Pacific Journal of Mathematics

CONDITIONS FOR COUNTABLE BASES IN SPACES OF COUNTABLE AND POINT-COUNTABLE TYPE

DAVID MICHAEL HENRY

Vol. 36, No. 1 November 1971

PACIFIC JOURNAL OF MATHEMATICS
Vol. 36, No. 1, 1971

CONDITIONS FOR COUNTABLE BASE IN SPACES OF COUNTABLE AND POINT-COUNTABLE TYPE

MICHAEL HENRY

A space X is of countable type if for every compact $C \subset X$, there exists a compact set K having a countable basis with $C \subset K$. X is of point-countable type if there exists a covering of compact subsets of X, each having a countable basis. It is shown that in a Hausdorff space of countable type, a compact set has a countable basis if and only if it is a G_δ-set. Similarly, for Hausdorff spaces of point-countable type, a point has a countable basis if and only if it is a G_δ-set.

1. Terminology. Notation and terminology will follow that of Dugundji [2]. By a neighborhood of a set A, we will mean an open set containing A.

If X is a space and $A \subset X$, a collection $\mathscr{D}$ of neighborhoods of A is called a *basis at* A if and only if for every neighborhood 0 of A, there exists $D \in \mathscr{D}$ with $A \subset D \subset 0$.

If X is a space and $A \subset X$, then A is said to be of *countable character* if and only if there exists a countable basis at A.

A space X is said to be of *countable type* if for every compact $C \subset X$, there exists a compact set K of countable character with $C \subset K$.

A space X is said to be of *point-countable type* if there exists a covering of compact subsets of X, each having countable character.

2. Discussion and theorems. Every first countable space, as well as every locally compact Hausdorff space, is of point-countable type, while spaces of point-countable type are, in turn, k-spaces. Compact spaces are trivially of countable type, but these two concepts are fairly far removed from each other since a metric space is of countable type.

The following lemmas will be needed. Lemma 2, which was first noted by Arhangel'skiĭ [1], can be verified by a slight modification of Wicke's proof of Lemma 1. The author is indebted to Howard Cook for some valuable suggestions.

LEMMA 1. (*Wicke*). *In a Hausdorff space X, the following properties are equivalent:*

(i) *X is of point-countable type.*

(ii) *If 0 is an open set in X and $x \in 0$, there exists a compact set B of countable character such that $x \in B$ and $B \subset 0$.*

LEMMA 2. (*Arhangel'skii*). *Suppose X is a Hausdorff space of* *countable type, U is an arbitrary compact subset, and* 0 *is any of its neighborhoods. Then there exists a compact set C of countable character such that* $U \subset C \subset 0$.

LEMMA 3. *Let X be a Hausdorff space and let U and V be compact subsets of countable character. Then $U \cap V$ is also a compact set of countable character.*

Proof. That $U \cap V$ is compact is obvious. Denote the members of the countable bases at U and V by U_n and V_n, respectively, and assume that the collections $\{U_n\}$ and $\{V_n\}$ are descending. It will be shown that the collection $\{U_n \cap V_n\}$ forms a local basis at $U \cap V$. Thus, let 0 be any neighborhood of $U \cap V$. Then $U - 0$ and $V - 0$ are disjoint compact sets, and hence there exist disjoint open sets U^* and V^* with $U - 0 \subset U^*$ and $V - 0 \subset V^*$. Since $U^* \cup 0$ is a neighborhood of U, there exists an integer m with $U \subset U_m \subset U^* \cup 0$. Similarly, there exists an integer n with $V \subset V_n \subset V^* \cup 0$. Letting $k = \max\{m, n\}$, it follows that $U \cap V \subset U_k \cap V_k \subset 0$; for if this is not true then there must exist a point $p \in U_k \cap V_k - 0$ which implies that $p \in U^* \cap V^*$, contradicting the disjointness of U^* and V^*.

For $n \geqq 1$, it follows from Lemma 2 that there exists a compact set C'_n of countable character such that $U \subset C'_n \subset G_n$. Let $C_n = \bigcap_{i=1}^{n} C'_i$. By Lemma 3, each C_n is also a compact set of countable character.

THEOREM 1. *Let X be a Hausdorff space of countable type and let U be any compact subset which is also a G_δ-set. Then U has a countable basis.*

Proof. By hypothesis, there exist neighborhoods G_n of U such that $U = \bigcap_{n=1}^{\infty} G_n$. Construct a sequence $\{C_n\}$ of compact sets in the following manner:

By Lemma 2, there exists a compact set C_1 of countable character such that $U \subset C_1 \subset G_1$. For $n > 1$, it also follows from Lemma 2 that there exists a compact set C'_n of countable character such that $U \subset C'_n \subset G_n$. Let $C_n = [\bigcap_{i=1}^{n-1} C_i] \cap C'_n$. From a previous remark, each C_n is also a compact set of countable character.

Let $\{U_{m,n}\}$ be a countable basis at C_m. Clearly, $U \subset U_{m,n}$ for every pair (m, n), and furthermore, $U \subset \bigcap_{m,n} U_{m,n} \subset \bigcap_{n=1}^{\infty} G_n = U$. Hence, $\bigcap_{m,n} U_{m,n} = U$. It will now be shown that the collection $\{U_{m,n}\}$ is a basis at U. Indeed, if it is not, then there exists a neighborhood K of U such that $U_{m,n} - K \neq \phi$ for every pair (m, n). This forces $C_m - K \neq \phi$ for each integer m because, if not, then $C_m \subset K$ for some m, and hence there exists an integer n with $C_m \subset U_{m,n} \subset K$ which is contrary to our assumption. Since $C_m - K$ is a decreasing sequence of nonempty compact sets, $\bigcap_{m=1}^{\infty}[C_m - K] \neq \phi$. But if $p \in \bigcap_{m=1}^{\infty}[C_m - K]$, then $p \in \bigcap_{m,n} U_{m,n}$ which implies that $p \in U$. This is impossible since $p \in X - K$ and $U \subset K$. Thus, $\{U_{m,n}\}$ is a basis at U, and the theorem is proved.

COROLLARY. *In a Hausdorff space of countable type, a compact set has a countable basis if and only if it is a G_δ-set.*

THEOREM 2. *Let X be a Hausdorff space of point-countable type, and let $p \in X$ be any point which is a G_δ-set. Then p has a countable basis.*

Proof. In the proof of Theorem 1, use Lemma 1 instead of Lemma 2 and substitute "point p" in place of U.

COROLLARY. *A Hausdorff space is first countable if and only if it is of point countable type and each point is a G_δ-set.*

COROLLARY. *A locally compact Hausdorff space is first countable if and only if each point is a G_δ-set.*

REFERENCES

1. A. V. Arhangel'skii, *On a class of spaces containing all metric and all locally bicompact spaces*, Soviet Math. Dokl. **4** (1963), 1051-1055.
2. James Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1966.
3. H. H. Wicke, *On the Hausdorff open continuous images of Hausdorff paracompact p-spaces*, Proc. Amer. Math. Soc. **22** (1969), 136-140.

Received May 4, 1970, and in revised form June 26, 1970. This research was supported by a TCU Research Fellowship and represents a portion of the author's doctoral dissertation, which was begun under the direction of the late Professor H. Tamano and completed under D. R. Traylor and Howard Cook.

TEXAS CHRISTIAN UNIVERSITY

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 36, No. 1 November, 1971