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**CONDITIONS FOR COUNTABLE BASES IN SPACES OF
COUNTABLE AND POINT-COUNTABLE TYPE**

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CONDITIONS FOR COUNTABLE BASE IN SPACES OF COUNTABLE AND POINT-COUNTABLE TYPE

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A space X is of countable type if for every compact $C \subset X$, there exists a compact set K having a countable basis with $C \subset K$. X is of point-countable type if there exists a covering of compact subsets of X , each having a countable basis. It is shown that in a Hausdorff space of countable type, a compact set has a countable basis if and only if it is a G_δ -set. Similarly, for Hausdorff spaces of point-countable type, a point has a countable basis if and only if it is a G_δ -set.

1. Terminology. Notation and terminology will follow that of Dugundji [2]. By a neighborhood of a set A , we will mean an open set containing A .

If X is a space and $A \subset X$, a collection \mathcal{S} of neighborhoods of A is called a *basis at A* if and only if for every neighborhood 0 of A , there exists $D \in \mathcal{S}$ with $A \subset D \subset 0$.

If X is a space and $A \subset X$, then A is said to be of *countable character* if and only if there exists a countable basis at A .

A space X is said to be of *countable type* if for every compact $C \subset X$, there exists a compact set K of countable character with $C \subset K$.

A space X is said to be of *point-countable type* if there exists a covering of compact subsets of X , each having countable character.

2. Discussion and theorems. Every first countable space, as well as every locally compact Hausdorff space, is of point-countable type, while spaces of point-countable type are, in turn, k -spaces. Compact spaces are trivially of countable type, but these two concepts are fairly far removed from each other since a metric space is of countable type.

The following lemmas will be needed. Lemma 2, which was first noted by Arhangel'skii [1], can be verified by a slight modification of Wicke's proof of Lemma 1. The author is indebted to Howard Cook for some valuable suggestions.

LEMMA 1. (Wicke). *In a Hausdorff space X , the following properties are equivalent:*

- (i) X is of point-countable type.
- (ii) If 0 is an open set in X and $x \in 0$, there exists a compact set B of countable character such that $x \in B$ and $B \subset 0$.

LEMMA 2. (Arhangel'skii). *Suppose X is a Hausdorff space of*

countable type, U is an arbitrary compact subset, and 0 is any of its neighborhoods. Then there exists a compact set C of countable character such that $U \subset C \subset 0$.

LEMMA 3. Let X be a Hausdorff space and let U and V be compact subsets of countable character. Then $U \cap V$ is also a compact set of countable character.

Proof. That $U \cap V$ is compact is obvious. Denote the members of the countable bases at U and V by U_n and V_n , respectively, and assume that the collections $\{U_n\}$ and $\{V_n\}$ are descending. It will be shown that the collection $\{U_n \cap V_n\}$ forms a local basis at $U \cap V$. Thus, let 0 be any neighborhood of $U \cap V$. Then $U - 0$ and $V - 0$ are disjoint compact sets, and hence there exist disjoint open sets U^* and V^* with $U - 0 \subset U^*$ and $V - 0 \subset V^*$. Since $U^* \cup 0$ is a neighborhood of U , there exists an integer m with $U \subset U_m \subset U^* \cup 0$. Similarly, there exists an integer n with $V \subset V_n \subset V^* \cup 0$. Letting $k = \max\{m, n\}$, it follows that $U \cap V \subset U_k \cap V_k \subset 0$; for if this is not true then there must exist a point $p \in U_k \cap V_k - 0$ which implies that $p \in U^* \cap V^*$, contradicting the disjointness of U^* and V^* .

For $n \geq 1$, it follows from Lemma 2 that there exists a compact set C'_n of countable character such that $U \subset C'_n \subset G_n$. Let $C_n = \bigcap_{i=1}^n C'_i$. By Lemma 3, each C_n is also a compact set of countable character.

THEOREM 1. Let X be a Hausdorff space of countable type and let U be any compact subset which is also a G_δ -set. Then U has a countable basis.

Proof. By hypothesis, there exist neighborhoods G_n of U such that $U = \bigcap_{n=1}^\infty G_n$. Construct a sequence $\{C_n\}$ of compact sets in the following manner:

By Lemma 2, there exists a compact set C_1 of countable character such that $U \subset C_1 \subset G_1$. For $n > 1$, it also follows from Lemma 2 that there exists a compact set C'_n of countable character such that $U \subset C'_n \subset G_n$. Let $C_n = [\bigcap_{i=1}^{n-1} C_i] \cap C'_n$. From a previous remark, each C_n is also a compact set of countable character.

Let $\{U_{m,n}\}$ be a countable basis at C_m . Clearly, $U \subset U_{m,n}$ for every pair (m, n) , and furthermore, $U \subset \bigcap_{m,n} U_{m,n} \subset \bigcap_{n=1}^\infty G_n = U$. Hence, $\bigcap_{m,n} U_{m,n} = U$. It will now be shown that the collection $\{U_{m,n}\}$ is a basis at U . Indeed, if it is not, then there exists a neighborhood K of U such that $U_{m,n} - K \neq \phi$ for every pair (m, n) . This forces $C_m - K \neq \phi$ for each integer m because, if not, then $C_m \subset K$ for some m , and hence there exists an integer n with $C_m \subset U_{m,n} \subset K$ which is contrary to our assumption. Since $C_m - K$ is a decreasing sequence of

nonempty compact sets, $\bigcap_{m=1}^{\infty} [C_m - K] \neq \phi$. But if $p \in \bigcap_{m=1}^{\infty} [C_m - K]$, then $p \in \bigcap_{m,n} U_{m,n}$ which implies that $p \in U$. This is impossible since $p \in X - K$ and $U \subset K$. Thus, $\{U_{m,n}\}$ is a basis at U , and the theorem is proved.

COROLLARY. *In a Hausdorff space of countable type, a compact set has a countable basis if and only if it is a G_δ -set.*

THEOREM 2. *Let X be a Hausdorff space of point-countable type, and let $p \in X$ be any point which is a G_δ -set. Then p has a countable basis.*

Proof. In the proof of Theorem 1, use Lemma 1 instead of Lemma 2 and substitute "point p " in place of U .

COROLLARY. *A Hausdorff space is first countable if and only if it is of point countable type and each point is a G_δ -set.*

COROLLARY. *A locally compact Hausdorff space is first countable if and only if each point is a G_δ -set.*

REFERENCES

1. A. V. Arhangel'skii, *On a class of spaces containing all metric and all locally bicomact spaces*, Soviet Math. Dokl. **4** (1963), 1051-1055.
2. James Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1966.
3. H. H. Wicke, *On the Hausdorff open continuous images of Hausdorff paracompact p -spaces*, Proc. Amer. Math. Soc. **22** (1969), 136-140.

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| | |
|--|-----|
| Norman Larrabee Alling, <i>Analytic and harmonic obstruction on nonorientable Klein surfaces</i> | 1 |
| Shimshon A. Amitsur, <i>Embeddings in matrix rings</i> | 21 |
| William Louis Armacost, <i>The Frobenius reciprocity theorem and essentially bounded induced representations</i> | 31 |
| Kenneth Paul Baclawski and Kenneth Kapp, <i>Topisms and induced non-associative systems</i> | 45 |
| George M. Bergman, <i>The index of a group in a semigroup</i> | 55 |
| Simeon M. Berman, <i>Excursions above high levels for stationary Gaussian processes</i> | 63 |
| Peter Southcott Bullen, <i>A criterion for n-convexity</i> | 81 |
| W. Homer Carlisle, III, <i>Residual finiteness of finitely generated commutative semigroups</i> | 99 |
| Roger Clement Crocker, <i>On the sum of a prime and of two powers of two</i> | 103 |
| David Eisenbud and Phillip Alan Griffith, <i>The structure of serial rings</i> | 109 |
| Timothy V. Fossum, <i>Characters and orthogonality in Frobenius algebras</i> | 123 |
| Hugh Gordon, <i>Rings of functions determined by zero-sets</i> | 133 |
| William Ray Hare, Jr. and John Willis Kenelly, <i>Characterizations of Radon partitions</i> | 159 |
| Philip Hartman, <i>On third order, nonlinear, singular boundary value problems</i> | 165 |
| David Michael Henry, <i>Conditions for countable bases in spaces of countable and point-countable type</i> | 181 |
| James R. Holub, <i>Hilbertian operators and reflexive tensor products</i> | 185 |
| Robert P. Kaufman, <i>Lacunary series and probability</i> | 195 |
| Erwin Kreyszig, <i>On Bergman operators for partial differential equations in two variables</i> | 201 |
| Chin-pi Lu, <i>Local rings with noetherian filtrations</i> | 209 |
| Louis Edward Narens, <i>A nonstandard proof of the Jordan curve theorem</i> | 219 |
| S. P. Philipp, Victor Lenard Shapiro and William Hall Sills, <i>The Abel summability of conjugate multiple Fourier-Stieltjes integrals</i> | 231 |
| Joseph Earl Valentine and Stanley G. Wayment, <i>Wilson angles in linear normed spaces</i> | 239 |
| Hoyt D. Warner, <i>Finite primes in simple algebras</i> | 245 |
| Horst Günter Zimmer, <i>An elementary proof of the Riemann hypothesis for an elliptic curve over a finite field</i> | 267 |