), y'(t), \lambda) \quad \text{or} \quad y''(t) = \gamma^*(y(t), y'(t), \lambda) .$$

If this is not the case, then (7.3) and the continuity of γ_*, γ^* imply that

$$0 < \gamma_*(y(t), y'(t), \lambda) < y''(t) < \gamma^*(y(t), y'(t), \lambda)$$

on any right interval of existence of y. But then $y(t)$ exists for $t \geqq 0$. Also, there exists a t-value t_0 such that $y'(t_0) \geqq 0$, so that

$$\gamma_*(y(t_0), y'(t_0), \lambda) = \gamma^*(y(t_0), y'(t_0), \lambda) .$$

This contradiction proves the sufficiency of (7.3).

(b) *Uniqueness.* Assume the continuity of γ_*, γ^* on (7.1). We shall show that this implies the uniqueness of the solution of (1.1)–(1.4) when $\beta = \beta_0$. If this is not the case, that is, if $\gamma_*(\alpha, \beta_0, \lambda) \not\equiv \gamma^*(\alpha, \beta_0, \lambda)$, then, by (7.2), Z contains a 2-dimensional open set. But this contradicts the fact that Z is homeomorphic to a line, and completes the proof of Proposition 7.2.

Appendix. This appendix will deal with lemmas on second order, linear differential equations used in the proofs of Theorem 1.1 and 2.1. They are simple variants of known results. They are given here for easy reference and the proofs will only be sketched or omitted.

LEMMA 1A. *Let $q(t) \geqq 0$, $p(t)$ be continuous, real-valued for $0 \leqq t < \infty$. Then*

$$h'' + p(t)h' - q(t)h = 0 \tag{1}$$

has a solution satisfying

$$h(0) = 1,\ h \geqq 0 \ \text{ and } \ h' \leqq 0 \qquad \text{for } t \geqq 0 , \tag{2}$$

while

$$h'(t_0) = 0 \ \text{ implies that } \ q(t) \equiv 0 \qquad \text{for } t \geqq t_0 , \tag{3}$$

so that, unless $q(t) \equiv 0$ for large t,

$$h'(t) < 0 \qquad \text{for } t \geqq 0 . \tag{4}$$

This solution is unique if and only if either

$$\int_0^\infty \exp\Big(-\int_0^t p(s)ds\Big)dt = \infty \quad \text{or} \quad \int_0^\infty q(t) \, \exp\Big(\int_0^t p(s)ds\Big)dt = \infty . \tag{5}$$

Finally, all solutions of (1), (2) *satisfy*

$$h(t) \longrightarrow 0 \quad as \quad t \longrightarrow \infty \tag{6}$$

if and only if

$$\int_0^\infty q(t)\Big[\exp\int_0^t p(r)dr\Big]\Big\{\int_0^t \exp\Big(-\int_0^s p(r)dr\Big)ds\Big\}\,dt = \infty\,; \tag{7}$$

in which case, $h(t)$ is unique.

REMARK. Note that if

$$q(t) \geqq \text{const.} > 0 \quad \text{and} \quad p(t) \geqq 0$$

for large t, then (5) and (7) hold, so that $h(t)$ is unique and satisfies (6).

The first part of this lemma concerning existence is due to A. Kneser; the part on uniqueness to Hartman and Wintner; and the part on (6)–(7) to Weyl; cf., e.g., [4, Chapter XI, § 6].

LEMMA 2A. *Let $P(t)$, $Q(t)$ be real-valued, continuous functions for $t \geqq 0$ such that*

$$H'' + P(t)H' - Q(t)H = 0 \tag{8}$$

has a solution satisfying

$$H(0) = 1,\ H > 0 \ \ and \ \ H' \leqq 0 \qquad for\ t \geqq 0\,. \tag{9}$$

Let $p(t)$, $q(t)$ be continuous for $t \geqq 0$ and

$$p(t) \geqq P(t)\,, \quad q(t) \geqq Q(t)\,. \tag{10}$$

Then (1) *has solution satisfying* (2) *and*

$$0 < h \leqq H \ \ and \ \ h'/h \leqq H'/H \leqq 0 \qquad for\ t \geqq 0\,. \tag{11}$$

This result can be obtained by introducing the new variable $v = h/H(t)$ in (1). By virtue of (9) and (10), Lemma 1A is applicable to the resulting differential equation for v,

$$v'' + (p + 2H'/H)v' - [(q-Q) - (p-P)H'/H]v = 0\,. \tag{12}$$

Hence there is a solution $v(t)$ satisfying

$$v(0) = 1,\ v > 0 \ \text{ and } \ v' \leqq 0 \qquad \text{for } t \geqq 0$$

This implies (11). Note that $v'(t) < 0$ for $t \geqq 0$ unless the coefficient of v in (12) vanishes for large t. In case $v'(t) < 0$ for $t \geqq 0$, (11) becomes

$$0 < h < H \ \text{ and } \ h'/h < H'/H \qquad \text{for } t > 0\,. \tag{13}$$

LEMMA 3A. *Let $q_n(t) \geqq 0$, $p_n(t)$ be continuous, real-valued for $t \geqq 0$, $n = 1, 2, \cdots$ and $n = \infty$ such that*

$$p_n \longrightarrow p_\infty \text{ and } q_n \longrightarrow q_\infty \quad as \quad n \longrightarrow \infty \tag{14}$$

uniformly on bounded t-intervals. For $n = 1, 2, \cdots$, *let* $h = h_n(t)$ *be the principal solution of*

$$h'' + p_n h' - q_n h = 0 \tag{15_n}$$

satisfying $h(0) = 1$, *hence* (2). *Then there exists a sequence of positive integers* $n(1) < n(2) < \cdots$ *such that*

$$h_\infty(t) = \lim_{k\to\infty} h_{n(k)}(t), \;\; h_\infty'(t) = \lim_{k\to\infty} h_{n(k)}'(t)$$

exist uniformly on bounded t-intervals and $h = h_\infty(t)$ *is a solution of* (15_∞) *satisfying* (2). *In particular, a selection of a subsequence is unnecessary if* (2) *determines a unique solution of* (15_∞).

This is a result of Hartman and Wintner; cf. [4, p. 360].

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